Complete Cototal Domination Number of Middle Graphs

K. Uma Samundesvari, J. Maria Regila Baby

Abstract: A total dominating set $D$ is said to be a complete cototal dominating set if the $(V-D)$ has no isolated nodes. The complete cototal domination number $\gamma_{cctd}(G)$ is the minimum cardinality of a complete cototal dominating set of $G$. Our aim is to determine the Complete Cototal Domination Number of Middle Graphs and its bounds.

Keywords: Domination number, Total domination number, Cototal domination number, Complete cototal domination number, Middle graph. MSC 2010 subject classification : 05C69

I. INTRODUCTION

Domination theory in graph was established by Claude Berge around 1960's with the problem of placing minimum number of queens on a $n \times n$ chess board to dominate every square by at least one queen. After that Oystein Ore established the concept dominating set and domination number [5]. A set $S$ of nodes of $G$ is a dominating set of $G$ if each node of $G$ is dominated by some node in $S$. The total domination in graphs which was presented by Cockayne, Dawes, and Hedetniemi [2,3]. A subset $D$ of $V$ is called a dominating set of $G$ if every node not in $D$ is adjacent to some node in $D$. A total dominating set for a graph $G$ is a dominating set $M$ for $G$ with the property that every node in $M$ has a neighbor in $G$. Note that total dominating sets are not defined for graphs with isolated nodes. The concept of cototal dominating set was presented by Kulli, Janakiram and Iyer [4]. This concept motivate us to do research under this topic. Throughout this paper we considered a simple connected graph the total number of nodes and edges are denoted by $p$ and $q$ respectively.

II. DEFINITIONS

Definition 2.1
A total dominating set $D$ is said to be a complete cototal dominating set ($\gamma_{cctd}$) if the $(V-D)$ has no isolated nodes. The complete cototal domination number $\gamma_{cc}(G)$ is the minimum cardinality of a complete cototal dominating set of $G$ [1].

Definition 2.2
Let $G$ be a connected graph. A subdivision graph $S(G)$ is said to be a middle graph $M(G)$ if the middle nodes lies on adjacent edges of $G$ should be adjacent.

III. MAIN RESULTS

Theorem 3.1 For a Path graph $P_n$, $\gamma_{cc}(M(P_n)) = \begin{cases} 3 & \text{if } n = 2 \\ 2n - 1 & \text{if } n \geq 3 \end{cases}$

Proof. The Middle Path graph $M(P_n)$ has $(2n - 1)$ nodes $v_1, v_2, ..., v_n, u_1, u_2, ..., u_{n-1}$ and $2n - 2$ edges. Here $u_1, u_2, ..., u_{n-1}$ be the middle nodes. Case (i) $n = 2$

The Middle Path graph $M(P_n)$ has three nodes $v_1, u_2, v_2$ and two edges $v_1u_1, u_1v_2$. Let us consider the total dominating set $\gamma_{td}(M(P_2)) = \{v_1, u_1\}$. Minimal cototal dominating set is obtained by $(V(M(P_2)) - \{v_1, u_1\}) \cap \{y\}$ where $y$ is $v_1$ or $v_2$ is an isolated node. Hence $\gamma_{cctd}(M(P_2)) = \{v_1, u_1\} \cup \{y\}$. Therefore $\gamma_{cc}(M(P_2)) = 3$.

Case (ii) $n \geq 3$

The Middle Path graph $M(P_n)$ has $(2n - 1)$ nodes and $(2n - 2)$ edges. Let us consider the total dominating set $\gamma_{td}(M(P_n)) = \{u_1, u_2, ..., u_{n-1}\}$. Minimal cototal dominating set is obtained by $(V(M(P_n)) - \{u_1, u_2, ..., u_{n-1}\}) \cap \{v_1, v_2, ..., v_n\}$. Hence $\gamma_{cctd}(M(P_n)) = \{u_1, u_2, ..., u_{n-1}\} \cup \{v_1, v_2, ..., v_n\}$. Therefore $\gamma_{cc}(M(P_n)) = 2n - 1$.

Theorem 3.2 For a Cycle graph $C_n$, $\gamma_{cc}(M(C_n)) = 2n - 3, n \geq 3$.

Proof. The Middle Cycle graph $M(C_n)$ has $2n$ nodes $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n$ and $3n$ edges. Here $v_1, v_2, ..., v_n$ be the outer nodes on cycle $C_n$ and $u_1, u_2, ..., u_n$ be the middle nodes on cycle $C_n$. Let us consider the total dominating set $\gamma_{td}(M(C_n)) = \{u_1, u_2, ..., u_{n-1}\}$. Minimal cototal dominating set is obtained by $(V(M(C_n)) - u_1u_2, ..., u_{n-1}u_{n-1}) \cap \{y\}$ where $y$ are the isolated nodes. Hence $\gamma_{cctd}(M(C_n)) = \{u_1, u_2, ..., u_{n-1}\} \cup \{y\}$. Therefore $\gamma_{cc}(M(P_n)) = 2n - 3$.

Theorem 3.3 For a Comb graph $P_n \otimes K_1$, $\gamma_{cc}(M(P_n \otimes K_1)) = 4n - 1, n \geq 2$.

Proof. The Middle Comb graph $M(P_n \otimes K_1)$ has $(4n - 1)$ nodes $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n, z_1, z_2, ..., z_{n-1}, x_1, x_2, ..., x_n$ and $(6n - 4)$ edges. Here $v_1, v_2, ..., v_n$ be the nodes on $P_n$ and $u_1, u_2, ..., u_n$ be the pendant nodes and $z_1, z_2, ..., z_{n-1}$ be the middle nodes on $P_n$ and $x_j$’s are the middle nodes on $v_ju_j$, $1 \leq i \leq n$. Let us consider the total dominating set $\gamma_{td}(M(P_n \otimes K_1)) = \{x_1, x_j\}$ where $1 \leq i \leq n, 1 \leq j \leq n - 1$. 

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Minimal cototal dominating set is obtained by \((V(M(P_n \odot K_1)) - \{x_i, z_j\}) \cap \{y\}\) where \(y\) are isolated nodes with cardinality \(2n\) so that \(\gamma_{c\text{cd}}(M(P_n \odot K_1)) = \{x_i, z_j\} \cup \{y\}\). Therefore \(\gamma_{c}(M(P_n \odot K_1)) = 4n - 1\).

**Theorem:** For a \(n\)-sunlet graph \(\gamma_{c}(M(n - \text{sunlet})) = 4n - 3, n \geq 3\).

**Proof:** The middle \(M(n - \text{sunlet})\) graph has \(4n\) nodes \(u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\), \(z_1, z_2, ..., z_n, x_1, x_2, ..., x_n\) and 7\(n\) edges. Here \(u_1, u_2, ..., u_n\) be the pendant nodes and \(v_1, v_2, ..., v_n\) be the nodes on cycle \(C_n\) and \(x_1\) and \(z_1\) be the middle nodes of \(u_i v_i\) and cycle \(C_n\) respectively. Let us consider the total dominating set \(\gamma_{td}(M(n - \text{sunlet})) = \{z_1, z_2, ..., z_{n-1}, x_1, x_2, ..., x_n\}\). Minimal cototal dominating set is obtained by \((V(M(n - \text{sunlet})) - \{z_1, z_2, ..., z_{n-1}, x_1, x_2, ..., x_n\}) \cap \{y\}\) where \(y\) are isolated nodes. Therefore \(\gamma_{c\text{cd}}(M(n - \text{sunlet})) = \{z_1, z_2, ..., z_{n-1}, x_1, x_2, ..., x_n\} \cup \{y\}\) so that \(\gamma_{c}(M(n - \text{sunlet})) = 4n - 3\).

**Theorem:** For a \(n\)-pan graph, \(\gamma_{c}(M(n - \text{pan})) = 2n - 1, n \geq 3\).

**Proof:** The middle \(M(n - \text{pan})\) graph has \((2n + 2)\) nodes \(v_1, v_2, ..., v_n, u, u_1, u_2, ..., u_n\) and \((3n + 4)\) edges. Here \(v_1, v_2, ..., v_n\) be the nodes on cycle \(C_n\) and \(u_1, u_2, ..., u_n\) be the middle nodes on cycle \(C_n\) and \(v\) be the pendant node and \(u\) be the middle node on the pendant node. Let us consider the total dominating set \(\gamma_{td} = \{u, u_1, u_2, ..., u_{n-1}\}\). Minimal cototal dominating set is obtained by \((V(M(n - \text{pan})) - \{u, u_1, u_2, ..., u_{n-1}\}) \cap \{y\}\) where \(y\) are isolated nodes. Therefore \(\gamma_{c\text{cd}}(M(n - \text{pan})) = \{u, u_1, u_2, ..., u_{n-1}\} \cup \{y\}\). Hence \(\gamma_{c}(M(n - \text{pan})) = 2n - 1\).

IV. **BOUNDS FOR \(\gamma_{c}(M(G))\)**

**Theorem:** Let \(M(G)\) be a connected graph, then \(\gamma_{c}(M(G)) > \left\lceil \frac{n}{\Delta(M(G))} \right\rceil\).

**Proof:** Let \(S \subseteq V(M(G))\) be a \(\gamma_{c\text{cd}}\) set in \(G\). Every node in \(S\) dominates at most \(\Delta(M(G)) - 1\) nodes of \(V(M(G)) - S\) and dominate at least one of the nodes in \(S\). Hence, \(|S|\Delta(M(G)) - 1| + |S| > n\). Since, \(S\) is an arbitrary \(\gamma_{c\text{cd}}\) - set, then \(\gamma_{c}(M(G)) > \left\lceil \frac{n}{\Delta(M(G))} \right\rceil\).

**Theorem:** If \(M(G)\) is a connected graph with the girth of length \(g(M(G)) \geq 3\) and \(\delta(M(G)) \geq 2\), then \(\gamma_{c}(M(G)) > n - \left\lceil \frac{g(M(G))}{2} \right\rceil + 1\).

**Proof:** Let \(M(G)\) be a connected graph with \(g(M(G)) \geq 3\) and let \(C\) be a cycle of length \(g(M(G))\). Remove \(C\) from \(M(G)\) to form a graph \(M(G')\). Suppose an arbitrary node \(v \in V(M(G'))\). Since \(\delta(M(G)) \geq 2\), \(v\) has at least two neighbors say \(w\) and \(z\). Let \(w, z \in C\). If \(d(w, z) \geq 3\), then replacing the path from \(w\) to \(z\) on \(C\) with the path \(w, v, z\), which reduces the girth of \(M(G)\), a contradiction. If \(d(w, z) \leq 2\), then \(w, z, v\) are on either \(C_3\) or \(C_4\) in \(M(G)\), contradicting the hypothesis that \(g(M(G)) \geq 3\). Hence, no node in \(M(G')\) has two or more neighbours on \(C\). Since \(\delta(M(G')) \geq 2\), the graph \(M(G')\) has minimum degree at least \(\delta(M(G)) - 1 \geq 1\). Then \(M(G)\) has no isolated nodes. Now let \(S'\) be a \(\gamma_{c\text{cd}}\) set for \(C\). Then \(S = S' \cup V(M(G'))\) is a \(\gamma_{c\text{cd}}\) set for \(M(G)\). Hence, \(\gamma_{c}(M(G)) > n - \left\lceil \frac{g(M(G))}{2} \right\rceil + 1\).

**Theorem:** Let \(M(G)\) be a graph without isolated nodes. Then \(\gamma_{c}(M(G)) > \left\lceil \frac{n}{2} \right\rceil\).

**Proof:** Let \(D \subseteq V(M(G))\) be a \(\gamma_{c\text{cd}}\) - set. Since \(M(G)\) has no isolated nodes, every \(v \in D\) has at least one neighbor in \(V - D\). This means that \(V - D\) is also a complete cototal dominating set. If \(|D| < \left\lfloor \frac{n}{2} \right\rfloor\), then \(V - D\) is the smallest \(\gamma_{c\text{cd}}\) set, contradicting the choice of \(D\) as a \(\gamma_{c\text{cd}}\) set. Thus \(\gamma_{c}(M(G)) = |D| > \left\lfloor \frac{n}{2} \right\rfloor\).

**Theorem:** Let \(M(G)\) be a graph with \(\text{diam}(M(G)) \geq 1\), then \(\gamma_{c}(M(G)) > \delta(M(G)) + 1\).

**Proof:** Let \(z \in V(M(G))\) and \(\deg(z) = \delta(M(G))\). Since \(\text{diam}(M(G)) \geq 1\), then \(N(z)\) is a total dominating set for \(M(G)\). Now \(S = N(z) \cup \{z\}\) is a \(\gamma_{c\text{cd}}\) set for \(M(G)\) and \(|S| = \delta(M(G)) + 1\). Hence, \(\gamma_{c}(M(G)) > \delta(M(G)) + 1\).

**Theorem:** For any graph \(M(G)\), \(\gamma_{c}(M(G)) > n - \Delta(M(G))\).

**Proof:** Let \(M(G)\) be a \(\gamma_{c}\) - set of \(M(G)\). Let \(l\) be a node of maximum degree \(\Delta(M(G))\). Then \(l\) dominates \(N[r]\) and the nodes in \(V - N[r]\) dominate themselves. Hence \(V - N[r]\) is a \(\gamma_{c\text{cd}}\) - set of cardinality \(n - \Delta(M(G))\), so \(\gamma_{c}(M(G)) > n - \Delta(M(G))\).
V. CONCLUSION

In this paper, Complete Cototal Domination Number of Middle Path graph, Middle Cycle graph, Middle Comb graph, Middle $n$-sunlet graph, Middle $n - pan$ graph are found and new bounds for $\gamma_{cc}(M(G))$ are obtained.

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