Even-odd Harmonious Labeling of Some Graphs

Dhvanik Zala, Narendra Chotaliya, Mehul Chaurasiya

Abstract: *Let $G = (V,E)$ be a graph, with $m = |V(G)|$ and $n = |E(G)|$. An injective mapping $f: V(G) \rightarrow \{1,3,5,\dots,2m-1\}$ is called an even-odd harmonious labeling of the graph G, if $\exists$ an induced edge mapping $f^*: E(G) \rightarrow \{0,2,4,\dots,2(n-1)\}$ such that (i) f^* is bijective mapping (ii) $f^*(e = pq) = (f(p) + f(q))(mod\ 2n)$. The graph acquired from this labeling is called even-odd harmonious graph. In this paper, we discovered some interesting results like H-graph, comb graph, bistar graph and $< k_{1,n}^{(1)}, k_{1,n}^{(2)}, k_{1,n}^{(3)} >$ graph for even-odd harmonious labeling.*

Keywords: *Comb graph, Even-odd harmonious labeling, H-graph, Injective mapping*

I. INTRODUCTION

Graph labeling is an engrossing and sizable research in the arena of graph theory. It trades in how vertices and edges of a graph are labelled corresponding to certain mathematical condition[1]. From Galian[5], the neoteric research development in the sphere of graph labeling is taken. The pioneer of graph labeling was A. Rosa[10]. Books of Harary[2] and D.B.west[8] are being used for the ideas and basic knowledge. The harmonius graph have instigated [3] in 1980. Z.liang etl[8] set forth the odd harmonious graphs in 2009 and in 2011 P.B.sarasijia etal [10] investigated the even harmonious graphs. Further Adalin Beatress and Sarasijia in [2] set forth the even odd harmonious graphs. We have manifested distinct graphs, as we have collected distinct types of graphs in forms of even - odd harmonious labeling.

II. BASIC TERMINOLOGY

Definition([6]) 2.1: A graph $G = (V(G), E(G))$ with n edges is called a harmonious, if there is an injective function $f: V(G) \rightarrow Z_n$ such that when each edge uv is alloted the label $\{f(u) + f(v)\}(mod\ n)$, the resulting edge labels are different. When the graph G is tree, exactly one edge label may be used on two vertices.

Definition([7]) 2.2: Let G = (V,E) be a graph, with $m = |V(G)|$ and $n = |E(G)|$. $f: V(G) \rightarrow \{1,3,5,\dots,2m-1\}$ is said to be an even-odd harmonious labeling of the graph G if $\exists$ an induced edge mapping $f^*: E(G) \rightarrow \{0,2,4,\dots,2(n-1)\}$ such that (i) f^* is bijective mapping (ii) $f^*(e = pq) = (f(p) + f(q))(mod\ 2n)$. The graph acquired from this labeling is called even-odd harmonious graph.

Definition([8]) 2.3: The H-graph of a path P_m is the graph secured from two copies of path P_m with vertices $u_1, u_2, u_3, \dots, u_m$ and $v_1, v_2, v_3, \dots, v_m$ by attaching the vertices $u_{\frac{m+1}{2}}$ and $v_{\frac{m+1}{2}}$ if m is odd and the vertices $u_{\frac{m}{2}+1}$ and $v_{\frac{m}{2}}$ if m is even.

Definition([8]) 2.4: A comb graph is a graph obtained by attaching a single pendent edge to each vertex of a path. The comb graph is defined as $P_m \odot K_1$. Where $P_m = \{v_1, \dots, v_m\}$ be the path **with** m vertices which contains $2m$ vertices and $2m - 1$ edges.

Definition([9]) 2.5: The bistar graph $B_{m,n}$ is a graph acquired from a path P_2 by attaching the root of stars S_m and S_n at the terminal vertices of P_2, the bistar graph is denoted by $B_{m,n}$.

Remark: In this paper, we use "EOHL" instead of "even-odd harmonious labeling", for the simplicity.

III. EVEN-ODD HARMONIOUS LABELING OF ACYCLIC GRAPH

Theorem 3.1: The H-graph of path P_n is an even-odd harmonious graph.

Proof: Let $G = H_n$ be the H -graph. Let $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ be the vertices of graph G and

$E(H_n) = \{u_i u_{i+1}, v_i v_{i+1} \mid i = 1,2,\dots,n-1\} \cup \{u_{\frac{n+1}{2}} v_{\frac{n+1}{2}} \text{ if } n \equiv 1(mod\ 2)\}$ (or) $\cup \{u_{\frac{n}{2}+1} v_{\frac{n}{2}} \text{ if } n \equiv 0(mod\ 2)\}$

Note that, $p = |V(G)| = 2n$ and $q = |E(G)| = 2n - 1$.

Define an injective function $f: V(G) \longrightarrow \{1, 3, 5, \dots, 4n - 1\}$ such that

Case-1: - When $n \equiv 1(mod\ 2)$

$$f(u_i) = \begin{cases} i & ; i \equiv 1(mod\ 2) \\ 2n + i - 1 ; & i \equiv 0(mod\ 2) \end{cases} \quad \text{and}$$

$$f(v_i) = \begin{cases} 3n + i - 1 & ; i \equiv 1(mod\ 2) \\ n + i & ; i \equiv 0(mod\ 2) \end{cases}$$

Case-2: - When $n \equiv 0(mod\ 2)$

$$f(u_i) = \begin{cases} i & ; i \equiv 1(mod\ 2) \\ 2n + i - 1 ; & i \equiv 0(mod\ 2) \end{cases} \quad \text{and}$$

$$f(v_i) = \begin{cases} 2n - i & ; i \equiv 1(mod\ 2) \\ 4n - i + 1 & ; i \equiv 0(mod\ 2) \end{cases}$$

and an induced edge function

$f^*: E(G) \longrightarrow \{0, 2, 4, \dots, 4n - 4\}$ such that

Case-1: - When $n \equiv 1(mod\ 2)$

$f^*(e_i) = f^*(u_i u_{i+1}) = (2n + 2i)(mod\ 2q), 1 \le i \le n - 1$

$f^*(e_j) = f^*(v_j v_{j+1}) = 2(2n + i)(mod\ 2q), 1 \le j \le n - 1$

and

$$f^*(e') = f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = (2q + 2)(mod\ 2q).$$

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Case-2: - When $n \equiv 0 (mod\ 2)$
$f^*(e_i) = f^*(u_i u_{i+1}) = (2n + 2i)(mod\ 2q), 1 \le i \le n-1$
$f^*(e_j) = f^*(v_j v_{j+1}) = (5n - 2(i-1) + 4)(mod\ 2q)$,
$1 \le j \le n-1$ and
$f^*(e') = f^*\left(u_{\frac{n}{2}+1} v_{\frac{n}{2}}\right) = (2q + 2)(mod\ 2q)$.

The above labeling pattern give rise EOHL to the graph G, as f is an injective mapping and f^* is one-one and onto. Thus, the H-graph is even-odd harmonious graph.

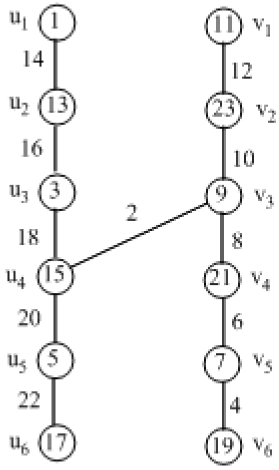

Fig.1 EOHL of H_6

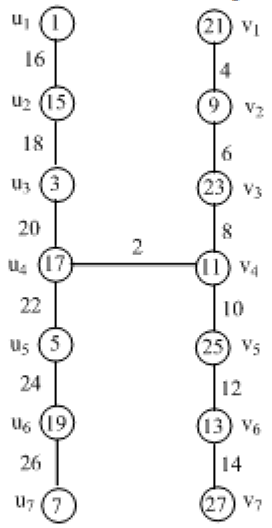

Fig.2 EOHL of H_7

Theorem 3.2: The comb graph C_{bn} is an even-odd harmonious graph, when $n \equiv 1(mod\ 2)$.

Proof: Let $G = C_{bn}$ be a comb graph, where $n \equiv 1(mod\ 2)$. Let $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ be the vertices of the graph G.

Also, $E(G) = \{ u_i v_i | \ i = 1,2, \dots, n \} \cup \{ u_i u_{i+1} \mid i = 1,2, \dots, n-1 \}$.

Here, $p = |V(G)| = 2n$ and $q = |E(G)| = 2n - 1$.

Define an injective function $f: V(G) \longrightarrow \{1, 3, 5, \dots, 4n - 1\}$ by

$$f(u_i) = \begin{cases} i & ; i \equiv 1(mod\ 2) \\ n + i & ; i \equiv 0(mod\ 2) \end{cases} \quad \text{and}$$

$$f(v_i) = \begin{cases} 3n + i - 1 & ; i \equiv 1(mod\ 2) \\ 2n + i - 1 & ; i \equiv 0(mod\ 2) \end{cases}$$

and the corresponding induced edge function $f^*: E(G) \longrightarrow \{0, 2, 4, \dots, 4n - 4\}$ by
$f^*(u_i u_{i+1}) = (n + 2i + 1)(mod\ 2q), 1 \le i \le n - 1$
$f^*(u_i v_i) = (5n + 2i - 1)(mod\ 2q), 1 \le i \le n - 1$

therefore, the function f^* is bijective. Thus, the graph $G = C_{bn}(n \equiv 1(mod\ 2))$ admits an EOHL and hence it is an even-odd harmonious graph.

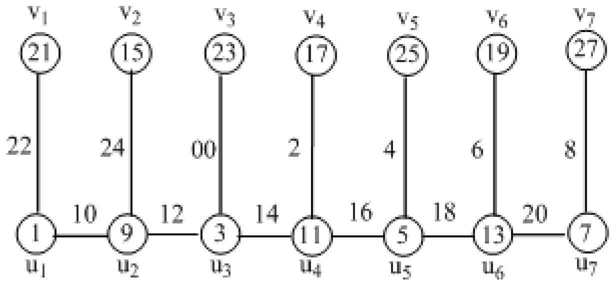

Fig.3 EOHL of comb graph C_{b7}

Theorem 3.3: The bistar graph $B_{m,n}$ is an even-odd harmonious graph.

Proof: Let $G = B_{m,n}$ be a bistar graph.

We know that, in bistar graph, $p = |V(G)| = m + n + 2$ and $q = |E(G)| = m + n + 1$.

Let $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ be the vertices of the graph G.

Also,
$E(G) = \{ uu_i \mid 1 \le i \le m \} \cup \{ uv \} \cup \{ vv_j \mid 1 \le j \le n \}$.

Now, an injective function $f: V(G) \longrightarrow \{1, 3, 5, \dots, 2m + 2n - 3\}$ is defined as follows:

$f(u) = 1, \qquad f(v) = 2p - 1$
$f(u_i) = 2i + 1, 1 \le i \le m$
$f(v_j) = 2m + 2j + 1, 1 \le j \le n$

and the corresponding edge labeling function $f^*: E(G) \longrightarrow \{0, 2, 4, \dots, 2m + 2n\}$ is defined as follows:

$f^*(e) = f^*(uv) = 2q + 2(mod\ 2q)$
$f^*(e_i) = f^*(uu_i) = 2(i + 1)(mod\ 2q), i \in \{1, 2, \dots, m\}$
$f^*(e_j') = f^*(vv_j) = m + n + 2j(mod\ 2q), j \in \{1, 2, \dots, n\}$

The above labeling pattern give rise EOHL to the graph $B_{m,n}$, as f is an injective mapping and f^* is one-one and onto function. Thus, the bistar graph $B_{m,n}$ is an even-odd harmonious graph.

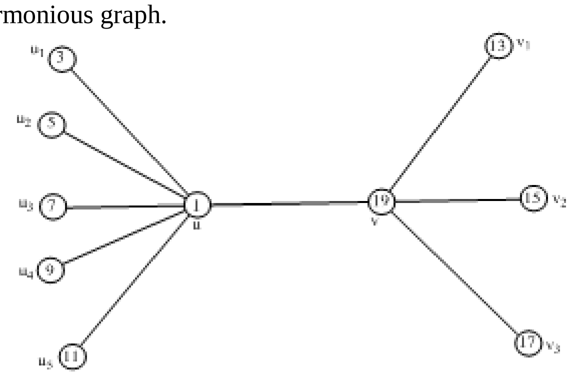

Fig.4 EOHL of the bistar graph $B_{5,3}$

Theorem 3.4: The graph $< k_{1,n}^{(1)}, k_{1,n}^{(2)}, k_{1,n}^{(3)} >$ is an even odd harmonious graph.

Proof: Let $G = < k_{1,n}^{(1)}, k_{1,n}^{(2)}, k_{1,n}^{(3)} >$. Let u_i be the apex vertex of $k_{1,n}^{(i)}$; $1 \le i \le 3$ and $v_1^{(i)}, v_2^{(i)}, \dots, v_n^{(i)}$ be the pendent vertices of $k_{1,n}^{(i)}$ for $1 \le i \le 3$.

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Here, apex vertices u_1 and u_2 are adjacent to the vertex w_1, u_1 and u_3 are adjacent to the vertex w_2.

So $p = |V(G)| = 3n + 5$ and $q = |E(G)| = 3n + 4$

Define an injective function $f: V(G) \rightarrow \{1,3,5,\dots,3(2n+3)\}$ such that

$f(u_i) = 2i - 1; i = 1,2,3$

$f(w_i) = 2i + 2^i n + 5; i = 1,2$

$f(v_j^{(i)}) = (2i + 3) + 2n(i - 2) + 2j; j = 1,2,\dots,n; i = 1,2,3$

and the corresponding edge labeling function is defined as follows:

$f^*(u_i w_j) = 2j + 5 + 2^j n + (2i - 1)(mod 2q)$; where $1 \le i \le 3, j = 1,2$

$f^*(u_i v_j^{(i)}) = 2(2i + 1) + 2n(i - 2) + 2j(mod\ 2q)$; where $1 \le i \le 3, 1 \le j \le n$

Hence f^* is bijective. Thus, the graph $G = < k_{1,n}^{(1)}, k_{1,n}^{(2)}, k_{1,n}^{(3)} >$ admits an EOHL, hence the graph $< k_{1,n}^{(1)}, k_{1,n}^{(2)}, k_{1,n}^{(3)} >$ is an even odd harmonious graph.

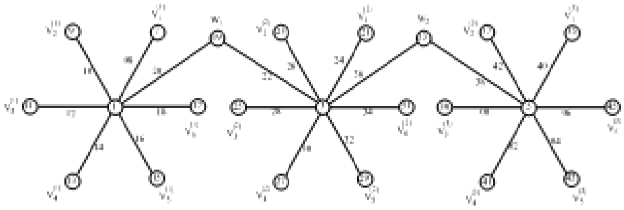

Fig.5 EOHL of $< k_{1,6}^{(1)}, k_{1,6}^{(2)}, k_{1,6}^{(3)} >$

IV. CONCLUSION

In this paper we investigated some bunch of graphs of acyclic graphs such as H-graph, comb graph, bistar graph and the graph $< k_{1,n}^{(1)}, k_{1,n}^{(2)}, k_{1,n}^{(3)} >$ which all are even-odd harmonious labeling. Also, we have proved that, comb graph C_{bn} is an even-odd harmonious graph, when n is even.

REFERENCES

1. B.D. Acharya, S.M.hegde, Arithmetic graphs. J.Graph Theory, Vol 14, No.3, 275-299(1990)
2. N. Adalin Beatress, P. B. Sarasija, Even-Odd Harmonious Graphs, International Journal of Mathematics and Soft Computing, Vol. 5, No. 1, 23-29, (2015)
3. R. L Graham, N. J. A. Sloane, On Additive Bases and Harmonious Graphs, SIAM Journal on Algebraic and Discrete Methods, Vol. 1, No. 4, 382-404, (1980).
4. F.Harary, Graph theory, Addison-Wesley, Reading Mass,(1972).
5. Joseph A. Gallian, A Dynamic Survey of Graph Labeling, The electronic journal of Combinatorics, DS6, (2018).
6. J.A.Gallian, Danielle Stewart, "Properly even harmonious labeling of disconnected graphs",AKCE International Journal Of Graphs and Combinatorics Vol.12(2015),193-203.
7. M.Kalaimathi, B.J.Balamurugan, "Computation of Even-Odd Harmonious Labeling of Certain Family of Acyclic Graphs", International journal of Engineering and Advanced Technology, Vol.9, Issue-1S3, Dec-2019, Pg No.414-419.
8. Dr.S. Meena, P.Kavtha ,"On some prime graphs" International journal of Scientific & Engineering Research,Vol.6,Issue 3,March-2015, Pg No.245-251.
9. R.Senthil Amutha, N.Murugesan,"Neighbourhood Prime labeling On Some Graphs", International journal of Innovative Research in Science, engineering and Technology,Vol.5,Issue 9, Sept.2016, Pg No.15998-16003.
10. D.B.West , Introduction to Graph theory, PHI Learning Private Limited, 2nd Edition, (2009).

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