International Journal of Innovative Technology and Exploring Engineering (IJITEE)
ISSN: 2278-3075 (Online), Volume-10 Issue-1, November 2020

A Modification of Quadratic Programming Algorithm

Poonam Kumari

Abstract: ***In the existing methods for solving Quadratic Programming Problems having linearly factorized objective function and linear constraints, all the linear factors of the objective function are supposed to be positive for all feasible solutions. Here, a modification of the existing methods is proposed and it has been proved that the modified method can be applied to find the optimal solution of the problem even if all the linear factors of the objective function are not necessarily positive for all feasible solutions. Moreover, the proposed method can be applied to find the optimal solution of the problem even if the basic solution at any stage is not feasible. If the initial basic solution is feasible, we use simplex method to find the optimal solution. If the basic solution at any stage is not feasible, we use dual simplex method to find the optimal solution. Numerical examples are given to illustrate the method and the results are compared with the results obtained by other methods.***

Keywords: ***Optimal Solution, Quadratic Programming Problem, Simplex Method***

I. INTRODUCTION

Quadratic Programming (QP) is the process of solving a special type of mathematical optimization problem, which maximizes (or minimizes) a quadratic objective function subject to some linear constraints and non-negative restrictions. Because of its wide range of applications in real life, quadratic programming is of considerable research and interest. In finance, QP is used in portfolio analysis; in agriculture, it is used in crop analysis; in statistics, in regression analysis; in electrical engineering, in signal processing; in industry, in planning and scheduling etc.
A quadratic programming problem can be written mathematically as follows:

$$\left.\begin{aligned} &\text{Maximize} \quad z = cx + \frac{1}{2}x^T Q\, x \quad (\text{or } z = cx + x^T D\, x) \\ &\text{subject to} \\ &\qquad Ax \le \mathbf{b} \text{ and } x \ge \mathbf{0} \end{aligned}\right\}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}_{1\times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}_{n\times 1},$$

$$Q = \begin{bmatrix} q_{11} & q_{12} & \ldots & \ldots & q_{1n} \\ q_{21} & q_{22} & \ldots & \ldots & q_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ q_{n1} & q_{n2} & \ldots & \ldots & q_{nn} \end{bmatrix}_{n\times n},$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_m \end{bmatrix}_{m\times 1}.$$

The matrix Q is a real symmetric matrix and the function $x^T Q\, x$ defines a quadratic form. The matrix Q is assumed negative definite (or negative semi-definite) if the problem is of maximization as given in equation (1). But if the problem is of minimization, the matrix Q is assumed positive definite (or positive semi-definite). This means z is strictly concave (or concave) for maximization and strictly convex (or convex) for minimization. Since the constraints are linear, it guarantees a convex solution space. Many researchers have considered quadratic programming problems in which the objective function z can be expressed as the product of linear factors and all the linear factors of the objective function are positive for all feasible solutions. In the present paper, a modification of the existing methods is proposed and it has been proved that the modified method can be applied to find the optimal solution of the problem even if all the linear factors of the objective function are not necessarily positive for all feasible solutions.

II. LITERATURE REVIEW

A number of methods have been developed for finding solution of such problems. Wolfe [1] proposed a modified simplex method for solving such problems. Phillips, Ravindran and Solberg [2] developed a complementary pivot method to solve convex quadratic programming problems.
Cabot and Francis [3] solved certain non-convex quadratic minimization problems by ranking extreme points. Konno [4] proposed two algorithms: one cutting plane and the other enumerative for maximization of a convex quadratic function under linear constraints. Frank and Wolfe [5] used finite iteration method for finding the optimal solution of quadratic programming problems. Swarup [6] developed a simplex type method for solving a special type of quadratic programming problems,

Revised Manuscript Received on November 30, 2020.
* Correspondence Author
Poonam Kumari*, Department of Mathematics, Magadh Mahila College, Patna University, Patna (Bihar), India. Email: poonamkumari1865@gmail.com

in which the objective function can be expressed as the product of linear factors.

The solution methods for optimization of quadratic programming problem proposed by Sharma and Singh [7] and Ghadle and Pawar [8] differ from Swarup's method [6] only in the criteria of selection of entering variable. Hasan [9] introduced a computational technique using computer algebra Mathematica to solve the quadratic programming problems.

Asadujjaman and Hasan [10] used objective separable method for solving quasi-concave quadratic programming problems with bounded variables. Beale [11] proposed an algorithm for minimizing a convex quadratic function subject to linear inequalities. Shetty [12] proposed a method for maximization (or minimization) of a quadratic functions of a certain form under linear restrictions and he used Wolfe's procedure [1] for quadratic programming with minor modifications. Fletcher [13] proposed a method for solving the general quadratic programming problem by generating a sequence of equality problems which differ only in the active constraints. Jensen and King [14] proposed a decomposition method for solving quadratic programming problems.

Whinston [15] proposed an algorithm to solve the bounded variable quadratic programming problem which is a direct extension of an earlier algorithm of H.

Wagner for a bounded variable linear programming problem. Cryer [16] solved the quadratic programming problems using systematic over relaxation.

Bunch and Kaufman [17] proposed a computational method for the indefinite quadratic programming problem. Apart from these, there are a number of papers [18], [19], [20], [21], [22], [23] on quadratic programming problem.

In this work, a modification of the existing methods for solving Quadratic Programming Problems having linearly factorized objective function and linear constraints is proposed and it has been proved that the modified method can be applied to find the optimal solution of the problem even if all the linear factors of the objective function are not necessarily positive for all feasible solutions.

Moreover, the proposed method can be applied to find the optimal solution of the problem even if the basic solution at any stage is not feasible.

The layout of the paper is as follows. In Section 3, the proposed method is presented.

The algorithm for the proposed method is given in Section 4. Validity of the proposed method is proved in Section 5 by comparing the results obtained for the numerical examples by the proposed method and the existing methods.

Finally, discussion for highlighting the importance of the proposed method is given in the last section.

III. PROPOSED METHOD

Consider the quadratic programming problem

$$\left.\begin{array}{l} \text{Max } z = z^1 z^2 = (c_0 + c_1x_1 + c_2x_2 + \ldots + c_nx_n)(d_0 + d_1x_1 + d_2x_2 + \ldots + d_nx_n) \\ \text{subject to} \\ \quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ \quad a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ \quad a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \end{array}\right\} \qquad (1a)$$

where $x_1, x_2, x_3, \ldots, x_n \ge 0.$

Introducing the slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$, the above constraints can be written as:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + x_{n+2} = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + x_{n+m} = b_m$$

where $x_1, x_2, x_3, \ldots, x_n, x_{n+1}, \ldots, x_{n+m} \ge 0.$

Let $c = \begin{bmatrix} c_1 & c_2 & \ldots & c_n & 0 & 0 & \ldots & 0 \end{bmatrix}_{1\times(m+n)}$,

$d = \begin{bmatrix} d_1 & d_2 & \ldots & d_n & 0 & 0 & \ldots & 0 \end{bmatrix}_{1\times(m+n)}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots. & a_{1n} & 1 & 0 & 0 & \ldots. & 0 \\ a_{21} & a_{22} & \ldots. & a_{2n} & 0 & 1 & 0 & \ldots. & 0 \\ \ldots. & \ldots. & \ldots. & \ldots. & \ldots. & \ldots. & \ldots. & \ldots. & \ldots. \\ a_{m1} & a_{m2} & \ldots. & a_{mn} & 0 & 0 & 0 & \ldots. & 1 \end{bmatrix}_{m\times(m+n)},$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \ldots. \\ x_n \\ x_{n+1} \\ \ldots. \\ x_{n+m} \end{bmatrix}_{(n+m)\times 1}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_m \end{bmatrix}_{m\times 1}.$$

Then the above problem can be written in the standard form as

$$\left.\begin{array}{l} \text{Maximize} \quad z = z^1 z^2 = (\boldsymbol{c}_0 + \boldsymbol{cx})(\boldsymbol{d}_0 + \boldsymbol{dx}) \\ \text{subject to} \\ \qquad \boldsymbol{Ax} = \mathbf{b} \text{ and } \boldsymbol{x} \ge \mathbf{0} \end{array}\right\} \qquad (1b)$$

Let B be any $m\times m$ sub-matrix of A formed from m linearly independent columns of A and let $x_B = \begin{bmatrix} x_{B_1} & x_{B_2} & \ldots & x_{B_m} \end{bmatrix}^T$ be an initial basic feasible solution of the above QP problem such that

$$B\,x_B = b, \text{ i.e., } x_B = B^{-1}b \qquad (2)$$

Also, let $z^1 = c_0 + c_B x_B$ and $z^2 = d_0 + d_B x_B$, where c_B and d_B are the vectors having their components associated with the basic variables in the numerator and the denominator of the objective function respectively.

If the columns of matrix A be denoted by $\alpha_1, \alpha_2, \ldots, \alpha_{n+m}$ and columns of sub-matrix B by $\beta_1, \beta_2, \ldots, \beta_m$, then $A = \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots.. & \alpha_{n+m} \end{bmatrix}$ and $B = \begin{bmatrix} \beta_1 & \beta_2 & \ldots. & \beta_m \end{bmatrix}$.

Let the new basic feasible solution be given by

$$x_B' = \left\{ x_{B_1} - x_{B_r}\frac{y_{1j}}{y_{rj}}, x_{B_2} - x_{B_r}\frac{y_{2j}}{y_{rj}}, \ldots, \frac{x_{B_r}}{y_{rj}}, \ldots, x_{B_m} - x_{B_r}\frac{y_{mj}}{y_{rj}} \right\} \qquad (3)$$

where

$$\frac{x_{B_r}}{y_{rj}} = \min_i \left\{ \frac{x_{B_i}}{y_{ij}} \right\} \quad (4)$$

and other non-basic components are zero.

Now we proceed to find the criterion to select the incoming vector $\alpha_j \in A$ such that the value of the objective function corresponding to the new basic feasible solution is improved. The value of the objective function for the original basic feasible solution is

$$z = z^1 z^2 = (c_0 + c_B x_B)(d_0 + d_B x_B)$$

$$= \left(c_0 + \sum_{i=1}^{m} c_{B_i} x_{B_i} \right)\left(d_0 + \sum_{i=1}^{m} d_{B_i} x_{B_i} \right) \quad (5)$$

The value of the objective function for the new basic feasible solution is

$$\bar{z} = \bar{z}^1 \bar{z}^2 = (c_0 + c_B' x_B')(d_0 + d_B' x_B')$$

$$= \left(c_0 + \sum_{i=1}^{m} c_{B_i}' x_{B_i}' \right)\left(d_0 + \sum_{i=1}^{m} d_{B_i}' x_{B_i}' \right) \quad (6)$$

But

$$c_{B_i}' = c_{B_i} \quad (i = 1, 2, \ldots, m,\ i \neq r),\quad c_{B_r}' = c_j \quad (7)$$

$$d_{B_i}' = d_{B_i} \quad (i = 1, 2, \ldots, m,\ i \neq r),\quad d_{B_r}' = d_j \quad (8)$$

Substituting the values of c_{B_i}' and d_{B_i}' from (7) and (8) in (6) and using (3), we get

$$\bar{z} = \left[c_0 + \sum_{\substack{i=1\\ i\neq r}}^{m} c_{B_i}\left(x_{B_i} - x_{B_r}\frac{y_{ij}}{y_{rj}} \right) + c_j \frac{x_{B_r}}{y_{rj}} \right].$$

$$\left[d_0 + \sum_{\substack{i=1\\ i\neq r}}^{m} d_{B_i}\left(x_{B_i} - x_{B_r}\frac{y_{ij}}{y_{rj}} \right) + d_j \frac{x_{B_r}}{y_{rj}} \right]$$

$$= \left[c_0 + \sum_{i=1}^{m} c_{B_i} x_{B_i} + \frac{x_{B_r}}{y_{rj}}\left(c_j - \sum_{i=1}^{m} c_{B_i} y_{ij} \right) \right].$$

$$\left[d_0 + \sum_{i=1}^{m} d_{B_i} x_{B_i} + \frac{x_{B_r}}{y_{rj}}\left(d_j - \sum_{i=1}^{m} d_{B_i} y_{ij} \right) \right]$$

$$= \left[z^1 + \frac{x_{B_r}}{y_{rj}}\left(c_j - z_j^1 \right) \right]\left[z^2 + \frac{x_{B_r}}{y_{rj}}\left(d_j - z_j^2 \right) \right],$$

where $z_j^1 = \sum_{i=1}^{m} c_{B_i} y_{ij}$ and $z_j^2 = \sum_{i=1}^{m} d_{B_i} y_{ij}$

Therefore,

$$\bar{z}^1 \bar{z}^2 = \left[z^1 + \frac{x_{B_r}}{y_{rj}}\left(c_j - z_j^1 \right) \right]\left[z^2 + \frac{x_{B_r}}{y_{rj}}\left(d_j - z_j^2 \right) \right] \quad (9)$$

It follows from (9) that $\bar{z}^1 \bar{z}^2 > z^1 z^2$ only if

$$\left[z^1 + \frac{x_{B_r}}{y_{rj}}\left(c_j - z_j^1 \right) \right]\left[z^2 + \frac{x_{B_r}}{y_{rj}}\left(d_j - z_j^2 \right) \right] > z^1 z^2$$

i.e., if

$$z^2\left(c_j - z_j^1 \right) + z^1\left(d_j - z_j^2 \right) + \frac{x_{B_r}}{y_{rj}}\left(c_j - z_j^1 \right)\left(d_j - z_j^2 \right) > 0$$

i.e., if

$$z^2\left(z_j^1 - c_j \right) + z^1\left(z_j^2 - d_j \right) - \frac{x_{B_r}}{y_{rj}}\left(z_j^1 - c_j \right)\left(z_j^2 - d_j \right) < 0 \quad (10)$$

It means that as soon as

$$z^2\left(z_j^1 - c_j \right) + z^1\left(z_j^2 - d_j \right) - \frac{x_{B_r}}{y_{rj}}\left(z_j^1 - c_j \right)\left(z_j^2 - d_j \right) \geq 0,$$

no further improvement is possible and the optimal solution is reached.

Also, it follows from (9) that

$$\bar{z}^1 \bar{z}^2 = z^1 z^2 - \frac{x_{B_r}}{y_{rj}}\left[z^2\left(z_j^1 - c_j \right) + z^1\left(z_j^2 - d_j \right) - \frac{x_{B_r}}{y_{rj}}\left(z_j^1 - c_j \right)\left(z_j^2 - d_j \right) \right] \quad (11)$$

Since $\frac{x_{B_r}}{y_{rj}} > 0$, it follows from (11) that $\bar{z}^1 \bar{z}^2$ is maximum if

$z^2\left(z_j^1 - c_j \right) + z^1\left(z_j^2 - d_j \right) - \frac{x_{B_r}}{y_{rj}}\left(z_j^1 - c_j \right)\left(z_j^2 - d_j \right)$ is minimum.

Therefore, we can conclude the following:
If

$$z^2\left(z_j^1 - c_j \right) + z^1\left(z_j^2 - d_j \right) - \frac{x_{B_r}}{y_{rj}}\left(z_j^1 - c_j \right)\left(z_j^2 - d_j \right) < 0,$$

then the non-basic vector $\alpha_j \in A$ corresponding to

$\min\left\{ z^2\left(z_j^1 - c_j \right) + z^1\left(z_j^2 - d_j \right) - \frac{x_{B_r}}{y_{rj}}\left(z_j^1 - c_j \right)\left(z_j^2 - d_j \right) \right\}$ is

selected as the incoming vector. Using simplex method, the outgoing vector is selected and a new basic feasible solution is obtained. The process is continued till the criterion of optimality is satisfied.
As soon as

$$z^2\left(z_j^1 - c_j \right) + z^1\left(z_j^2 - d_j \right) - \frac{x_{B_r}}{y_{rj}}\left(z_j^1 - c_j \right)\left(z_j^2 - d_j \right) \geq 0$$

for all the non-basic vectors, no further improvement is possible and the optimal solution is reached.

Now, we consider the case of infeasible solution.
If the basic solution obtained at any stage is infeasible, then we proceed as follows.

We compute $\min\left\{ x_{B_i} : x_{B_i} < 0 \right\}$, where x_{B_i} denotes all basic infeasible solutions.

Let us suppose that $\min\left\{ x_{B_i} : x_{B_i} < 0 \right\} = x_{B_r}$.

Then the basis vector corresponding to x_{B_r} will leave the basis.

To find the incoming vector, we compute

$$\max\left\{ \frac{\Delta_j}{y_{rj}} : y_{rj} < 0 \right\}.$$

Let us suppose that

$$\max\left\{\frac{\Delta_j}{y_{rj}} : y_{rj} < 0\right\} = \frac{\Delta_k}{y_{rk}}.$$

Then the non-basis vector α_k will enter the basis.
We continue the process till the criterion of optimality is satisfied.

IV. ALGORITHM FOR THE PROPOSED METHOD

Step1. Find an initial basic solution of the given quadratic programming problem.

Step 2. Check whether the initial basic solution obtained in step 1 is feasible or infeasible.

If the initial basic solution is feasible, then go to step 3, otherwise step 4.

Step 3. Calculate

$$\Delta_j = z^2\left(z_j^1 - c_j\right) + z^1\left(z_j^2 - d_j\right) - \frac{x_{B_r}}{y_{rj}}\left(z_j^1 - c_j\right)\left(z_j^2 - d_j\right)$$

for all the non-basic vectors.

If $\Delta_j \geq 0$ for all the non-basic vectors, then no further improvement is possible and the optimal solution is reached.

If $\Delta_j < 0$ for some non-basic vectors, then find $\min \Delta_j$. In this case, the non-basic vector $\alpha_j \in A$ corresponding to $\min \Delta_j$ is selected as the incoming vector. Using simplex method, the outgoing vector is selected and a new basic feasible solution is obtained. The process is continued till the criterion of optimality is satisfied.

Step 4. Compute $\min\{x_{B_i} : x_{B_i} < 0\}$, where x_{B_i} denotes all basic infeasible solutions.

If $\min\{x_{B_i} : x_{B_i} < 0\} = x_{B_r}$, then the basis vector corresponding to x_{B_r} will leave the basis.

To find the incoming vector, we compute

$$\max\left\{\frac{\Delta_j}{y_{rj}} : y_{rj} < 0\right\}.$$

If $\max\left\{\frac{\Delta_j}{y_{rj}} : y_{rj} < 0\right\} = \frac{\Delta_k}{y_{rk}}$, then the non-basis vector α_k will enter the basis.

The process is continued till the criterion of optimality is satisfied.

V. NUMERICAL EXAMPLES

Example 1.

Max $z = (2x_1 + 4x_2 + x_3 + 1)(x_1 + x_2 + 2x_3 + 2)$

subject to

$x_1 + 3x_2 \leq 4,\ 2x_1 + x_2 \leq 3,\ x_2 + 4x_3 \leq 3,\ x_1, x_2, x_3 \geq 0.$ **S**

Solution: After adding slack variables, the above problem can be written in the standard form as follows:

Max $(2x_1 + 4x_2 + x_3 + 1)(x_1 + x_2 + 2x_3 + 2)$

subject to

$$x_1 + 3x_2 + x_4 = 4$$
$$2x_1 + x_2 + x_5 = 3$$
$$x_2 + 4x_3 + x_6 = 3$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

After computing $z^1, z^2, z_j^1 - c_j$ and $z_j^2 - d_j$, the initial basic feasible solution is given in Table I (a).

Table I(a): Initial Table for Example 1

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	$\frac{x_{B_i}}{y_{ij}}$
y_4	0	0	4	1	(3)	0	1	0	0	$\frac{4}{3}$
y_5	0	0	3	2	1	0	0	1	0	3
y_6	0	0	3	0	1	4	0	0	1	3
z^1=1			$z_j^1 - c_j$	-2	-4	-1	0	0	0	
z^2=2			$z_j^2 - d_j$	-1	1	-2	0	0	0	
z = 2			Δ_j	-8	-(43/3)	-(11/2)	0	0	0	
					↑		↓			

Since $\Delta_j < 0$ for all the non-basic vectors, we have not reached the optimal solution. Therefore, we continue the process till the optimal solution is reached.

Retrieval Number: 100.1/ijitee.A81441110120
DOI: 10.35940/ijitee.A8144.1110120
Journal Website: www.ijitee.org

Published By:
Blue Eyes Intelligence Engineering and Sciences Publication (BEIESP)

Table I(b): Intermediate Table for Example 1

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	$\frac{x_{B_i}}{y_{ij}}$
y_2	4	1	4/3	1/3	1	0	1/3	0	0	-
y_5	0	0	5/3	5/3	0	0	-(1/3)	1	0	-
y_6	0	0	5/3	-(1/3)	0	(4)	-(1/3)	0	1	$\frac{5}{12}$
z^1 =19/3			$z_j^1 - c_j$	-(2/3)	0	-1	4/3	0	0	
z^2 =10/3			$z_j^2 - d_j$	-(2/3)	0	-2	1/3	0	0	
z = 190/9			Δ_j	-(62/9)	0	-(101/6)	43/9	0	0	
						↑			↓	

Table I(c): Intermediate Table for Example 1

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	$\frac{x_{B_i}}{y_{ij}}$
y_2	4	1	4/3	1/3	1	0	1/3	0	0	4
y_5	0	0	5/3	(5/3)	0	0	-(1/3)	1	0	1
y_3	1	2	5/12	-(1/12)	0	1	-(1/12)	0	1/4	-
z^1 =27/4			$z_j^1 - c_j$	-(3/4)	0	0	5/4	0	1/4	
z^2 =25/6			$z_j^2 - d_j$	-(5/6)	0	0	1/6	0	1/2	
z =225/8			Δ_j	-(75/8)	0	0	11/2	0	101/24	
				↑				↓		

Table I(d): Final Table for Example 1

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
y_2	4	1	1	0	1	0	2/5	-(1/5)	0
y_1	2	1	1	1	0	0	-(1/5)	3/5	0
y_3	1	2	1/2	0	0	1	-(1/10)	1/20	1/4
z^1 = 15/2			$z_j^1 - c_j$	0	0	0	11/10	9/20	1/4
z^2 = 5			$z_j^2 - d_j$	0	0	0	0	1/2	1/2
z = 75/2			Δ_j	0	0	0	11/2	45/8	19/4

Now the criterion of optimality is satisfied, therefore the optimal solution of the given QP problem is reached, which is given by $x_1 = 1,\ x_2 = 1,\ x_3 = 1/2$ and $\max z = 75/2$.

Example 2. Max $z = (2x_1 + 3x_2 + 2)(x_2 - 5)$

subject to $x_1 + x_2 \le 1, \quad 4x_1 + x_2 \ge 2, \quad x_1, x_2 \ge 0$

Solution: After adding slack variables, the above problem can be written in the standard form as follows:

Therefore, we drop y_4 and enter y_1 to obtain the following dual simplex table:

$$\text{Max}\ (2x_1 + 3x_2 + 2)(x_2 - 5)$$

subject to

$$x_1 + x_2 + x_3 = 1$$

$$-4x_1 - x_2 + x_4 = -2$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

The initial basic solution is $x_3 = 1, x_4 = -2$, which is infeasible.

After computing $z^1, z^2, z_j^1 - c_j$ and $z_j^2 - d_j$, we get the following initial dual simplex table:

Table II(a): Initial Table for Example 2

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4
y_3	0	0	1	1	1	1	0
y_4	0	0	- 2	(- 4)	- 1	0	1
$z^1 = 2$, $z^2 = -5$, $z = -10$			$z_j^1 - c_j$	-2	-3	0	0
			$z_j^2 - d_j$	0	-1	0	0
			Δ_j	10	10	0	0
				↑			↓

Since $\Delta_j \geq 0 \;\; \forall\, j$ and $x_{B_1} = y_3 = 1, x_{B_2} = y_4 = -2$, an optimal but infeasible solution has been attained. In order to obtain a feasible optimal solution, we select a basis vector to leave the basis and a non-basis vector to enter the basis.

To find the outgoing vector, we compute

$$\min\{x_{B_1}, x_{B_2}\} = \min\{y_3, y_4\} = \min\ \{1, -2\} = -2 = x_{B_2}$$

i.e., the basis vector corresponding to $x_{B_2} = y_4$ is the outgoing vector.

To find the incoming vector, we compute

$$\max\left\{\frac{\Delta_j}{y_{2j}} : y_{2j} < 0\right\} = \max\left\{\frac{\Delta_1}{y_{21}}, \frac{\Delta_2}{y_{22}}\right\}$$

$$= \max\left\{\frac{10}{-4}, \frac{10}{-2}\right\} = \frac{10}{-4} = \frac{\Delta_1}{y_{21}}$$

i.e., the non basic vector corresponding to y_1 is the incoming vector.

Table II(b): Final Table for Example 2

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4
y_3	0	0	1/2	0	3/4	1	1/4
y_1	2	0	1/2	1	1/4	0	-(1/4)
$z^1 = 3$, $z^2 = -5$, $z = -15$			$z_j^1 - c_j$	0	- (5/2)	0	- (1/2)
			$z_j^2 - d_j$	0	-1	0	0
			Δ_j	0	47/6	0	5/2

Now the criterion of optimality is satisfied, therefore the optimal solution of the given QP problem is reached, which is given by $x_1 = \frac{1}{2}$, $x_2 = 0$ and $\max z = -15$.

Example 3. Max $z = (2x_1 + 3x_2 + 12)\,(x_1 + 3x_2 + 6)$

subject to

$$x_1 + 2x_2 \geq 10, \quad 2x_1 + 3x_2 \leq 60,$$

$$5 \leq x_1 \leq 15, \quad 4 \leq x_2 \leq 30, \quad x_1, x_2 \geq 0.$$

Solution: After adding slack variables, the above problem can be written in the standard form as follows:

Max $z = (2x_1 + 3x_2 + 12)\,(x_1 + 3x_2 + 6)$

subject to

$$-x_1 - 2x_2 + x_3 = 10$$

$$2x_1 + 3x_2 + x_4 = 60$$

$$-x_1 + x_5 = -5$$

$$x_1 + x_6 = 15$$

$$-x_2 + x_7 = -4$$

$$x_2 + x_8 = 30$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0.$$

After computing $z^1, z^2, z_j^1 - c_j$ and $z_j^2 - d_j$, the initial basic solution is given in Table 3(a).

It can be seen from Table 3(a) that the solution obtained is not feasible and the condition of optimality is not satisfied. Therefore, we introduce the following additional constraint:

$$x_1 + x_2 \leq \mathrm{M} \quad (\mathrm{M} > 0)$$

The next step is to eliminate

This $\Rightarrow x_1 + x_2 + x_3 = \mathrm{M}$

$$\Rightarrow x_2 = \mathrm{M} - x_1 - x_3 \qquad (1)$$

Table III(a): Initial Table for Example 3

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
y_3	0	0	10	-1	-2	1	0	0	0	0	0
y_4	0	0	60	2	3	0	1	0	0	0	0
y_5	0	0	-5	-1	0	0	0	1	0	0	0
y_6	0	0	15	1	0	0	0	0	1	0	0
y_7	0	0	- 4	0	-1	0	0	0	0	1	0
y_8	0	0	30	0	1	0	0	0	0	0	1
z^1 =12 z^2 =6 z = 72			$z_j^1 - c_j$	-2	-3	0	0	0	0	0	0
			$z_j^2 - d_j$	-1	-3	0	0	0	0	0	0
			Δ_j	-54	-234	0	0	0	0	0	0

Now, we eliminate x_2. Therefore, the problem reduces to

Max $z = (2x_1 + 3(M - x_1 - x_3) + 12)\,(x_1 + 3(M - x_1 - x_3) + 6)$

subject to

$$-x_1 - 2(M - x_1 - x_3) \le -10$$

$$2x_1 + 3(M - x_1 - x_3) \le 60$$

$$5 \le x_1 \le 15$$

$$4 \le (M - x_1 - x_3) \le 30$$

$$x_1, x_2, x_3 \ge 0.$$

After adding slack variables, the above problem can be written in the standard form as follows:

Max $z = (2x_1 + 3(M - x_1 - x_3) + 12)\,(x_1 + 3(M - x_1 - x_3) + 6)$

subject to

$$-x_1 - 2(M - x_1 - x_3) + x_4 = -10$$

$$2x_1 + 3(M - x_1 - x_3) + x_5 = 60$$

$$-x_1 + x_6 = -5$$

$$x_1 + x_7 = 15$$

$$-(M - x_1 - x_3) + x_8 = -4$$

$$(M - x_1 - x_3) + x_9 = 30$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \ge 0,$$

i.e.,

Max $z = (3M + 12 - x_1 - 3x_3)\,(3M + 6 - 2x_1 - 3x_3)$ (2)

subject to

$$x_1 + 2x_3 + x_4 = 2M - 10 \quad (3)$$

$$-x_1 - 3x_3 + x_5 = -3M + 60 \quad (4)$$

$$-x_1 + x_6 = -5 \quad (5)$$

$$x_1 + x_7 = 15 \quad (6)$$

$$x_1 + x_3 + x_8 = M - 4 \quad (7)$$

$$-x_1 - x_3 + x_9 = -M + 30 \quad (8)$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \ge 0.$$

Now, we compute $z^1, z^2, z_j^1 - c_j$ and $z_j^2 - d_j$ and follow the proposed method to obtain Table III (b).

It can be seen from Table III (b) that the solution obtained is optimal but infeasible. In order to obtain a feasible optimal solution, we select a basis vector to leave the basis and a non-basis vector to enter the basis.

To find the outgoing vector, we compute

$$\begin{aligned} &\min\{x_{B_i} : x_{B_i} < 0\} \\ &= \min\{x_{B_2}, x_{B_3}, x_{B_6}\} \\ &= \min\{y_5, y_6, y_9\} \\ &= \min\{-3M + 60, -5, -M + 30\} \\ &= -3M + 60 \\ &= y_5 \\ &= x_{B_2} \end{aligned}$$

i.e., the basis vector corresponding to $x_{B_2} = y_5$ is the outgoing vector.

To find the incoming vector, we compute

$$\max\left\{\frac{\Delta_j}{y_{2j}} : y_{2j} < 0\right\} = \max\left\{\frac{9M}{-1}, \frac{9M + 99}{-3}\right\} = \frac{9M + 99}{-3}$$

i.e., the non basic vector corresponding to y_3 is the incoming vector.

Therefore, we drop y_5 and enter y_3 to obtain Table III (c). The solution obtained at this stage is optimal but infeasible. Therefore, we continue the process to obtain a feasible optimal solution.

Table III(b): Intermediate Table for Example 3

Basis	c_B	d_B	X_B	y_1	y_3	y_4	y_5	y_6	y_7	y_8	y_9
y_4	0	0	2M-10	1	2	1	0	0	0	0	0
y_5	0	0	-3M + 60	-1	- 3	0	1	0	0	0	0
y_6	0	0	-5	-1	0	0	0	1	0	0	0
y_7	0	0	15	1	0	0	0	0	1	0	0
y_8	0	0	M- 4	1	1	0	0	0	0	1	0
y_9	0	0	-M + 30	-1	-1	0	0	0	0	0	1
z^1 = 3M +12			$z_j^1 - c_j$	1	3	0	0	0	0	0	0
z^2 = 3M + 6			$z_j^2 - d_j$	2	3	0	0	0	0	0	0
$z = 9M^2 + 54M + 72$			Δ_j	9M	9M+99	0	0	0	0	0	0
					↑		↓				

Table III(c): Intermediate Table for Example 3

Basis	c_B	d_B	X_B	y_1	y_3	y_4	y_5	y_6	y_7	y_8	y_9
y_4	0	0	30	1/3	0	1	2/3	0	0	0	0
y_3	-3	-3	M - 20	1/3	1	0	-(1/3)	0	0	0	0
y_6	0	0	-5	- 1	0	0	0	1	0	0	0
y_7	0	0	15	1	0	0	0	0	1	0	0
y_8	0	0	16	2/3	0	0	1/3	0	0	1	0
y_9	0	0	10	-(2/3)	0	0	-(1/3)	0	0	0	1
z^1 = 72			$z_j^1 - c_j$	0	0	0	1	0	0	0	0
z^2 = 66			$z_j^2 - d_j$	1	0	0	1	0	0	0	0
z = 4752			Δ_j	72	0	0	93	0	0	0	0
				↑				↓			

To find the outgoing vector, we compute

$$\min\{x_{B_i} : x_{B_i} < 0\} = -5 = y_6 = x_{B_3}$$

i.e., the basis vector corresponding to $x_{B_3} = y_6$ is the outgoing vector.

To find the incoming vector, we compute

$$\max\left\{\frac{\Delta_j}{y_{3j}} : y_{3j} < 0\right\} = \frac{\Delta_1}{y_{31}} = \frac{72}{-1}$$

i.e., the non basic vector corresponding to y_1 is the incoming vector. Therefore, we drop y_6 and enter y_1 to obtain Table III (d).

Now the solution obtained is optimal and feasible, therefore, we stop the process.

The optimal value of the objective function is 4392 and the optimal solution is

$$x_1 = 5,\ x_3 = \frac{3M - 65}{3}$$

$$\text{i.e., } x_1 = 5,\ x_2 = M - x_1 - x_3 = M - 5 - \frac{3M - 65}{3} = \frac{50}{3}.$$

Table III(d): Final Table for Example 3

Basis	c_B	d_B	X_B	y_1	y_3	y_4	y_5	y_6	y_7	y_8	y_9
y_4	0	0	85/3	0	0	1	2/3	1/3	0	0	0
y_3	-3	-3	$\frac{3M-65}{3}$	0	1	0	-(1/3)	1/3	0	0	0
y_1	-1	-2	5	1	0	0	0	-1	0	0	0
y_7	0	0	10	0	0	0	0	1	1	0	0
y_8	0	0	38/3	0	0	0	1/3	2/3	0	1	0
y_9	0	0	40/3	0	0	0	-(1/3)	-(2/3)	0	0	1
$z^1 = 72$			$z_j^1 - c_j$	0	0	0	1	0	0	0	0
$z^2 = 61$			$z_j^2 - d_j$	0	0	0	1	1	0	0	0
z = 4392			Δ_j	0	0	0	95	72	0	0	0

VI. COMPARISON OF THE NUMERICAL RESULTS

The following table shows the comparison between the proposed method and other optimization methods:

Table IV: Comparison of the Numerical Results

Example	Reference	Optimal solution	Optimal value
Ex.1	Proposed	(1, 1, 0.5)	37.5
	Ref.[9]	(1, 1, 0.5)	37.5
	Ref.[10]	(1, 1, 0.5)	37.5
	Ref.[22]	(1.5, 0, 0.75)	23.75
Ex.2	Proposed	(0.5, 0)	-15
	Ref.[23]	(0.5, 0)	-15
Ex.3	Proposed	(5, 16.66)	4392
	Ref.[8]	(5, 16.66)	4392

It can be seen that the results obtained by the proposed method are the same as those obtained by other methods for almost all the examples, which proves the validity of the proposed method. For Example 1, the optimal solution obtained by Jayalakshmi[22] is different from that obtained by the proposed method and I claim that the optimal solution obtained by the proposed method is the correct one.

For solving Example 2, Jain and Mangal[23] have used that $a \le b$ and $c \le d \Rightarrow a - c \le b - d$. But , this is not always true, although the optimal solution obtained by them is the same.

Asadujjaman and Hasan[8] have constructed seven simplex tables for solution of Example 3, but only four tables have been constructed to solve the same problem by the proposed method.

Moreover, no single optimization method exists, which can solve different type of QP problems like examples 1, 2 and 3 given above. For each particular type of QP problem, a particular method has been developed. But the proposed method serves this purpose. Comparison of the computational steps by the proposed method with existing methods shows that the proposed method helps to save our time.

VII. CONCLUSION

The optimization method proposed in this article provides an easy method to find the optimal solution for all quadratic programming problems that have linearly factorized objective function and linear constraints. This method is applicable to all problems regardless of the existence of a feasible solution. Additionally, we save time during computation because the proposed method has fewer steps than the existing methods.

REFERENCES

1. Wolfe, P. (1959), "The Simplex Method for Quadratic Programming," Econometrica, 27(3), 382-398.
2. Phillips, Don T., Ravindran, A. and Solberg, James J. (1976), "Operations Research: Principles and Practice", John Wiley & Sons, New York, 526-539.
3. Cabot A. and Francis, R.L.(1970), "Solving Certain Non-convex Quadratic Minimization Problems by ranking Extreme Points", Operations Research ,18(1), 82-86.
4. Konno, H.(1976), Maximization of a Convex Quadratic Function under Linear Constraints", Mathematical Programming, (11), 117-127.
5. Frank, M., and Wolfe, P.(1956), "An Algorithm for Quadratic Programming", Naval Resesarch Logistics Quarterly 3 (1-2), 95-110.
6. Swarup, K. (1966), "Quadratic Programming" CCERO (Belgium), 8(2), 132-136.
7. Hasan, M.B. (2012), "A Technique for Solving Special Type Quadratic Programming Problems", The Dhaka University Journal of Science, 60(2) 209-215.
8. Asadujjaman, M. and Hasan, M.B.(2016),"A Proposed Technique for Solving Quasi-Concave Quadratic Programming Problems with Bounded Variables by Objective Separable Method", The Dhaka University Journal of Science, 64(1), 51-58.
9. Sharma, K. C. and Singh, Jitendra (1913), "Solution Methods for Linear Factorized Quadratic Optimization and Quadratic Fractional Optimization Problem", IOSR Journal of Mathematics, 8(3), 81-86.
10. Ghadle, Kirtiwant P. and Pawar, Tanaji S. (2015), "Quadratic Programming Problems–Special Case- Alternative Method", International Journal of Latest Technology in Engineering, Management & Applied Science, 4(6), 49-51.

11. Beale, E.M.L., (1959), On Quadratic Programming, Naval Research Logistics Quarterly, 6 (3), 227-243.
12. Shetty, C.M. (1963), A Simplified Procedure for Quadratic Programming, Operations Research, 11, 248-260.
13. Fletcher, R. (1971), A general quadratic programming algorithm, J. Inst., Maths. Applics, 7, 76-91.
14. Jensen, D.L. and King A.J.(1992), A decomposition method for quadratic programming, IBM Systems Journal, 31, 39-48.
15. Whinston, A.(1956), The Bounded Variable Problem- An Application of The Dual Method For Quadratic Programming, Naval Research Logistics Quarterly, 12(2), 173-179.
16. Cryer, C.W.(1971), " The Solution of a Quadratic Programming Problem using Systematic Over relaxation ", SIAM Journal on Control,1-22.
17. Bunch, J.R. and Kaufman, L. (1980), "A Computational Method for the Indefinite Quadratic Programming Problem" Linear Algebra and its Applications, Elsevier North Holland, 341-370.
18. Moraru, V. (1997), "An Algorithm for Solving Quadratic Programming Problems" Computer Science Journal of Moldova, vol.5, no.2(14), 223-235.
19. Cottle, R.W., Dantzig G. B.(1968), "Complementary Pivot Theory of Mathematical Programming", Linear Algebra and Applications, No.1, 103–125.
20. Flecher, R.A. (1971), "General Quadratic Programming Algorithm", J. Inst. Maths. Applics, No.7, p.76–91.
21. Lemke, C.E. (1965), "Bimatrix Equilibrium Points and Mathematical Programming", Management Science, No.11, 681–689.
22. Jayalakshmi M. & Pandian, P. (2014), "A method for solving quadratic programming problems having linearly factorized objective function", International Journal Of Modern Engineering Research (IJMER), 4(8), 20-24.
23. Jain, S., Mangal, A. and Sangeeta (2019), "Modelling of Gauss Elimination Technique for Quadratic Optimization", Advances in Modelling and Analysis A, Vol. 56, No. 2-4, December, 2019 (31-34).

AUTHORS PROFILE

Dr. Poonam Kumari is a faculty in the Department of Mathematics, Magadh Mahila College, Patna University, Patna. She is working in the field of Operations research. Her current research interest includes optimization techniques for linear and non-linear programming.