Stability Analysis of Mathematical Model Comparing Solute Kinetics in Low & High Hemodialysis Patients

R. Hussain, A. Ali, N. Arif

Abstract. This paper is about the stability analysis of model, which we have taken from “The mathematical model comparing solute kinetics in low- and high-BMI hemodialysis patients” [2]. The purpose of this study is to check the stability of three types of patients i.e small medium and large during dialysis and in between the dialysis treatment. In all cases we get the stable solution for above model presented in [2].

Index Terms—Stability analysis, hemodialysis, mathematical modeling.

I. INTRODUCTION

The model is a four compartmental model taken from [2]. The block diagram of model is given below:

\[ \frac{dx_m}{dt} = G - \frac{dx_m}{dt} \text{ (1)} \]

\[ \frac{dx_e}{dt} = \frac{dx_m}{dt} \text{ (2)} \]

\[ \frac{dx_e}{dt} = \frac{dx_m}{dt} \text{ (3)} \]

\[ \frac{dx_e}{dt} = \frac{dx_m}{dt} \text{ (4)} \]

The above set of equations can be written as:

\[ \frac{dx_m}{dt} = G - \frac{dx_m}{dt} \text{ (5)} \]

\[ \frac{dx_e}{dt} = \frac{dx_m}{dt} \text{ (6)} \]

\[ \frac{dx_e}{dt} = \frac{dx_m}{dt} \text{ (7)} \]

\[ \frac{dx_e}{dt} = \frac{dx_m}{dt} \text{ (8)} \]

Where

\[ \text{R clearance} = \begin{cases} & \text{ during dialysis} \\ & \text{ between dialysis} \end{cases} \]

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</thead>
<tbody>
<tr>
<td>35</td>
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<td>35</td>
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</tbody>
</table>

Table 1: Initial Condition

<table>
<thead>
<tr>
<th>R clearance</th>
<th>R(1)</th>
<th>R(2)</th>
<th>R(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.045</td>
<td>0.03177</td>
<td>0.0193</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 2: Rate Constants

<table>
<thead>
<tr>
<th>Size</th>
<th>( v_e )</th>
<th>( v_{in} )</th>
<th>( v_{out} )</th>
<th>( v_{sc} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large</td>
<td>58</td>
<td>21</td>
<td>6.54</td>
<td>4.46</td>
</tr>
<tr>
<td>Medium</td>
<td>40.6</td>
<td>17.22</td>
<td>2.93</td>
<td>2.25</td>
</tr>
<tr>
<td>Small</td>
<td>23.2</td>
<td>11.22</td>
<td>1.486</td>
<td>0.034</td>
</tr>
</tbody>
</table>

Table of Parameter

For stability analysis of model, we divide it into two sections. Section 1 is for the inter-dialytic interval. And in second section we determine the Stability analysis for dialytic interval.

1. Stability of Model in between dialysis (\( \theta = 0 \))

1.1 LARGE PATIENTS

First we choose equation (5) and (8)

\[ \frac{dx_m}{dt} = \frac{dx_m}{dt} \text{ (9)} \]

\[ \frac{dx_e}{dt} = \frac{dx_m}{dt} \text{ (10)} \]

After putting values from table in equations (9) and (10), we get

Critical points (\( \theta = 0 \)) = (16.098, 9.420)

Eigenvalue (-0.0032, -0.0006)

As both eigenvalues are –ve so the solution is stable and is presented in Fig 1.1.
Secondly we choose equations (6) and (8)

\[
\begin{align*}
\text{Equation (6)} & : \frac{dc_m}{dt} = c_m v_m \left( \frac{k_{in} c_{in}}{K_{m} + c_{in}} + k_{out} c_{out} - k_{rec} c_m \right) \\
\text{Equation (8)} & : \frac{dc_{out}}{dt} = -c_{out} v_m \left( \frac{k_{in} c_{in}}{K_{m} + c_{in}} + k_{out} c_{out} - k_{rec} c_m \right) - \frac{1}{v_m} (k_{in} c_{in} + k_{out} c_{out} + k_{rec} c_m) c_{out} + k_{rec} c_m 
\end{align*}
\]

After putting values from table in equations (11) and (12), we get Critical points \((c_{mst}, c_{oout}) = (34.964, 34.964)\)

Eigenvalue \((-0.0055, -0.0010)\)

Both eigenvalues are –ve so solution is stable and is presented in Fig 1.2.

Third we choose equations (7) and (8)

\[
\begin{align*}
\text{Equation (7)} & : \frac{dc_m}{dt} = c_m v_m \left( \frac{k_{in} c_{in}}{K_{m} + c_{in}} + k_{out} c_{out} - k_{rec} c_m \right) \\
\text{Equation (8)} & : \frac{dc_{out}}{dt} = -c_{out} v_m \left( \frac{k_{in} c_{in}}{K_{m} + c_{in}} + k_{out} c_{out} - k_{rec} c_m \right) - \frac{1}{v_m} (k_{in} c_{in} + k_{out} c_{out} + k_{rec} c_m) c_{out} + k_{rec} c_m 
\end{align*}
\]

After putting values from table in equations (13) and (14), we get Critical points \((c_{mst}, c_{oout}) = (20.421, 20.421)\)

Eigenvalue \((-0.0048, -0.0012)\)

Again both eigenvalues are –ve so solution is stable and is presented in Fig 1.3

The same pair of equations have been used for medium and small patients.

1.2 MEDIUM PATIENTS

After putting values from table in equations (9) and (10), we get Critical points \((c_{mst}, c_{oout}) = (47.500, 40.825)\)

Eigenvalue \((-0.0042, -0.0008)\)

Both eigenvalues are –ve so the solution is stable and is presented in Fig 1.4.

After putting values from table in equations (11) and (12), we get Critical points

\[(c_{mst}, c_{oout}) = (34.862, 34.862)\]

Eigenvalue \((-0.0117, -0.0015)\)

As Both eigenvalues are –ve so the solution is stable and is presented in Fig 1.5.

After putting values from table in equations (13) and (14), it gives Critical points \((c_{mst}, c_{oout}) = (35.016, 35.016)\)

Eigenvalue \((-0.0092, -0.0018)\)

Both eigenvalues are –ve so the solution is stable and is presented in Fig 1.6.
1.3 SMALL PATIENTS

After putting values from table in equations (9) and (10), it gives Critical points $(c_{min}, c_{max}) = (47.567, 40.9)$

Eigenvalue $(-0.0013, -0.0068)$

Again all eigenvalues are $–ve$ so the solution is stable and is presented in Fig 1.7

After putting values from table in equations (11) and (12), it gives Critical points $(c_{min}, c_{max}) = (35.022, 35.022)$

Eigenvalue $(-0.0229, -0.0026)$

As both eigenvalues are $–ve$ so the solution is stable and is presented in Fig 1.8

After putting values from table in equations (13) and (14), it gives Critical points $(c_{min}, c_{max}) = (34.991, 34.991)$

Eigenvalue $(-0.5685, -0.0033)$

Again all eigenvalues are $–ve$ so the solution is stable and is presented in Fig 1.9

II. STABILITY OF MODEL DURING DIALYSIS ($\theta = 1$)

Using the same pair of equations which we have already used for in between dialysis for three cases (large, medium and small).

2.1 LARGE PATIENTS

After putting values from table in equations (9) and (10), it
Stability Analysis of Mathematical Model Comparing Solute Kinetics in Low & High Hemodialysis Patients

gives Critical points \((c_{\text{low}}, c_{\text{e}}) = (20.535, 13.858)\)

Eigenvalue \((-0.0013, -0.0042)\)

Solution of above equations are stable as eigenvalues are \(-ve\) and is presented in Fig 2.1

After putting values from table in equations (11) and (12), it gives Critical points \((c_{\text{low}}, c_{\text{e}}) = (13.714, 13.714)\)

Eigenvalue \((-0.0059, -0.0023)\)

Solution of above pair of equations is again stable due to \(-ve\) eigenvalues and is presented in Fig 2.2

After putting values from table in equations (13) and (14), it gives Critical points \((c_{\text{low}}, c_{\text{e}}) = (8.905, 8.905)\)

Eigenvalue \((-0.0051, -0.0026)\)

Again stable solution due to \(-ve\) eigenvalues and is presented in Fig 2.3

2.2 MEDIUM PATIENTS

After putting values from table in equations (9) and (10), it gives Critical points

\((c_{\text{med}}, c_{\text{e}}) = (20.502, 13.828)\)

Eigenvalue \((-0.0017, -0.0058)\)

Due to \(-ve\) eigenvalues solution of above model is stable and is shown in Fig 2.4

After putting values from table in equations (11) and (12), it gives Critical points

\((c_{\text{med}}, c_{\text{e}}) = (13.686, 13.686)\)

Eigenvalue \((-0.0120, -0.0037)\)

Again solution presented in Fig 2.5 is stable due to \(-ve\) eigenvalues

After putting values from table in equations (13) and (14), it gives Critical points \((c_{\text{med}}, c_{\text{e}}) = (15.214, 15.214)\)

Eigenvalue \((-0.0095, -0.0039)\)

Stable solution presented in Fig 2.6
Figure 2.5: Stability of \( \phi_{old} \) vs \( \phi_{new} \) for medium patients

2.3 SMALL PATIENTS

After putting values from table in equations (9) and (10), it gives Critical points

\[ (\phi_{old}, \phi_{new}) = (20.488, 13.822) \]

Eigenvalue \((-0.0027, -0.0098)\)

Stable solution presented in Fig 2.7 for small patients.

After putting values from table in equations (11) and (12), it gives Critical points

\[ (\phi_{old}, \phi_{new}) = (13.702, 13.702) \]

Eigenvalue \((-0.0233, -0.0065)\)

Stable solution for small patients is presented in Fig 2.8

After putting values from table in equations (13) and (14), it gives Critical points

\[ (\phi_{old}, \phi_{new}) = (15.199, 15.199) \]

Eigenvalue \((-0.5685, -0.0076)\)

Stable solution for small patients is presented in Fig 2.9

Figure 2.6: Stability of \( \phi_{old} \) vs \( \phi_{new} \) for medium patients

Figure 2.7: Stability of \( \phi_{old} \) vs \( \phi_{new} \) for small patients

Figure 2.8: Stability of \( \phi_{old} \) vs \( \phi_{new} \) for small patients

Figure 2.9: Stability of \( \phi_{old} \) vs \( \phi_{new} \) for small patients
III. CONCLUSION

Stability analysis of the model presented in [2] was done for different types of patients (large, medium, and small) during dialysis and in between the dialysis. We conclude that in both cases the model solution was stable.

REFERENCES


[6] P. Howard, Modelling with ODE, Available at www.math.tamu.edu/~phoward/M442.html


