International Journal of Innovative Technology and Exploring Engineering (IJITEE)
ISSN: 2278-3075 (Online), Volume-11 Issue-2, December 2021

Comparative Study of Various Iterative Numerical Methods for Computation of Approximate Root of the Polynomials

Roopa K M, Venkatesha P

Abstract: ***The aim of this article is to present a brief review and a numerical comparison of iterative methods applied to solve the polynomial equations with real coefficients. In this paper, four numerical methods are compared, namely: Horner's method, Synthetic division with Chebyshev method (Proposed Method), Synthetic division with Modified Newton Raphson method and Birge-Vieta method which will helpful to the readers to understand the importance and usefulness of these methods.***

Keywords: ***Horner's method, Synthetic division, Chebyshev method, Modified Newton Raphson method and Birge-Vieta method.***

I. INTRODUCTION

Numerical techniques are very important in the applications of Scientific and Engineering fields. It is also broadly used in all the major scientific disciplines. This article provides a brief on the comparative study of different methods for Numerical method to find the positive and negative roots of the polynomial equations by using various iterative techniques such as Horner's method, Synthetic division, Chebyshev method, Modified Newton-Raphson method and Birge-Vieta method

II. METHODOLOGY

1. Synthetic Division method:

Synthetic division is one of the method used to manually perform division of polynomials. The division of polynomials can also be done using the long division method. But, compared to the method of long division of polynomials, synthetic division requires less writing and fewer calculations. This means that the synthetic division is the shortest method compared to the traditional long division of a polynomial for the special cases were the division by a linear factor.

Step-1: Consider the given polynomial say $P(x)$

Step-2: Arrange the coefficient of give polynomial equation in decreasing order

Step-3: Perform the synthetic division with initial value p_0

Suppose, $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots \dots . + a_n$.

Then synthetic division process is as follows:

p_0	a_0	a_1	a_2	……….	a_{n-1}	a_n
		p_0a_0	p_0b_1	……….	p_0b_{n-2}	p_0b_{n-1}
	b_0	b_1	b_2	………….	b_{n-1}	b_n
		p_0b_0	p_0c_1	……….	p_0c_{n-2}	
	c_0	c_1	c_2	………….	c_{n-1}	

2. Horner's method:

This is one of the traditional method to finding approximate values of both rational and irrational roots of the polynomial equations. Horner's method consists in diminution of the root of any equation by successive digits occurring in the roots. First, we have to find the integer part and then find decimal part to any desired decimal place of accuracy.

The step-by-step procedure for diminution is given below:

(i) Let $f(x)$ denotes the given polynomial.

Find (a, b) for which $f(x)$ at $x = a$ and $x = b$ are the opposite signs.

(ii)Using a we have to diminish the root of the equation $f(x) = 0$ to get new equation $f_1(x) = 0$ with root $0 \cdot d_1d_2d_3d_4 \dots \dots \dots$

(iii) The root of the equation $f_1(x) = 0$ is multiply by 10, so that now equation $g(x) = 0$ is obtained with one of the root as $d_1 \cdot d_2d_3d_4 \dots \dots \dots$ which lies between 0 and 10. Thus first figure after the decimal place is d_1.

(iv) Now, d_1 is found such that $g(x)$ at $x = d_1$ and $x = (d_1 + 1)$ are of the opposite signs. Then, the root of the equation $g(x) = 0$ is diminished by d_1 and the resulting equation $g_1(x) = 0$ with root $0 \cdot d_2d_3d_4 \dots \dots \dots$

(v) The root of the equation $g_1(x) = 0$ is multiply by 10, so that now equation $h(x) = 0$ is obtained with one of the root as $d_2 \cdot d_3d_4 \dots \dots \dots$ which lies between 0 and 10. Thus the second figure after the decimal place is d_2.

Continue this procedure to gain the approximate root of the equation to any desired degree of accuracy digit by digit.

3. Synthetic Division with Chebyshev method (Proposed Method):

This Proposed method is used to determine a real root or extract linear factor of the polynomial equation with minimum iterations while compared to other three methods.

Algorithm for Synthetic Division with Chebyshev Method:

Follow Step-1 to Step-3 of synthetic division method

Manuscript received on November 23, 2021.
Revised Manuscript received on November 29, 2021.
Manuscript published on December 30, 2021.

* Correspondence Author

Dr. Roopa K M*, Department of Mathematics, Bangalore Institute of Technology, Bengaluru, India. Email: roopakm@bit-bangalore.edu.in

Venkatesha P, Department of Mathematics, Sri Sairam College of Engineering, Bengaluru, India. Email: venky.preethu743@ gmail.com

Step-4: Find the next approximate root by

$$x_{k+1} = x_k - \left\{\frac{f(x_k)}{f'(x_k)}\right\} - \frac{1}{2}\left[\frac{\{f(x_k)\}^2 f''(x_k)}{\{f'(x_k)\}^3}\right] \quad for\ k = 0,1,2,3 \ldots.$$

Step-5: Repeat the Step-3 and Step-4 until x reach desired accuracy.

4. Synthetic division with Modified Newton Raphson method:

It produces successively better approximations to the roots of a function.

Algorithm for Synthetic Division with Modified Newton Raphson Method:

Follow Step-1 to Step-3 of synthetic division method

Step-4: Find the next approximate root by

$$x_{k+1} = x_k - \left\{\frac{f(x_k)f'(x_k)}{\{f'(x_k)\}^2 - f(x_k)f''(x_k)}\right\} \quad for\ k = 0,1,2,3 \ldots.$$

Step-5: Repeat the Step-3 and Step-4 until x reach desired accuracy.

5. Birge-Vieta method:

This is an iterative method for finding real root or extract linear factor of the polynomial equation.

Algorithm for Birge-Vieta Method:

Follow Step-1 to Step-3 of synthetic division method

Step-4: Find the next approximate root by

$$p_{k+1} = p_k - \frac{b_n}{c_{n-1}}, \quad k = 0,1,2,3 \ldots.$$

Step-5: Repeat the Step-3 and Step-4 until x reach desired accuracy.

III. CASE STUDY

Case Study 1: Compute the root of the equation $x^3 + 24x - 50 = 0$.

Solution:

Horner's method: Here $f(x) = x^3 + 24x - 50 = 0$ then, $f(1) = -25 < 0$ and $f(2) = 6 > 0$

$\Rightarrow f(x) = 0$ having one of the root in the interval $(1,2)$.

So, the integral (integer) part of the root of $f(x) = 0$ is 1.

I iteration: Now, using 1 we have to diminish the root of the equation $f(x) = 0$, so that the root of the transformed equation $0 \cdot d_1\, d_2\, d_3\, d_4 \ldots\ldots\ldots$

1	1	0	24	-50
	-	1	1	25
1	1	1	25	-25
	-	1	2	
1	1	2	27	
	-	1		
	1	3		

Thus, the transformed equation $f_1(x) = x^3 + 3x^2 + 27x - 25 = 0$ has a root $0 \cdot d_1\, d_2\, d_3\, d_4 \ldots\ldots\ldots$ which occurs in the interval (0,1). Now, multiply the root of $f_1(x) = 0$ by 10 so that the changed equation $g(x) = x^3 + 30x^2 + 2700x - 25000 = 0$ has one of its roots as $d_1 \cdot d_2\, d_3\, d_4 \ldots\ldots\ldots$ We see that $g(8) < 0$ and $g(9) > 0$ so that $d_1 = 8$

II iteration: Now, using 8 we have to diminish the root of the equation $g(x) = 0$

8	1	30	2700	-25000
	-	8	304	24032
8	1	38	3004	-968
	-	8	368	
8	1	46	3372	
	-	8		
	1	54		

Thus, the transformed equation $f_2(x) = x^3 + 54x^2 + 3372x - 968 = 0$ has a root $0 \cdot d_2\, d_3\, d_4 \ldots\ldots\ldots$ which occurs in the interval (0,1). Now, multiply the root of $f_2(x) = 0$ by 10 then, the resultant equation $h(x) = x^3 + 540x^2 + 337200x - 968000 = 0$ has one of its roots as $d_2 \cdot d_3\, d_4 \ldots\ldots\ldots$ We can see that $h(2) < 0$ and $h(3) > 0$ so that $d_2 = 2$.

III iteration: Now, using 2 we have to diminish the root of the equation $h(x) = 0$

2	1	540	337200	-968000
	-	2	1084	676568
2	1	542	338284	-291432
	-	2	1088	
2	1	544	339372	
	-	2		
	1	546		

Thus, the transformed equation $f_3(x) = x^3 + 546x^2 + 339372x - 291432 = 0$ has a root $0 \cdot d_3\, d_4 \ldots\ldots\ldots$ which occurs in the interval (0,1). Now, multiply the root of $f_3(x) = 0$ by 10 then, the resultant equation $p(x) = x^3 + 5460x^2 + 33937200x - 291432000 = 0$ has one of its roots as $d_2 \cdot d_4 \ldots\ldots\ldots$ We can see that $p(8) < 0$ and $p(9) > 0$ so that this root is $8d_4 \ldots\ldots$

IV iteration: Now, using 8 we have to diminish the root of the equation $p(x) = 0$

8	1	5460	33937200	-291432000
	-	8	43744	271847552
8	1	5468	33980944	-19584448
	-	8	43808	
8	1	5476	34024752	
	-	8		
	1	5884		

Thus, the transformed equation $f_4(x) = x^3 + 5884x^2 + 34024752x - 19584448 = 0$ has a root $0 \cdot d_4 \ldots\ldots\ldots$ which occurs in the interval (0,1). Now, multiply the root of $f_4(x) = 0$ by 10 then the resultant equation $q(x) = x^3 + 58840x^2 + 3402475200x - 19584448000 = 0$

In $q(x) = 0$, $\left|\frac{\text{constant term}}{\text{coefficient of x}}\right| = \left|\frac{-19584448000}{3402475200}\right|$=5.756

So, we may guess that a root of $f_4(x) = 0$ has between 5 and 6.

Hence, the required root of the given equation corrects to 3rd place of decimals is 1.8285≅1.829.

Synthetic division with Chebyshev method:

Here, $f(x) = x^3 + 24x - 50 = 0 \Rightarrow f'(x) = 3x^2 + 24$ and $f''(x) = 6x$

We know that $f(x) = 0$ having one of the root in the interval $(1,2)$.

Retrieval Number: 100.1/ijitee.B96351211221
DOI: 10.35940/ijitee.B9635.1211221
Journal Website: www.ijitee.org

Published By:
Blue Eyes Intelligence Engineering and Sciences Publication (BEIESP)

I iteration:
From the first iteration of Horner's method,
$f(1) = -25,\ f'(1) = 27,\ \frac{1}{2!}f''(1) = 3 \Rightarrow f''(1) = 6$
Chebyshev method: $x_1 = x_0 - \left\{\frac{f(x_0)}{f'(x_0)}\right\} - \frac{1}{2}\left[\frac{\{f(x_0)\}^2 f''(x_0)}{\{f'(x_0)\}^3}\right]$
$= 1 - \left\{\frac{-25}{27}\right\} - \frac{1}{2}\left[\frac{\{-25\}^2(6)}{\{27\}^3}\right] = 1 + 0.926 - 0.095 = 1.831$

II iteration:

1.831	1	0	24	-50
	-	1.831	3.353	50.083
1.831	1	1.831	27.353	0.083
	-	1.831	6.705	
1.831	1	3.662	34.058	
	-	1.831		
	1	5.493		

$f(1.831) = 0.083,\ f'(1.831) = 34.058,\ \frac{1}{2!}f''(1.831) = 5.493$

$\Rightarrow f''(1.831) = 10.986$
Chebyshev method: $x_2 = x_1 - \left\{\frac{f(x_1)}{f'(x_1)}\right\} - \frac{1}{2}\left[\frac{\{f(x_1)\}^2 f''(x_1)}{\{f'(x_1)\}^3}\right]$
$= 1.831 - \left\{\frac{0.083}{34.058}\right\} - \frac{1}{2}\left[\frac{\{0.083\}^2(10.986)}{\{34.058\}^3}\right]$
$= 1.831 - 0.002 - 0.000$
$\Rightarrow x_2 = 1.829$

III iteration:

1.829	1	0	24	-50
	-	1.829	3.345	50.014
1.829	1	1.829	27.345	0.014
	-	1.829	6.690	
1.829	1	3.658	34.035	
	-	1.829		
	1	5.487		

$f(1.829) = 0.014,\ f'(1.829) = 34.035,\ \frac{1}{2!}f''(1.829) = 5.487 \Rightarrow f''(1.831) = 10.974$

Chebyshev method: $x_3 = x_2 - \left\{\frac{f(x_2)}{f'(x_2)}\right\} - \frac{1}{2}\left[\frac{\{f(x_2)\}^2 f''(x_2)}{\{f'(x_2)\}^3}\right]$
$= 1.829 - \left\{\frac{0.014}{34.035}\right\} - \frac{1}{2}\left[\frac{\{0.014\}^2(10.974)}{\{34.035\}^3}\right]$
$= 1.829 - 0.000 - 0.000 \Rightarrow x_3 = 1.829$

Synthetic division with Modified Newton Raphson method:
Here $f(x) = x^3 + 24x - 50 = 0 \Rightarrow f'(x) = 3x^2 + 24$ and $f''(x) = 6x$.
We know that $f(x) = 0$ having one of the root in the interval $(1,2)$.

I iteration: From the I iteration of Horner's method,
$f(1) = -25,\ f'(1) = 27,\ \frac{1}{2!}f''(1) = 3 \Rightarrow f''(1) = 6$
From Modified Newton Raphson method,
$x_1 = x_0 - \left\{\frac{f(x_0)f'(x_0)}{\{f'(x_0)\}^2 - f(x_0)f''(x_0)}\right\} = 1 - \left\{\frac{(-25)(27)}{\{27\}^2 - (-25)(6)}\right\}$
$= 1 + 0.768 = 1.768$ `

II iteration:

1.768	1	0	24	-50
	-	1.768	3.126	47.959
1.768	1	1.768	27.126	-2.041
	-	1.768	6.252	
1.768	1	3.536	33.378	
	-	1.768		
	1	5.304		

$f(1.768) = -2.041,\ f'(1.768) = 33.378,\ \frac{1}{2!}f''(1.768) = 5.304 \Rightarrow f''(1.768) = 10.908$

From Modified Newton Raphson method,
$x_2 = x_1 - \left\{\frac{f(x_1)f'(x_1)}{\{f'(x_1)\}^2 - f(x_1)f''(x_1)}\right\} = 1 - \left\{\frac{(-2.041)(33.378)}{\{33.378\}^2 - (-2.041)(10.908)}\right\}$

$= 1.768 + 0.06 = 1.828$

III iteration:

1.828	1	0	24	-50
	-	1.828	3.342	49.981
1.828	1	1.828	27.342	-0.019
	-	1.828	6.684	
1.828	1	3.656	34.026	
	-	1.828		
	1	5.484		

$f(1.828) = -0.019,\ f'(1.828) = 34.026,\ \frac{1}{2!}f''(1.828) = 5.484 \Rightarrow f''(1.828) = 10.968$

From Modified Newton Raphson method,
$x_3 = x_2 - \left\{\frac{f(x_2)f'(x_2)}{\{f'(x_2)\}^2 - f(x_2)f''(x_2)}\right\} = 1 - \left\{\frac{(-0.019)(34.026)}{\{34.026\}^2 - (-0.019)(10.968)}\right\}$

$= 1.828 + 0.001$
$\Rightarrow x_3 = 1.829$

IV iteration:

1.829	1	0	24	-50
	-	1.829	3.345	50.014
1.829	1	1.829	27.345	0.014
	-	1.829	6.690	
1.829	1	3.658	34.035	
	-	1.829		
	1	5.487		

$f(1.829) = 0.014,\ f'(1.829) = 34.035,\ \frac{1}{2!}f''(1.829) = 5.487 \Rightarrow f''(1.829) = 10.974$

From Modified Newton Raphson method,
$x_4 = x_3 - \left\{\frac{f(x_3)f'(x_3)}{\{f'(x_3)\}^2 - f(x_3)f''(x_3)}\right\} = 1 - \left\{\frac{(0.014)(34.035)}{\{34.035\}^2 - (0.014)(10.974)}\right\}$

$= 1.829 + 0.000$
$\Rightarrow x_4 = 1.829$

Retrieval Number: 100.1/ijitee.B96351211221
DOI: 10.35940/ijitee.B9635.1211221
Journal Website: www.ijitee.org

Published By:
Blue Eyes Intelligence Engineering and Sciences Publication (BEIESP)

Birge-Vieta Method:

Here $f(x) = x^3 + 24x - 50 = 0 \Rightarrow f'(x) = 3x^2 + 24$

We know that $f(x) = 0$ having one of the root in the interval $(1,2)$.

I iteration:

From the I iteration of Horner's method,

$f(1) = -25,\ f'(1) = 27$

From Newton Raphson method,

$x_1 = x_0 - \left\{\frac{f(x_0)}{f'(x_0)}\right\} = 1 - \left\{\frac{-25}{27}\right\} = 1 + 0.926 \Rightarrow x_1 = 1.926$

II iteration:

1.926	1	0	24	-50
	-	1.926	3.709	53.368
1.926	1	1.926	27.709	3.368
	-	1.926	7.419	
	1	3.852	35.128	

$\Rightarrow f(1.926) = 3.368,\ f'(1.926) = 35.128$

From Newton Raphson method,

$x_2 = x_1 - \left\{\frac{f(x_1)}{f'(x_1)}\right\} = 1.926 - \left\{\frac{3.368}{35.128}\right\} = 1.926 - 0.096$

$\Rightarrow x_2 = 1.83$

III iteration:

1.83	1	0	24	-50
	-	1.83	3.349	50.213
1.83	1	1.83	27.349	0.213
	-	1.83	6.698	
	1	3.66	34.137	

$\Rightarrow f(1.83) = 0.213,\ f'(1.83) = 34.137$

From Newton Raphson method,

$x_3 = x_2 - \left\{\frac{f(x_2)}{f'(x_2)}\right\} = 1.83 - \left\{\frac{0.213}{34.137}\right\} = 1.83 - 0.006$

$\Rightarrow x_3 = 1.824$

IV iteration:

1.824	1	0	24	-50
	-	1.824	3.327	49.844
1.824	1	1.824	27.327	-0.156
	-	1.824	6.654	
	1	3.648	33.981	

$\Rightarrow f(1.824) = -0.156,\ f'(1.824) = 33.981$

From Newton Raphson method,

$x_4 = x_3 - \left\{\frac{f(x_3)}{f'(x_3)}\right\} = 1.824 - \left\{\frac{-0.156}{33.981}\right\} = 1.824 + 0.005$

$\Rightarrow x_4 = 1.829$

V iteration:

1.829	1	0	24	-50
	-	1.829	3.345	50.014
1.829	1	1.829	27.345	0.014
	-	1.829	6.690	
	1	3.658	34.035	

$\Rightarrow f(1.829) = 0.014,\ f'(1.829) = 34.035$

From Newton Raphson method,

$x_5 = x_4 - \left\{\frac{f(x_4)}{f'(x_4)}\right\} = 1.83 - \left\{\frac{0.014}{34.035}\right\} = 1.829 - 0.000$

$\Rightarrow x_5 = 1.829$

Case Study 2: Compute the root of the equation $x^3 + x^2 + 3x + 4 = 0$.

Solution:

Horner's method: Here

$f(x) = x^3 + x^2 + 3x + 4 = 0 \cdots\cdots\cdots (1)$

Then, $f(-2) = -6 < 0$ and $f(-1) = 1 > 0$

$\Rightarrow f(x) = 0$ having one of the root in the interval $(-2,-1)$. So, change x to $-x$ in $f(x)$.

we get $-x^3 + x^2 - 3x + 4 = 0 \Rightarrow x^3 - x^2 + 3x - 4 = 0$

Let $g(x) = x^3 - x^2 + 3x - 4 = 0 \cdots\cdots\cdots (2)$

Now negative root of (1) is equal to the positive root of (2) with the sign changed$g(x) = x^3 - x^2 + 3x - 4 = 0$.

Then, $g(1) = -1 < 0$ and $g(2) = 6 > 0$

$\Rightarrow g(x) = 0$ having one of the root in the interval $(1,2)$.

Hence, the integral (integer) part of the root of $g(x) = 0$ is 1.

I iteration: Now, using 1 we have to diminish the root of the equation $g(x) = 0$.

1	1	-1	3	-4
	-	-1	0	-3
1	1	0	3	-1
	-	1	1	
1	1	1	4	
	-	1		
	1	2		

Thus, the transformed equation $g_1(x) = x^3 + 2x^2 + 4x - 1 = 0$

$\Rightarrow g_1(x) = 0$ having a root in the interval $(0,1)$.

Now, multiply the root of $g_1(x) = 0$ by 10 so that the changed equation $g_1(x) = x^3 + 20x^2 + 400x - 1000 = 0$.

We see that $g_1(2) < 0$ and $g_1(3) > 0$

So, the root of $g_1(x) = 0$ which occurs between 2 and 3. Hence the first figure after the decimal place is 2

II iteration: Now, using 2 we have to diminish the root of the equation $g_1(x) = 0$

2	1	20	400	-1000
	-	2	44	888
2	1	22	444	-112
	-	2	48	
2	1	24	492	
	-	2		
	1	26		

Thus, the transformed equation $g_2(x) = x^3 + 26x^2 + 492x - 112 = 0$

$\Rightarrow g_2(x) = 0$ having a root in the interval $(0,1)$. Now, multiply the root of $g_2(x) = 0$ by 10. then, the resultant equation $g_2(x) = x^3 + 260x^2 + 49200x - 112000 = 0$.

Consider the last two terms by omitting the remaining terms of $g_2(x) = 0$.

We get, $49200x - 112000 = 0 \Rightarrow x = \frac{112000}{49200} = 2.28$

We can see that $g_2(2) < 0$ and $g_2(3) > 0$. So, the root of $g_2(x) = 0$ which occurs between 2 and 3

Hence the second figure after the decimal place is 2.

Retrieval Number: 100.1/ijitee.B96351211221
DOI: 10.35940/ijitee.B9635.1211221
Journal Website: www.ijitee.org

Published By:
Blue Eyes Intelligence Engineering and Sciences Publication (BEIESP)

III iteration: Now, using 2 we have to diminish the root of the equation $g_2(x) = 0$

2	1	260	49200	-112000
	-	2	524	99448
2	1	262	49724	-12552
	-	2	528	
2	1	264	50252	
	-	2		
	1	266		

Thus, the transformed equation $g_3(x) = x^3 + 266x^2 + 50252x - 12552 = 0$
$\Rightarrow g_3(x) = 0$ having a root in the interval $(0,1)$. Now, multiply the root of $g_3(x) = 0$ by 10 then, the equation $g_3(x) = x^3 + 2660x^2 + 5025200x - 12552000 = 0$.
Consider the last two terms by omitting the remaining terms of $g_3(x) = 0$, we get $5025200x - 12552000 = 0$
$\Rightarrow x = \frac{12552000}{5025200} = 2.50$
We can see that $g_3(2) < 0$ and $g_3(3) > 0 \Rightarrow$ the root of $g_3(x) = 0$ occurs between 2 and 3
Hence the third figure after the decimal place is 2.

IV iteration: Now, using 2 we have to diminish the root of the equation $g_3(x) = 0$

2	1	2660	5025200	-12552000
	-	2	5324	10061048
2	1	2662	5030524	-2490952
	-	2	5328	
2	1	2664	5035852	
	-	2		
	1	2666		

Thus, the transformed equation $g_4(x) = x^3 + 2666x^2 + 5035852x - 2490952 = 0$
$\Rightarrow g_4(x) = 0$ having one of the root in the interval $(0,1)$.
Now, multiply the root of $g_4(x) = 0$ by 10 then, the resultant equation
$g_4(x) = x^3 + 26660x^2 + 503585200x - 2490952000 = 0$
Consider the last two terms by omitting the remaining terms of $g_4(x) = 0$.
We get, $503585200x - 2490952000 = 0$
$\Rightarrow x = \frac{2490952000}{503585200} = 4.95$
We can see that $g_4(4) < 0$ and $g_4(5) > 0$. The root of $g_4(x) = 0$ which occurs in the interval $(4,5)$
So, the fourth figure after the decimal place is 4.
Hence, the root of $g(x) = 0$ in the three places of decimals is 1.2224≅1.222.
Therefore, the root of the given equation $f(x) = 0$ is -1.222

Synthetic division with Chebyshev method:
Here, $f(x) = x^3 + x^2 + 3x + 4 = 0 \cdots\cdots\cdots (2)$
Then, $f(-2) = -6 < 0$ and $f(-1) = 1 > 0$
$\Rightarrow f(x) = 0$ having a negative root in the interval $(-2, -2)$.
So, change x to $-x$ in $f(x)$
we get $-x^3 + x^2 - 3x + 4 = 0 \Rightarrow x^3 - x^2 + 3x - 4 = 0$
Let $g(x) = x^3 - x^2 + 3x - 4 = 0 \cdots\cdots\cdots (3)$
Now negative root of (3) is equal to the positive root of (4) with the sign changed

I iteration: $g(x) = x^3 - x^2 + 3x - 4 = 0$
$\Rightarrow g'(x) = 3x^2 - 2x + 3$ and $g''(x) = 6x - 2$

$g(1) = -1,\ g'(1) = 4,\ \frac{1}{2!}g''(1) = 2 \Rightarrow g''(1) = 4$
Chebyshev method, $x_1 = x_0 - \left\{\frac{g(x_0)}{g'(x_0)}\right\} - \frac{1}{2}\left[\frac{\{g(x_0)\}^2 g''(x_0)}{\{g'(x_0)\}^3}\right]$
$= 1 - \left\{\frac{-1}{4}\right\} - \frac{1}{2}\left[\frac{\{-1\}^2(4)}{\{4\}^3}\right]$
$= 1 + 0.25 - 0.031 \Rightarrow x_1 = 1.219$

II iteration:

1.219	1	1	3	-4
	-	1.219	0.270	3.986
1.219	1	0.219	3.270	-0.014
	-	1.219	1.753	
1.219	1	1.438	5.023	
	-	1.219		
	1	2.657		

$g(1.219) = -0.014,\ g'(1.219) = 5.023,$
$\frac{1}{2!}g''(1.219) = 2.657 \Rightarrow g''(1.219) = 5.314$
Chebyshev method: $x_2 = x_1 - \left\{\frac{g(x_1)}{g'(x_1)}\right\} - \frac{1}{2}\left[\frac{\{g(x_1)\}^2 g''(x_1)}{\{g(x_1)\}^3}\right]$
$= 1.219 - \left\{\frac{-0.014}{5.023}\right\} - \frac{1}{2}\left[\frac{\{-0.014\}^2(5.314)}{\{5.023\}^3}\right]$
$= 1.219 + 0.003 - 0.000$
$\Rightarrow x_2 = 1.222$

Synthetic division with Modified Newton Raphson method:
Here $f(x) = x^3 + x^2 + 3x + 4 = 0$
$\Rightarrow f'(x) = 3x^2 + 2x + 3$ and $f''(x) = 6x + 2$
We know that $f(x) = 0$ having a root in the interval $(-2, -1)$

I iteration:
From the I iteration of Horner's method,
$f(-1) = 1,\ f'(-1) = 4,\ \frac{1}{2!}f''(-1) = -2 \Rightarrow f''(-1) = -4$
From Modified Newton Raphson method,
$x_1 = x_0 - \left\{\frac{f(x_0)f'(x_0)}{\{f'(x_0)\}^2 - f(x_0)f''(x_0)}\right\} = -1 - \left\{\frac{(1)(4)}{\{4\}^2 - (1)(-4)}\right\}$
$= -1 - 0.2 \Rightarrow x_1 = -1.2$

II iteration:

-1.2	1	1	3	4
	-	-1.2	0.24	-3.888
-1.2	1	-0.2	3.24	0.112
	-	-1.2	1.68	
-1.2	1	-1.4	4.92	
	-	-1.2		
	1	-2.6		

$f(-1.2) = 0.112,\ f'(-1.2) = 4.92,\ \frac{1}{2!}f''(-1.2) = -2.6 \Rightarrow f''(-1.2) = -5.2$

From Modified Newton Raphson method,
$x_2 = x_1 - \left\{\frac{f(x_1)f'(x_1)}{\{f'(x_1)\}^2 - f(x_1)f''(x_1)}\right\} = 1 - \left\{\frac{(0.112)(4.92)}{\{4.92\}^2 - (0.112)(-5.2)}\right\}$
$= -1.2 - 0.022$
$\Rightarrow x_2 = -1.222$

III iteration:

-1.222	1	1	3	4
	-	-1.222	0.271	-3.997
-1.222	1	-0.222	3.271	0.003
	-	-1.222	1.765	
-1.222	1	-1.444	5.036	
	-	-1.222		
	1	-2.666		

$f(-1.222) = 0.003,\ f'(-1.222) = 5.036,$
$\frac{1}{2!}f''(-1.222) = -2.666 \Rightarrow f''(-1.222) = -5.332$

From Modified Newton Raphson method,

$$x_3 = x_2 - \left\{\frac{f(x_2)f'(x_2)}{\{f'(x_2)\}^2 - f(x_2)f''(x_2)}\right\} = 1 - \left\{\frac{(0.003)(5.036)}{\{5.036\}^2 - (0.003)(-5.332)}\right\}$$

$$= -1.222 - 0.001$$

$\Rightarrow x_3 = -1.223$

IV iteration:

-1.223	1	1	3	4
	-	-1.223	0.273	-4.003
-1.223	1	-0.223	3.273	-0.003
	-	-1.223	1.768	
-1.223	1	-1.446	5.041	
	-	-1.223		
	1	-2.669		

$f(-1.223) = -0.003,\ f'(-1.223) = 5.041,$
$\frac{1}{2!}f''(-1.223) = -2.669 \Rightarrow f''(-1.223) = -5.338$

From Modified Newton Raphson method,

$$x_4 = x_3 - \left\{\frac{f(x_3)f'(x_3)}{\{f'(x_3)\}^2 - f(x_3)f''(x_3)}\right\} = 1 - \left\{\frac{(-0.003)(5.041)}{\{5.041\}^2 - (-0.003)(-5.338)}\right\}$$

$$= -1.223 - 0.001$$

$\Rightarrow x_4 = -1.222$

Birge-Vieta Method:

Here $f(x) = x^3 + x^2 + 3x + 4 = 0 \Rightarrow f'(x) = 3x^2 + 2x + 3$

We know that $f(x) = 0$ having one of the root in the interval $(-2, -1)$.

I iteration: From the I iteration of Horner's method,

$f(-1) = 1,\ f'(-1) = 4$

From Newton Raphson method:

$$x_1 = x_0 - \left\{\frac{f(x_0)}{f'(x_0)}\right\} = -1 - \left\{\frac{1}{4}\right\} = -1.25 \Rightarrow x_1 = -1.25$$

II iteration:

-1.25	1	1	3	4
	-	-1.25	0.313	-4.141
-1.25	1	-0.25	3.313	-0.141
	-	-1.25	1.875	
	1	-1.50	5.188	

$\Rightarrow f(-1.25) = -0.141,\ f'(-1.25) = 5.188$

From Newton Raphson method,

$$x_2 = x_1 - \left\{\frac{f(x_1)}{f'(x_1)}\right\} = -1.25 - \left\{\frac{-0.141}{5.188}\right\} = -1.25 + 0.027$$

$\Rightarrow x_2 = -1.223$

III iteration:

-1.223	1	1	3	4
	-	-1.223	0.273	-4.003
-1.223	1	-0.223	3.273	-0.003
	-	-1.223	1.768	
	1	-1.446	5.041	

$\Rightarrow f(-1.223) = -0.003,\ f'(-1.223) = 5.041$

From Newton Raphson method,

$$x_3 = x_2 - \left\{\frac{f(x_2)}{f'(x_2)}\right\} = -1.223 - \left\{\frac{-0.003}{5.041}\right\} = -1.223 + 0.001$$

$\Rightarrow x_3 = -1.223$

IV. COMPARISON TABLE

Table I: Number of iterations performed to compute the approximate root of the polynomial equations in the above case study

	Number of iterations			
Case	**Horner's method**	**Chebyshev method**	**Modified Newton Raphson method**	**Birge-Vieta Method**
1	4	3	4	5
2	4	2	4	3

V. CONCLUSION

While computing the approximate root of the polynomial equations using many methods in the above case study, it is observed that **Synthetic Division with Chebyshev Method (Proposed Method)** is converged quickly compared to other three methods. Also, it requires less time consumption in performing the iterations to compute approximate root of the polynomial equations.

REFERENCES

1. Comparative analysis of polynomial root finding techniques; STUDENT THESES AND DISSERTATIONS, 1967: Charles Ray O'Daniel
2. Mathematical Model Formulation and Comparison Study of Various Methods of Root- Finding Problems; IOSR Journal of Mathematics (IOSR-JM); Arup Dey; e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 2 Ver. III (Mar - Apr. 2015), PP 64-71 www.iosrjournals.org
3. Comparative Study of Bisection, Newton-Raphson and Secant Methods of Root- Finding Problems; Ehiwario, J.C., Aghamie, S.O.; IOSR Journal of Engineering (IOSRJEN) ISSN (e): 2250-3021, ISSN (p): 2278-8719
4. Text Book-Numerical methods by R. K Jain and S.R.K Iyengar

AUTHORS PROFILE

Dr. Roopa KM, M.Sc., M.Phil., Ph.D. Professor, Department of Mathematics, Dean for Skill Development, Bangalore Institute of Technology, Bengaluru, Published 10 Papers in National and International Journals.

Venkatesha P, M.Sc., (Ph.D). Assistant Professor, Department of Mathematics, Sri Sairam College of Engineering, Bengaluru.

Retrieval Number: 100.1/ijitee.B96351211221
DOI: 10.35940/ijitee.B9635.1211221
Journal Website: www.ijitee.org

Published By:
Blue Eyes Intelligence Engineering and Sciences Publication (BEIESP)