Derivation of Fundamental Solution of Heat Equation by using Symmetry Reduction

Kahsay Godifey Wubneh, Teklay Hailay Tsegay

Abstract: The objective of this article is to present the fundamental solution of heat equation using symmetry of reduction which is associated with partial derivatives of heat equations through its initial conditions (ICs). To emphasize our main results, we also consider some important way of solving of partial differential equation. The main results of our paper are quite general in nature and yield a very large interesting fundamental solution of heat equation and it is used for problems of differential mathematics and mathematical physics special in the area of thermodynamics.

Keywords: Partial differential equation, Heat equation, fundamental solution of heat equation.

I. INTRODUCTION AND PRELIMINARY

Heat equation is superposition of solutions and therefore from a stock of simple solutions it is possible to build solutions to complex problems. Thus, the temperature distribution in a body can be considered to be due to the additive influence of the various external and boundary agents affecting the heat flow. There are in fact special solutions to the heat equation which are sufficiently fundamental that solutions to very broad categories of heat conduction problems can be written immediately in terms of these fundamental solutions to the differential equation. Even in cases where this is not possible, these solutions generally play an essential role in determining the solution. These basic solutions correspond to the temperature distribution due to an external "pulse" (i.e. an concentrated instant source) of heat as we increase its heat pressure.

We recall, an equation containing the derivatives or differentials of one or more dependent variables in relation to one or more independent variables is called a differential equation (abbreviated to DE). If the unknown function is a function of a variable, the differential equation is ordinary, otherwise it is partial. Note that the order of a partial differential equation is the degree of the highest order derivatives in the equation. For instance, if there are two independent variables (x, y), a partial differential equation of second order has the general form

$$F(u_{xx}, u_{yy}, u_{xy}, u_x, u_y, u, x, y) = 0 \qquad (1)$$

In Multi direction notation, these can be written as

$$F(D^2 u, Du, u, x, y) = 0 \text{ or } F(Du, u, x, y) = 0 \qquad (2)$$

When solving a partial differential equation, we will need value problems to get the particular solution. But what conditions do we need?

If we look at the heat equation, there is only a first time derivative of u. So it needs only one initial condition (i.e. an initial condition is a condition at $t = 0$) usually such a condition takes the form $u(x, 0) = f(x)$.

However; the heat equation contains a second derived from X and so we used two boundary conditions (i.e. boundary condition is a condition at two different values of t).

II. HEAT EQUATION

Heat Equation is generally defined as

$$u_t = \Delta u \qquad (3)$$

and the corresponding non homogeneous equation is

$$u_t = \Delta u - f \qquad (4)$$

Subject to appropriate initial and boundary conditions hence $t > 0$ and $x \in U$, where $U \in IR^n$ is open. The unknown is $u = u(x, t)$ and the Laplachian delta or symbolically Δ is taken with respect to the spatial variable,

$$x = (x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots\ldots.x_n) \qquad (5)$$

$$\Delta u = \Delta_x u = \sum_{i=1}^{n} u_{x_i x_i} \ (in\ the\ function\ f : u \times [0, \infty) \to \mathbb{R}\ is\ given) \ (6)$$

A guiding principle is any assertion about harmonic functions yields an analogues (but more complicated) statement about solution of the heat equation. Accordingly our development will largely parallel with the corresponding theory for Laplace's equation.

III. INTERPRETATION OF HEAT EQUATION

The heat equation also known as the diffusion equation describes in typical application the evolution in time of the density U of some quantity such as heat chemical concentration etc. ∂v is any smooth sub region the rate of change of the total energy within v equals the negative of the net flux through ∂v.

$$\frac{d}{dt} \int_u u dx = -\int_{\partial v} F. vds \ (where\ f\ is\ the\ flux\ density) \ (7)$$

$$u_t = -divF \ (Where\ div\ is\ the\ divergence\ of\ F) \qquad (8)$$

As V was arbitrary function in many situations F is proportional to the gradient of u but points in the opposite direction (since the flow is from the region of higher to lower concentration)

$$F = -\omega Du \qquad (9)$$

Substituting equation (3) in to (9), we obtain the partial differential equations

$$u_t = \omega div(Du) = \omega Du \qquad (10)$$

Which for $\omega = 1$, equation (3) is the heat equation and heat equation appears as well in the Brownian equation.

IV. FUNDAMENTAL SOLUTIONS OF HEAT EQUATION

The Fundamental Solution is the heart of the theory of infinite domain problems of a solution of partial derivatives of a differential equation.

The fundamental solution also has to do with bounded domains of the equations.

Observe that the heat equation is a linear equation. Therefore, if u and v are both solutions to the heat equation, then so is any linear combination of u and v. This fact will be used frequently in our analysis. Here we define a very special solution which allows us to construct solutions to initial value problems. The fundamental solution for the heat equation is the function

$$\varphi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} \tag{11}$$

Defined by $t > 0$. We have already observed this function in our derivation of the relation. Now we take this function as the starting point and show how it can be used to solve the heat equation. This function is also called the heat kernel.

IV. 1 Derivation of Fundamental Solution of Heat Equation using symmetry reduction

Theorem 1: Let $u(x,t)$ be any solution of $u_t = \Delta u$. then given a parameter $\lambda \neq 0$, $u(\lambda x, \lambda^2 t)$ is also a solution (just plug in the latter function in the equation and see that it satisfies it, regardless of the value of λ!). This suggests we might attempt to find solutions that depend on the ratio r^2/t $(r = |x|)$ instead of on the pair (x,t). Therefore, we let $u(x,t)$ be of the flowing form:

$$u(x,t) = v\left(\frac{r^2}{t}\right) = v\left(\frac{x^2}{t}\right), (t > 0, x \in \mathbb{R}^n) \tag{12}$$

For some function v as yet determined

It is quicker to seek a solution having the special structure

Although this approach eventually leads to what we went (see proposition 1 below) it is quicker to seek a solution U having the special structure.

Remark 4.1.1: If U is a solution, so is $\lambda^\alpha U(\lambda^\beta, \lambda t)$. Therefore we search solution of the form

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right), (x \in \mathbb{R}^n, t > 0) \tag{13}$$

Where the constant α and β and the function $V: \mathbb{R}^n \to \mathbb{R}$ must be found we come to equation (12) if we look for a solution u of the heat equation invariant under the dilation scaling

$$u(x,t) \longmapsto \lambda^\alpha u(\lambda^\beta x, \lambda t) \tag{14}$$

That is we ask

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t) \tag{15}$$

For all $\lambda > 0, x \in \mathbb{R}^n, t > 0$, seating $\lambda = t^{-1}$ we derive equation (12) for $v(y) = u(y,1)$

Let us insert equation (12) in to equation (3) and there after compute

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} + t^{-(\alpha+2\beta)} \Delta v(y) = 0 \tag{16}$$

For $y = t^{-\beta} x$ in order to transform equation (16) in to an expression involving the variable y alone we take $\beta = \frac{1}{2}$. Then the terms with t are identities and so equation (16) reduces to

$$\alpha v + \frac{1}{2} y . Dv + \Delta v = 0 \tag{17}$$

We simplify further by guessing v to be radial; i.e. $v(y) = w(|y|)$, for some $w: \mathbb{R} \to \mathbb{R}$. there up on equation (17)

Remark 4.1.2: If $v(y) = w(r)$ where $x \in \mathbb{R}^n$ and $r = |x|$, then

$$\Delta v(y) = \frac{d^2 w}{dr^2} + \frac{n-1}{r} \frac{dw}{dr} \tag{18}$$

Proof: let

$$\frac{\partial r}{\partial y_j} = \frac{y_j}{r} \tag{19}$$

We have

$$\Delta v(y) = \sum_{n=1}^{\infty} \partial_j \left[\frac{y_j}{r} w'(r)\right] \tag{20}$$

$$= \sum_{n=1}^{\infty} \left[\frac{y_j^2}{r^2} \frac{dw}{dr} + \frac{1}{r} \frac{dw}{dr} - \frac{y_j^2}{r^3} \frac{dw}{dr}\right] \tag{21}$$

$$= \frac{d^2 w}{dr^2} + \frac{n}{r} \frac{dw}{dr} - \frac{1}{r} \frac{dw}{dr} \tag{22}$$

$$\alpha w + \frac{1}{2} r \frac{dw}{dr} + w'' + \frac{n-1}{r} \frac{dw}{dr} = 0 \tag{23}$$

For $r = |x|, ' = \frac{d}{dr}$. Now we set $\alpha = \frac{n}{2}$, this simplifies to read

$$\frac{d\left(r^{n-1}\frac{dw}{dr}\right)}{dr} + \frac{1}{2} \frac{d\left(r^n w\right)}{dr} = 0, (i.e \text{ integrating both sides we get}) \tag{24}$$

Thus

$$r^{n-1} \frac{dw}{dr} + r^n \mathcal{W} = \alpha \tag{25}$$

For some constant α assuming $\lim_{r \to \infty} \mathcal{W} \frac{dw}{dr}$, we conclude the following result

$$\frac{dw}{dr} = -\frac{1}{2} r \mathcal{W} \tag{26}$$

$$\frac{\frac{dw'}{dr}}{W} = -\frac{1}{2} r, (\text{where the differential equation is separable}) \tag{27}$$

Taking integration both sides

$$\int \frac{\frac{dw}{dr}}{W} dr = -\int \frac{1}{2} r \, dr \tag{28}$$

$$\ln W = -\frac{1}{4} r^2 + \ln b, \text{ (where b is any constant } (b \in \mathbb{R})) \tag{29}$$

$$\ln w - \ln b = -\frac{1}{4} r^2 \tag{30}$$

$$\ln\left(\frac{w}{b}\right) = -\frac{1}{4} r^2 \tag{31}$$

$$\frac{w}{b} = e^{-\frac{1}{4} r^2}, (\text{where } b \neq 0) \tag{32}$$

$$\mathcal{W}(r) = b e^{\frac{-r^2}{4}} \tag{33}$$

Combining equation (12) and equation (33) and our choice α and β, we conclude that $\frac{b}{t^{\frac{n}{2}}} e^{\frac{-|x|^2}{4t}}$ solves the heat equation (16). This computation motivates the following.

Theorem 2: The function

$$\varphi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0 \end{cases} \tag{34}$$

is the required fundamental solution of the heat equation.

Proposition 1: For $t > 0, \phi(x,t) > 0$ is an infinitely differentiable function of x and t.

Proposition 2: $\phi_t = \Delta\phi$ for all $x \in \mathbb{R}^n$ and $t > 0$

Proposition 3: We choose $c_{1=1}/{(4\pi)^{\frac{n}{2}}}$ then we have

$$\int_{-\infty}^{\infty} \phi(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{-\infty}^{\infty} e^{\frac{-|x|^2}{4t}} dx =$$
$$\frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-|y|^2} (4t)^{\frac{n}{2}} dy = \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-|y|^2} dy = 1 \tag{35}$$

Where we have set $y = x/2\sqrt{t}$ and use that
Proposition 1 is easy if slightly tediously to verify directly by taking derivatives of $\phi(x,t)$. Property proposition 3 says that the integral of ϕ is invariant in t (remember, no heat created or destroyed). This is easy to verify by using a change of variables and the following basic fact:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \left[\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2\right]^{\frac{1}{2}} \tag{36}$$
$$=$$
$$\left(\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy\right)^{\frac{1}{2}}, \left(\text{Where } \int_{-\infty}^{\infty} e^{-x^2} \text{ is not possible to integrate }\right) \tag{37}$$
$$= \left(\int_0^{\infty} \int_0^{2\pi} re^{-r^2} d\theta dr\right)^{\frac{1}{2}} \tag{38}$$
$$= \left(2\pi \int_0^{2\pi} re^{-r^2} dr\right)^{\frac{1}{2}} = \sqrt{\pi} \tag{39}$$

So that we get

$$\int_{-\infty}^{\infty} \phi(x,t) dx = \frac{1}{(4\pi t)^{n/2}} \int_{-\infty}^{\infty} e^{-\frac{|x|^2}{4t}} dx \tag{40}$$
$$= \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-y^2} dy = \frac{1}{\pi^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-y_1^2-y_1^2-,\ldots\ldots-y_n^2} dy \tag{41}$$
$$= \frac{1}{\pi^{\frac{n}{2}}} \left(\int_{-\infty}^{\infty} e^{-z^2} dz\right)^n \tag{42}$$
$$= 1$$

Theorem 3: Assume $n = 1$ and $u(x,t) = \left(\frac{x^2}{t}\right)$
show $u_t = u_{xx}$ if and only if
$$4z \frac{d^2v(z)}{dz^2}(z) + (2+z)\frac{dv(z)}{dz} = 0 \; (z > 0) \tag{43}$$
show that the general solution of $(*)$
$$v(z) = k_1 \int_0^z e^{-s/4} S^{1/2} ds + k_2 \tag{44}$$

Differentiate $v\left(\frac{x^2}{t}\right)$ with respect to x and select the constant C properly, so as to obtain the fundamental solution $\phi(x), for \; n = 1$
Proof (a): suppose that
$$4z \frac{d^2v(z)}{dz^2}(z) + (2+z)\frac{dv(z)}{dz} = 0 \tag{45}$$
Let $V \subset U$ is any smooth sub region then the rate of change of the total quantity with in V equal to the negative of the net flux through ∂V
$$\frac{d}{dt} \int_V u dx = -\int_{\partial V} F. vd \tag{46}$$
F being the flux density. Thus
$$u_t = -divF, (\text{where } v \text{ is any arbitrary variable }) \tag{47}$$
In many situation $F \propto U$ but opposite direction (since the flow is from the region of higher to lower)
$$F = -au_X \; (a > 0) \tag{48}$$
Substituting in to equation (46) we obtain the partial differential equation
$$u_t = adiv(u_x) = au_{xx} \tag{49}$$
Which for $a = 1$, is the heat equation
Therefore
$$u_t = u_{xx} \tag{50}$$
Conversely: suppose that the equation $u_t = \Delta u$, is heat equation

A common technique for finding special solutions of partial differential equation is to reduce the equation to an ordinary differential equation. In the present case we observe that the one-dimensional heat equation
$$u_t = u_{xx} \tag{51}$$
is invariant under the linear transformation, $y = \lambda x \; \Gamma = \lambda^2 t, \lambda \epsilon \mathbb{R}$
And the fact that this transformation leaves invariant the ratio $(z = x^2/t)$ suggests to look for a solution u of equation (46) in the form
$$u(x,t) = v\left(\frac{x^2}{t}\right) \tag{52}$$
Then we need to show
$$4z \frac{d^2v(z)}{dz^2}(z) + (2+z)\frac{dv(z)}{dz} = 0 \tag{53}$$

Since the solution of (1) is of the form $u(x,t) = v\left(\frac{x^2}{t}\right)$ now we have to find u_t and u_{xx}

In the fact we the following partial derivatives of $u(x,t) = v\left(\frac{x^2}{t}\right)$

$$u_t = -\frac{x^2}{t^2}\frac{dv(z)}{dz} \text{ and } u_x = \frac{2x}{t}\frac{dv(z)}{dz} \tag{54}$$
$$u_{xx} = \frac{4x^2}{t^2}\frac{d^2v(z)}{dz^2} + \frac{2}{t}\frac{dv(z)}{dz}, -\infty < x < \infty \text{ and } t > 0 \tag{55}$$
Substituting this in to equation (46)
$$-\frac{x^2}{t^2}\frac{dv(z)}{dz} = \frac{4x^2}{t^2}\frac{dv(z)}{dz} + \frac{2}{t}\frac{dv(z)}{dz} \tag{56}$$
$$-z\frac{dv(z)}{dz} = 4\frac{d^2v(z)}{dz^2} + 2\frac{dv(z)}{dz} = 0 \tag{57}$$
$$\frac{d^2v(z)}{dz^2} + (2+z)\frac{dv(z)}{dz} = 0 \tag{58}$$
show that the general solution of equation (43) is
$$v(z) = k_1 \int_0^z e^{-\frac{1}{4}S} S^{\frac{1}{2}} ds + k_2 \tag{59}$$
Proof:

$$4Z\frac{d^2v(z)}{dz^2} + (2+z)\frac{dv(z)}{dz} = 0 \tag{60}$$
$$4Z\frac{d^2v(z)}{dz^2} = -(2+z)\frac{dv(z)}{dz} \tag{61}$$
$$\frac{d^2v(z)}{dz^2} = -\frac{(2+z)}{4z}\frac{dv(z)}{dz} \tag{62}$$

$$\frac{\frac{d^2v(z)}{dz^2}}{\frac{dv(z)}{dz}} = -\frac{(2+z)}{4z}, (\text{where the differential equation is separable }) \tag{63}$$

$$\frac{\frac{d^2v(z)}{dz^2}}{\frac{dv(z)}{dz}} = -\left(\frac{1}{4} + \frac{1}{2z}\right), (\text{integrating both sides we get the following result }) \tag{64}$$

$$\int_0^z \frac{\frac{d^2v(z)}{dz^2}}{\frac{dv(z)}{dz}} dz = -\int_0^z \left(\frac{1}{4} + \frac{1}{2z}\right) dz \tag{65}$$
$$ln\left(\frac{dv(z)}{dz}\right) = -\frac{1}{4}z - ln z^{\frac{1}{2}} + ln k_1 \tag{66}$$

$$ln\left(\frac{dv\,(z)}{dz}\right) - \ln k_1 = -\frac{1}{4}z - \ln z^{\frac{1}{2}} \tag{67}$$

$$\ln\frac{\frac{dv\,(z)}{dz}}{k} = -\frac{1}{4}z - \ln z^{\frac{1}{2}} \tag{68}$$

$$\frac{\frac{dv\,(z)}{dz}}{k} = e^{\left(-\frac{1}{4}z - \ln z^{\frac{1}{2}}\right)} \tag{69}$$

$$\frac{dv\,(z)}{dz} = k_1 e^{\left(-\frac{1}{4}z - \ln z^{\frac{1}{2}}\right)} \tag{70}$$

$$\frac{dv\,(z)}{dz} = k_1 z^{\frac{1}{2}} e^{-\frac{1}{4}z} \tag{71}$$

By integrating both sides from with the interval of integration $0\ to\ z$ we get the general solution of the form

$$v(z) = k_1 \int_0^z z^{\frac{1}{2}} e^{-\frac{1}{2}z}\,dz + k_2 \tag{72}$$

$$v(z) = k_1 \int_0^z e^{-\frac{1}{4}S} S^{\frac{1}{2}}\,ds + k_2 \tag{73}$$

case (1)

Since we have $u(x,t) = v\left(\frac{x^2}{t}\right)$

For some appropriate function v, yet to be determined. In this case,

$$u_t = -\frac{x^2}{t^2}\frac{dv}{dt}\left(\frac{x^2}{t}\right)\ and\ u_x = \frac{2x}{t}\frac{dv}{dt}\left(\frac{x^2}{t}\right)\ and\ u_{xx} = \frac{2}{t}\frac{dv}{dt}\left(\frac{x^2}{t}\right) + \frac{4x^2}{t^2}\frac{dv}{dt}\left(\frac{x^2}{t}\right) \tag{74}$$

And the function v should satisfy the following ODE:

$$\frac{4x^2}{t}\frac{d^2v}{dt^2} + \left(\frac{1}{t} + \frac{x^2}{t^2}\right)\frac{dv}{dt} = 0 \tag{75}$$

Or write $y = \frac{x^2}{t}$; $\frac{d^2v}{dt^2} + \left(\frac{1}{2y} + \frac{1}{4}\right)\frac{dv}{dt} = 0 \tag{76}$

Integrating once we find that

$$\frac{dv\,(y)}{dt} = k_1 e^{-\int\left(\frac{1}{2y} + \frac{1}{4}\right)dy} = k_1 y^{-\frac{1}{2}} e^{-\frac{y}{4}} \tag{77}$$

And integrating once more that

$$v(y) = k_1 \int_0^z Z^{-\frac{1}{2}} e^{-\frac{z}{4}}dz + k_2 \tag{78}$$

Thus, the diffusion equation $u_t = u_{xx}$ has general solution

$$U(x,t) = v\left(\frac{x^2}{t}\right) = k_1 \int_0^{\frac{x^2}{t}} z^{-\frac{1}{2}} e^{-z/4}dz + k_2 \tag{79}$$

With two integration constants k and k.

We now observe that if u (x; t) is a solution of the diffusion equation, then so is $u_x(x,t)$; by linearity of the equation. For the general solution found above this yields another solution $U(x,t) = u_x(x,t)$;

$$U(x,t) = k_1 \frac{2x}{t}\left(\frac{x^2}{t}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} = k_1 \frac{2}{\sqrt{t}} e^{-\frac{x^2}{4t}} \tag{80}$$

The integration constant k is chosen such that $U(x,t)$ satisfies:

$$\int_{-\infty}^{\infty} U(x,t)\,dx = 1 \tag{81}$$

For all $t > 0$.

This constraint is motivated by the fact that it also holds for $t = 0$:

$$\int_{-\infty}^{\infty} U_0(x,t)\,dx = \int_{-\infty}^{\infty} \delta_0(x)\,dx = 1 \tag{82}$$

and that the diffusion equation models movement of individuals (not deaths or births), so that the total population should not change over time.

Thus, to find C_1, we let

$$1 = \int_{-\infty}^{\infty} \frac{2k}{\sqrt{t}} e^{-\frac{x^2}{4t}}dx \tag{83}$$

$$= \frac{4k_1}{\sqrt{t}} \int_0^{+\infty} e^{-\left(\frac{x}{2\sqrt{t}}\right)^2} dx \tag{84}$$

$$= 8k \int_0^{+\infty} e^{-z^2}\,dz \tag{85}$$

$$= 8C_1 \frac{\sqrt{\pi}}{2} \tag{86}$$

In the first step we used the fact that the integrand is an even

function (so the integral equals twice the integral of the function over the interval $[0, +\infty]$). In the second step we used the substitution $z = x/(2\sqrt{t})$ and in the last step we used the famous integral.

$$\int_0^{+\infty} e^{-z^2}\,dz = \frac{\sqrt{\pi}}{2} \tag{87}$$

Finding the values of c_1 we get

$$k_1 = \frac{1}{4\sqrt{\pi}} \tag{88}$$

And plugging this back into the formula of $U(x,t)$, we finally arrive at the fundamental solution.

$$U(x,t) = k\frac{2}{\sqrt{t}} e^{-\frac{x^2}{4t}},\ \text{Where } k_1 = \frac{1}{4\sqrt{\pi}} U(x,t) = \frac{1}{4\sqrt{\pi}}\frac{2}{\sqrt{t}} e^{-\frac{x^2}{4t}} \tag{89}$$

Then the required fundamental solution will be:

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \tag{90}$$

Case (2); integrating

$$\frac{d^2v\,(z)}{dz^2}(z) + (2 + z)\frac{dv\,(z)}{dz} = 0 \tag{91}$$

$$\frac{dv\,(z)}{dz} = k_1 z^{\frac{1}{2}} e^{-\frac{1}{4}z} \tag{92}$$

$$v(z) = k_1 \int_0^z e^{-\frac{1}{4}S} S^{\frac{1}{2}}\,ds + k \tag{93}$$

So the function

$$U(x,t) = \frac{1}{4\sqrt{\pi}} \int_0^z e^{-\frac{1}{4}S} S^{\frac{1}{2}}\,ds \tag{94}$$

is a solution of equation (3). Since equation (3)is a linear homogeneous equation the derivative of this function with respect to x, becomes that is the function

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \tag{95}$$

Equation (95) is the required fundamental solution of homogenous heat equation by using symmetry of ruction.

V.CONCLUSION

This study developed fundamental solutions based Symmetry Reduction for a heat equation. A solution of heat equation was constructed using superposition principles through combining simple solutions of the homogenous equation. Therefore, the fundamental solution of heat equation derived based on Symmetry Reduction and to derive this we have used different concepts as theorem and proposition.

However, in this study we considered a symmetry reduction for homogeneous heat equation and it simple and easy to derive. Moreover, this procedure can be applied for non-homogeneous heat equations.

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