Analytical Solution of Non-Linear DPL Bioheat Transfer Model for Temperature Dependent Metabolic Heat Source During Thermal Therapy

Neha Sharma, Surjan Singh, Dinesh Kumar

Abstract: *In this research paper, the simulation based modelling of heat transfer in tissue under periodic boundary condition has been considered. The dual-phase-lag bioheat transfer (DPLBHT) model is implemented for computation of the temperature based thermal therapy treatment. The elements of volumetric heat source such as perfusion of blood, metabolism heat source and external heat source are considered in non-linear DPL model. In this paper we have taken three cases for metabolic heat source namely, constant, linear and exponential. The combined two numerical methods which are based on finite difference scheme and Runge - Kutta (4, 5) scheme are exerted to solve the non-linear problem. We compute the exact solution for particular case. Numerical conclusions which are computed by numerical hybrid method are equated with exact result. It has been found that non-linear DPL model with exponential metabolic heat source is closed to exact solution. We also expressed the effect of different parameters such as relaxation time, perfusion rate, metabolic heat source parameter, associated blood perfusion heat, heat source because of heat flux and temperature gradient etc.*

Keywords: Bioheat, Hybrid method, Metabolic heat source, Non-linear, Runge - Kutta Method, Thermal therapy, Tissues.

I. INTRODUCTION

Heat is caused by perfusion of blood and metabolism in living body. The heat which is produced during metabolic process is defined as metabolic heat such as growth and produced of energy in the living biological system. Thermal therapy is one of the most important therapies for tumor analysis. In thermal therapy, heat is used to Damage the tissue in therapeutic treatment. It involves the procedure of heating of tissues for their removal. Blood perfusion and metabolic heat generation have an important effect on heat transfer in tissues. The thermal therapy is determined by temperature which is based on bioheat transfer. The mathematical modelling of bioheat transfer with different cases of metabolic heat source is explained in this article.

Manuscript received on 30 June 2022 | Revised Manuscript received on 11 July 2022 | Manuscript Accepted on 15 July 2022 | Manuscript published on 30 July 2022.
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Nomenclature:

q	heat flux, W/m^2
r	space coordinate, m
t	time, s
k	thermal conductivity of tissue, $W/m℃$
T	temperature of tissue, $℃$
τ_q	phase lag due to heat flux, s
τ_T	phase lag due to temperature gradient, s
ρ	density of living tissue, kg/m^3
c	specific heat of tissue, $J/kg℃$
w_b	blood perfusion rate, s^{-1}
ρ_b	density of blood, kg/m^3
c_b	specific heat of blood, $J/kg℃$
T_b	arterial blood temperature, $℃$
w_{bo}	reference blood perfusion term, s^{-1}
Q_{mo}	reference metabolic heat generation, W/m^3
β	associated metabolism constant, $℃$
S	antenna constant associated with transmitted power, kg^{-1}
P	transmitted power, W
a	antenna constant associated with location of probe region, m^{-1}
r_p	length of probe region, m
l	length of tissue, m
T_w	wall temperature of outer boundary, $℃$

Non-dimensional variables

x	space coordinate
F_o	time
F_{oq}	phase lag due to heat flux
F_{oT}	phase lag due to temperature gradient
θ	local tissue temperature
P_f	blood perfusion coefficient
P_m	metabolic heat source coefficient
P_r	external heat source coefficient
α	associated metabolism constant
θ_w	wall temperature at boundary

Abbreviation

PBHT = Pennes bioheat transfer
SPL = Single phase lag
SPLBHT = Single-phase-lag bioheat transfer
DPL = Dual phase lag
DPLBHT = Dual-phase-lag bioheat transfer
RKM = Runge- Kutta Method
FDM = Finite Difference Method

Pennes model of bioheat transfer [5] is used. This model is based on Fourier law of heat conduction with infinite velocity of thermal signal. Fourier law of heat conduction is given as:

$$q(r,t) = -k\nabla T(r,t), \quad (1)$$

where $q(r,t)$ is heat flux; k is proportionality constant which is thermal conductivity and ∇ is temperature gradient. Negative sign shows that the distance increases the temperature decreases. In sense of finite speed propagation of thermal wave, the non-Fourier law of heat conduction is the next modification of Fourier's law.

The modification is in sense of finite speed propagation of thermal wave. There are two types of constitutive relation i.e. SPL and DPL which are formulated under non-Fourier law of heat conduction.

The relation between heat flux and relaxation time because of heat flux is SPL constitutive relation which is independently observed by Cattaneo [1] and Vernotte [7] and is defined as:

$$q(r,t+\tau_q) = -k\nabla T(r,t), \quad (2)$$

The next modification of SPL constitutive relation is dual-phase-lag constitutive relation (DPL) given by Tzou [2] which involves two kind of lagging time because of heat flux and temperature gradient respectively i.e.

$$q(r,t+\tau_q) = -k\nabla T(r,t+\tau_T), \quad (3)$$

The DPLBHT model gives an accurate temperature, when DPL constitutive relation combined with energy balance equation.

We used the first order Taylor's series expansion of DPL model in our problem. Therefore the first order Taylor's series expansion of DPL model (3) can be written as

$$q(r,t) + \tau_q \frac{\partial T}{\partial r} = -k\left[\nabla T(r,t) + \tau_T \frac{\partial \nabla T}{\partial t}\right], \quad (4)$$

Several researchers studied mathematical modelling related to bioheat transfer. Wissler [10] took the physiological parameters to describe the mathematical modelling to simulate the physical characteristics of the human thermal system in the transient state. Finite-difference technique has been used to solve the heat conduction equation. Eberhart [11] gave the solution of inverse problem with the prediction of w_b and q from the solution of bioheat transfer equation. Singh [12] observed that metabolism has an important role in biological tissue with cryosurgery. They studied the effect of metabolic heat generation on the freezing of biological tissue by using Quasi-steady approximation. Shitzer [13] presented the analysis of thermal profile of living tissues taking boundary conditions of general type. It is shown that in areas where convection by perfusion of blood is dominant near the boundaries. Saxena [14] introduced the unsteady state temperature profile in the human body where subcutaneous tissues are absent. The metabolic heat generation is taken as temperature dependent.

The problem has been solved by using Laplace transform and Bessel functions. Rai [15] considered the metabolic heat source and blood perfusion are temperature dependent in the tissue. They observed that variable metabolic heat generation and blood perfusion instead of constant metabolic heat generation and blood perfusion in the PBHT equation gives accurate explanation about the process of the heat transfer in the tissue.

Moradi [16] gave the algorithm depend on the linearly temperature-dependent enthalpy method to solve the solidification of biological tissues DPL model. They studied the effect of blood perfusion and metabolic heat source in the freezing process of biological tissue. Kumar [17] developed the PBHT, SPL, DPL models in biological tissues. The results are compared with experimental data with three models obtained by using Laplace transform. They also evaluated the heat source in first, second and third BCs of thermal therapy.

In this research paper, the DPLBHT model is analyzing by applying periodic boundary condition. The whole paper is solved and written in non-dimensional form. The problem is converted into non-linear ordinary differential equation of second order by discretizing space coordinate with central difference formula. Again the non-linear ordinary differential equation of second order is converted into non-linear ODEs of first order with initial value problem. Then initial value problem is solved by Runge-Kutta (4, 5) scheme. In this research paper, we solved the DPLBHT model with temperature dependent metabolic heat source taking as a constant, linear and exponential in different cases. The effect of various parameters is shown with dimensionless temperature and dimensionless time. In particular case, we conclude the exact solution by using Laplace and Inverse Laplace technique and compared it with present numerical method. This shows that exponential metabolic heat source gives good result with exact solution.

II. FORMULATION OF THE PROBLEM

The one-dimensional non-homogeneity of inner structure of living tissue of l length and taking initial temperature T_0 is shown in Fig.1.

The one-dimensional bioheat energy balanced equation is usually given by the known PBHT [5] i.e.

$$\rho c \frac{\partial T}{\partial t} = -\nabla q + Q_b + Q_m + Q_e, \quad (5)$$

where ρ, c are density and specific heat of material. Q_b, Q_m are the temperature dependent blood perfusion term and metabolic heat source respectively. Q_e is the external heat source in PBHT model.

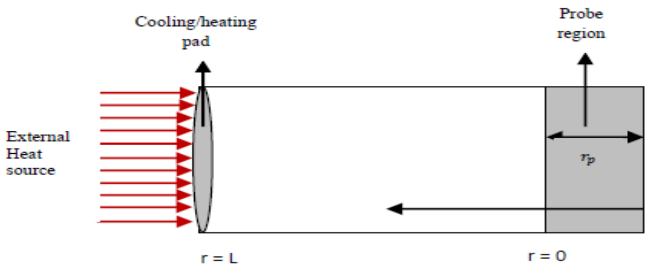

Fig.1. Schematic geometry of the skin tissue with the probe region.

Blood perfusion term Q_b manage to transfer the oxygen, nutrients, and waste products from the body with the help of flow of blood and mathematically expressed by Zhang et.al. [9] as

$$Q_b = w_b(T)\rho_b c_b(T_b - T), \quad (6)$$

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where temperature dependent blood perfusion rate is $w_b(T)$, ρ_b is density of blood; c_b is specific heat of blood; T_b and T are the arterial blood temperature and local tissue temperature rate respectively.

There are many physiological processes due to which heat is generated in living biological tissue. The metabolism term Q_m indicates heat generated in tissue due to physical activities. The three different cases of metabolic heat source are taken in this article and as follows:

$$Q_m(T) = \left\{\begin{matrix} Q_{m0} \\ Q_{m0} \times \left(1 + \beta\left(\frac{T - T_0}{10}\right)\log 2\right) \\ Q_{m0} \times e^{\beta\left(\frac{T-T_0}{10}\right)} \end{matrix}\right\}, \tag{7}$$

where Q_{m0} is reference metabolic heat source and β is associated metabolic constant.

The term Q_e indicates externally heat generated per unit volume of local tissue due to electromagnetic radiation absorbed and it is given by Kumar et al. [4] which is as follows:

$$Q_e = \rho S P e^{a(\bar{r} - r_p)}, \tag{8}$$

where S and a are antenna constant associated with transmitted power and antenna constant associated with location of probe region respectively. P is transmitted power; $\bar{r}(= l - r)$ is distance of tissue from outer surface and r_p is length of probe region.

Now for non-linear DPLBHT equation, we combine one-dimensional energy balanced equation (5) and first order Taylor's series expansion of DPL constitutive relation (4), then the non-linear

DPLBHT equation becomes:

$$\rho c \tau_q \frac{\partial^2 T}{\partial t^2} + \rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial r^2} + k\tau_T \frac{\partial^3 T}{\partial t \partial r^2} + Q_b + Q_m + Q_e + \tau_q \frac{\partial Q_b}{\partial t} + \tau_q \frac{\partial Q_m}{\partial t} + \tau_q \frac{\partial Q_e}{\partial t}, \tag{9}$$

subject to initial condition

$$T(r, 0) = T_0, \quad \frac{\partial T(r, 0)}{\partial t} = 0. \tag{10}$$

Our assumptions we consider fluctuating temperature i.e. periodic BC, which is given by Kumar et al. [3] as follows:

$$T(L, t) = T_w + A\cos\omega t, \tag{11}$$

and inner surface is insulated so that the symmetric condition at $r = 0$ is given as follows

$$\frac{\partial T(0, t)}{\partial r} = 0. \tag{12}$$

III. CONVERSION OF THE PROBLEM INTO NON-DIMENSIONAL FORM

The dimensional DPLBHT model is converted into non-dimensional quantities through which we can solve mathematical model easily.

Dimensionless variables and similarity criteria which to convert the problem $(9) - (12)$ into non-dimensional variable are as follows:

$x = \frac{r}{l}, F_o = \frac{kt}{\rho c L^2}, F_{oq} = \frac{k\tau_q}{\rho c L^2}, F_{oT} = \frac{k\tau_T}{\rho c L^2},\ \theta = \frac{T - T_0}{T_0}, P_f^2 = \frac{w_{bo} c_b \rho_b}{k} L^2, P_m = \frac{Q_{m0} L^2}{k T_0}, P_r = \frac{\rho S P}{k T_0} L^2, \alpha = 0.1 \times T_0 \times \beta \times$ log 2,

$a1 = a \times l,$ $x = 1 - x$

(13)

The solution of the problem for different metabolic heat source is discussed in next three cases.

IV. HYBRID NUMERICAL METHOD

To solve the present problem, we used the hybrid numerical method which combine two schemes, first is discretized by finite difference scheme [6] which converts our problem into a system of second order non-linear ordinary differential equation (ODEs) with initial conditions, and it is converted into system of first order non-linear ordinary differential equation (ODEs) with initial conditions. Then we apply second scheme, Runge-Kutta (4, 5) [8] which solved initial value problem, easily. This hybrid method is used in every case.

Case-1:

Using the non-dimensional parameters of (13) and by taking constant temperature metabolic heat source from (7), then $(9) - (12)$ becomes

$$F_{oq} \frac{\partial^2 \theta}{\partial F_o^2} = -\left[1 + F_{oq} P_f^2\right] \frac{\partial \theta}{\partial F_o} + \frac{\partial^2 \theta}{\partial x^2} + F_{oT} \frac{\partial^3 \theta}{\partial F_o \partial x^2} + P_f^2(\theta_b - \theta) + P_m + P_r \exp\left(a_1(\bar{x} - x_p)\right), \tag{14}$$

Subject to initial conditions

$$\theta(x, 0) = 0 \text{ and } \frac{\partial \theta(x, 0)}{\partial F_o} = 0, \tag{15}$$

boundary condition

$$\theta(1, F_o) = \theta_w + A\cos(\omega F_o), \tag{16}$$

and symmetric condition

$$\frac{\partial \theta(0, F_o)}{\partial x} = 0. \tag{17}$$

Using hybrid numerical scheme in the problem (14-17) is as follows:

Spatial discretization scheme:

Taking $x_{i+1} = x_i + h$ where h is the spatial step length, Discretized the space coordinate domain $[0,1]$ into $l + 1$ subintervals of equal length h i.e, $0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_i < \cdots x_l < x_{l+1} = 1$. Central finite difference formula of second order derivative is applied for the solution i.e.,

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$$\left.\frac{\partial^2\theta(x,F_o)}{\partial x^2}\right|_{x=x_i} = \frac{\theta_{i+1}(F_o) - 2\theta_i(F_o) + \theta_{i-1}(F_o)}{h^2}, \quad 1 \le i \le l.$$

Using above, (14)-(17) are reduced in following forms

$$F_{oq}\frac{d^2\theta_1}{dF_o^2} = -\left[1 + F_{oq}P_f^2\right]\frac{d\theta_1}{dF_o} + \frac{(-21\theta_1+38\theta_2-9\theta_3)}{21h^2} + F_{oT}\frac{d}{dF_o}\frac{(-21\theta_1+38\theta_2-9\theta_3)}{21h^2} + P_f^2(\theta_b - \theta_1) + P_m + P_r\exp\left(a_1(\overline{x_1} - x_p)\right), \quad (18)$$

$$F_{oq}\frac{d^2\theta_i}{dF_o^2} = -\left[1 + F_{oq}P_f^2\right]\frac{d\theta_i}{dF_o} + \frac{(\theta_{i+1}-2\theta_i+\theta_{i-1})}{h^2} + F_{oT}\frac{d}{dF_o}\frac{(\theta_{i+1}-2\theta_i+\theta_{i-1})}{h^2} + P_f^2(\theta_b - \theta_i) + P_m + P_r\exp\left(a_1(\overline{x_i} - x_p)\right), \quad (19)$$

$$F_{oq}\frac{d^2\theta_n}{dF_o^2} = -\left[1 + F_{oq}P_f^2\right]\frac{d\theta_n}{dF_o} + \frac{(\theta_w+A\cos(\omega F_o)-2\theta_n+\theta_{n-1})}{h^2} + F_{oT}\frac{d}{dF_o}\left(\frac{\theta_w+A\cos(\omega F_o)-2\theta_n+\theta_{n-1}}{h^2}\right) + P_f^2(\theta_b - \theta_n) + P_m + P_r\exp\left(a_1(\overline{x_n} - x_p)\right), \quad (20)$$

with initial conditions

$$\theta_i(0) = 0 \; and \; \frac{d\theta_i(0)}{dF_o} = 0, \qquad 1 \le i \le n. \qquad (21)$$

Runge-Kutta (4, 5) scheme:

Suppose that

$$\frac{d}{dF_o}\left(F_{oq}\frac{d\theta_i}{dF_o}\right) = \frac{d\varphi_i}{dF_o} \; and \; \frac{d\theta_i(F_o)}{dF_o} = \frac{\varphi_i(F_o)}{F_{oq}}, \quad 1 \le i \le n,$$

Thus, (18)-(21) are reduced in the form, i.e.

$$\frac{d\varphi_1}{dF_o} = -\left[1 + F_{oq}P_f^2\right]\frac{\varphi_1}{F_{oq}} + \frac{(-21\theta_1+38\theta_2-9\theta_3)}{21h^2} + \frac{F_{oT}}{21h^2F_{oq}}(-21\varphi_1 + 38\varphi_2 - 9\varphi_3) + P_f^2(\theta_b - \theta_1) + P_m + P_r\exp\left(a_1(\overline{x_1} - x_p)\right), \quad (22)$$

$$\frac{d\varphi_i}{dF_o} = -\left[1 + F_{oq}P_f^2\right]\frac{\varphi_i}{F_{oq}} + \frac{(\theta_{i+1}-2\theta_i+\theta_{i-1})}{h^2} + \frac{F_{oT}}{h^2F_{oq}}(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}) + P_f^2(\theta_b - \theta_i) + P_m + P_r\exp\left(a_1(\overline{x_i} - x_p)\right), \quad (23)$$

$$\frac{d\varphi_n}{dF_o} = -\left[1 + F_{oq}P_f^2\right]\frac{\varphi_n}{F_{oq}} + \frac{(\theta_w+A\cos(\omega F_o)-2\theta_n+\theta_{n-1})}{h^2} + \frac{F_{oT}}{h^2F_{oq}}(-\omega A\sin(\omega F_o) - 2\varphi_n + \varphi_{n-1}) + P_f^2(\theta_b - \theta_i) + P_m + P_r\exp\left(a_1(\overline{x_i} - x_p)\right). \quad (24)$$

With initial conditions

$$\theta_i(0) = 0 \; and \; \varphi_i(0) = 0, \quad 1 \le i \le n. \qquad (25)$$

Case-2:

Using the non-dimensional parameters of (13) then (9)-(12) by taking linear temperature depended metabolic heat source from (7) can be written in the following form

$$F_{oq}\frac{\partial^2\theta}{\partial F_o^2} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha)\right]\frac{\partial\theta}{\partial F_o} + \frac{\partial^2\theta}{\partial x^2} + F_{oT}\frac{\partial^3\theta}{\partial F_o\partial x^2} + P_f^2(\theta_b - \theta) + P_m(1 + \alpha\theta) + P_r\exp\left(a_1(\bar{x} - x_p)\right), \quad (26)$$

Subject to initial conditions

$$\theta(x,0) = 0 \; and \; \frac{\partial\theta(x,0)}{\partial F_o} = 0, \qquad (27)$$

boundary condition

$$\theta(1,F_o) = \theta_w + A\cos(\omega F_o), \qquad (28)$$

and symmetric condition

$$\frac{\partial\theta(0,F_o)}{\partial x} = 0. \qquad (29)$$

Using hybrid numerical method in (26)-(29), this is explained as below:

Spatial discretization scheme:

Taking $x_{i+1} = x_i + h$ where h is the spatial step length, Discretized the space coordinate domain $[0,1]$ into $l + 1$ subintervals of equal length h i.e,

$0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_i < \cdots x_l < x_{l+1} = 1$.

Central finite difference formula of second order derivative is applied for the solution i.e.,

$$\left.\frac{\partial^2\theta(x,F_o)}{\partial x^2}\right|_{x=x_i} = \frac{\theta_{i+1}(F_o) - 2\theta_i(F_o) + \theta_{i-1}(F_o)}{h^2}, \quad 1 \le i \le l.$$

Using above, (26)-(29) are reduced in following forms

$$F_{oq}\frac{d^2\theta_1}{dF_o^2} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha)\right]\frac{d\theta_1}{dF_o} + \frac{(-21\theta_1+38\theta_2-9\theta_3)}{21h^2} + F_{oT}\frac{d}{dF_o}\frac{(-21\theta_1+38\theta_2-9\theta_3)}{21h^2} + P_f^2(\theta_b - \theta_1) + P_m(1 + \alpha\theta_1) + P_r\exp\left(a_1(\overline{x_1} - x_p)\right), \quad (30)$$

$$F_{oq}\frac{d^2\theta_i}{dF_o^2} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha)\right]\frac{d\theta_i}{dF_o} + \frac{(\theta_{i+1}-2\theta_i+\theta_{i-1})}{h^2} + F_{oT}\frac{d}{dF_o}\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{h^2} + P_f^2(\theta_b - \theta_i) + P_m(1 + \alpha\theta_i) + P_r\exp\left(a_1(\overline{x_i} - x_p)\right), \quad (31)$$

$$F_{oq}\frac{d^2\theta_n}{dF_o^2} = -\left[1 + F_{oq}(P_f^2 - P_m\alpha)\right]\frac{d\theta_n}{dF_o} + \frac{(\theta_w+A\cos(\omega F_o)-2\theta_n+\theta_{n-1})}{h^2} + F_{oT}\frac{d}{dF_o}\left(\frac{\theta_w+A\cos(\omega F_o)-2\theta_n+\theta_{n-1}}{h^2}\right) + P_f^2(\theta_b - \theta_n) + P_m(1 + \alpha\theta_n) + P_r\exp\left(a_1(\overline{x_n} - x_p)\right), \quad (32)$$

with initial conditions

$$\theta_i(0) = 0 \; and \; \frac{d\theta_i(0)}{dF_o} = 0, \quad 1 \le i \le n. \qquad (33)$$

Runge-Kutta (4, 5) scheme:

Suppose that

$$\frac{d}{dF_o}\left(F_{oq}\frac{d\theta_i}{dF_o}\right) = \frac{d\varphi_i}{dF_o} \text{ and } \frac{d\theta_i(F_o)}{dF_o} = \frac{\varphi_i(F_o)}{F_{oq}}, \quad 1 \le i \le n,$$

Thus, (30)-(33) are reduced in the form, i.e.

$$\frac{d\varphi_1}{dF_o} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha)\right]\frac{\varphi_1}{F_{oq}} + \frac{(-21\theta_1+38\theta_2-9\theta_3)}{21h^2} + \frac{F_{oT}}{21h^2F_{oq}}(-21\varphi_1 + 38\varphi_2 - 9\varphi_3) + P_f^2(\theta_b - \theta_1) + P_m(1 + \alpha\theta_1) + P_r\exp\left(a_1(\overline{x_1} - x_p)\right), \quad (34)$$

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$\frac{d\varphi_i}{dF_o} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha)\right]\frac{\varphi_i}{F_{oq}} + \frac{(\theta_{i+1}-2\theta_i+\theta_{i-1})}{h^2} + \frac{F_{oT}}{h^2F_{oq}}(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}) + P_f^2(\theta_b - \theta_i) + P_m(1 + \alpha\theta_i) + P_r\exp\left(a_1(\overline{x_i} - x_p)\right),$ (35)

$\frac{d\varphi_n}{dF_o} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha)\right]\frac{\varphi_n}{F_{oq}} + \frac{(\theta_w+A\cos(\omega F_o)-2\theta_n+\theta_{n-1})}{h^2} + \frac{F_{oT}}{h^2F_{oq}}(-\omega A\sin(\omega F_o) - 2\varphi_n + \varphi_{n-1}) + P_f^2(\theta_b - \theta_n) + P_m(1 + \alpha\theta_n) + P_r\exp\left(a_1(\overline{x_i} - x_p)\right).$
(36)

with initial conditions

$$\theta_i(0) = 0 \text{ and } \varphi_i(0) = 0,\ 1 \le i \le n. \quad (37)$$

Case-3:
Using the non-dimensional parameters of (13) then (9)-(12) by taking linear temperature depended metabolic heat source from (7) can be written in the following form

$F_{oq}\frac{\partial^2\theta}{\partial F_o^2} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha\log(2)\times 2^{\alpha\theta})\right]\frac{\partial\theta}{\partial F_o} + \frac{\partial^2\theta}{\partial x^2} + F_{oT}\frac{\partial^3\theta}{\partial F_o\partial x^2} + P_f^2(\theta_b - \theta) + P_m\times 2^{\alpha\theta} + P_r\exp\left(a_1(\bar{x} - xp,\right.$ (38)

Subject to initial conditions

$$\theta(x,0) = 0 \text{ and } \frac{\partial\theta(x,0)}{\partial F_o} = 0, \quad (39)$$

boundary condition

$$\theta(1,F_o) = \theta_w + A\cos(\omega F_o), \quad (40)$$

and symmetric condition

$$\frac{\partial\theta(0,F_o)}{\partial x} = 0. \quad (41)$$

Using hybrid numerical method in (38)-(41), this is explained as below:

Spatial discretization scheme:
Taking $x_{i+1} = x_i + h$ where h is the spatial step length, Discretized the space coordinate domain [0,1] into $l+1$ subintervals of equal length h i.e,

$$0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_i < \cdots x_l < x_{l+1} =.$$

Central finite difference formula of second order derivative is applied for the solution i.e.,

$$\left.\frac{\partial^2\theta(x,F_o)}{\partial x^2}\right|_{x=x_i} = \frac{\theta_{i+1}(F_o) - 2\theta_i(F_o) + \theta_{i-1}(F_o)}{h^2},\quad 1 \le i \le l.$$

Using above, (26)-(29) are reduced in following forms

$F_{oq}\frac{d^2\theta_1}{dF_o^2} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha\log(2)\times 2^{\alpha\theta_1})\right]\frac{d\theta_1}{dF_o} + \frac{(-21\theta_1+38\theta_2-9\theta_3)}{21h^2} + F_{oT}\frac{d}{dF_o}\frac{(-21\theta_1+38\theta_2-9\theta_3)}{21h^2} + P_f^2(\theta_b - \theta_1) + P_m\times 2^{\alpha\theta_1} + P_r\exp\left(a_1(\overline{x_1} - x_p)\right),$ (42)

$F_{oq}\frac{d^2\theta_i}{dF_o^2} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha\log(2)\times 2^{\alpha\theta_i})\right]\frac{d\theta_i}{dF_o} + \frac{(\theta_{i+1}-2\theta_i+\theta_{i-1})}{h^2} + F_{oT}\frac{d}{dF_o}\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{h^2} + P_f^2(\theta_b - \theta_i) + P_m\times 2^{\alpha\theta_i} + P_r\exp\left(a_1(\overline{x_i} - x_p)\right),$
(43)

$F_{oq}\frac{d^2\theta_n}{dF_o^2} = -\left[1 + F_{oq}(P_f^2 - P_m\alpha\log(2)\times 2^{\alpha\theta_n})\right]\frac{d\theta_n}{dF_o} + \frac{(\theta_w+A\cos(\omega F_o)-2\theta_n+\theta_{n-1})}{h^2} + F_{oT}\frac{d}{dF_o}\left(\frac{\theta_w+A\cos(\omega F_o)-2\theta_n+\theta_{n-1}}{h^2}\right) + P_f^2(\theta_b - \theta_n) + P_m\times 2^{\alpha\theta_n} + P_r\exp\left(a_1(\overline{x_n} - x_p)\right),$ (44)

with initial conditions

$$\theta_i(0) = 0 \text{ and } \frac{d\theta_i(0)}{dF_o} = 0,\ 1 \le i \le n. \quad (45)$$

Runge-Kutta (4, 5) scheme:
Suppose that

$$\frac{d}{dF_o}\left(F_{oq}\frac{d\theta_i}{dF_o}\right) = \frac{d\varphi_i}{dF_o} \text{ and } \frac{d\theta_i(F_o)}{dF_o} = \frac{\varphi_i(F_o)}{F_{oq}},\ 1 \le i \le n,$$

Thus, (30)-(33) are reduced in the form, i.e.

$\frac{d\varphi_1}{dF_o} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha\log(2)\times 2^{\alpha\theta_1})\right]\frac{\varphi_1}{F_{oq}} + \frac{(-21\theta_1+38\theta_2-9\theta_3)}{21h^2} + \frac{F_{oT}}{21h^2F_{oq}}(-21\varphi_1 + 38\varphi_2 - 9\varphi_3) + P_f^2(\theta_b - \theta_1) + P_m\times 2^{\alpha\theta_1} + P_r\exp\left(a_1(\overline{x_1} - x_p)\right),$ (46)

$\frac{d\varphi_i}{dF_o} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha\log(2)\times 2^{\alpha\theta_i})\right]\frac{\varphi_i}{F_{oq}} + \frac{(\theta_{i+1}-2\theta_i+\theta_{i-1})}{h^2} + \frac{F_{oT}}{h^2F_{oq}}(\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}) + P_f^2(\theta_b - \theta_i) + P_m\times 2^{\alpha\theta_i} + P_r\exp\left(a_1(\overline{x_i} - x_p)\right),$ (47)

$\frac{d\varphi_n}{dF_o} = -\left[1 - F_{oq}(P_f^2 - P_m\alpha\log(2)\times 2^{\alpha\theta_n})\right]\frac{\varphi_n}{F_{oq}} + \frac{(\theta_w+A\cos(\omega F_o)-2\theta_n+\theta_{n-1})}{h^2} + \frac{F_{oT}}{h^2F_{oq}}(-\omega A\sin(\omega F_o) - 2\varphi_n +$ $\varphi n-1+Pf2\theta b-\theta n+Pm\times 2\alpha\theta n+Pr\exp a1xi-xp.$
(48)

with initial conditions

$$\theta_i(0) = 0 \text{ and } \varphi_i(0) = 0,\ 1 \le i \le n. \quad (49)$$

Runge-Kutta (4, 5) method is used for the system of non-linear ordinary differential equation. This method is very helpful to solve the non-linear ODEs [8]. The dimensionless temperature in the skin tissue is obtained by the solution of system of non-linear ODE.

V. EXACT SOLUTION

The exact solution is required to validate the result. Supposed the associate blood perfusion constant $a = 0$, metabolism constant $\alpha = 0$ and external heat source $P_r = 0$ in the non-linear DPLBHT model (9), then this model is reduced in the following form i.e.

$$F_{oq}\frac{\partial^2\theta}{\partial F_o^2} + \left[1 + F_{oq}P_f^2\right]\frac{\partial\theta}{\partial F_o} = \frac{\partial^2\theta}{\partial x^2} + F_{oT}\frac{\partial^3\theta}{\partial F_o\partial x^2} + P_f^2(\theta_b - \theta) + P_m, \qquad (50)$$

Subject to initial conditions

$$\theta(x,0) = 0 \text{ and } \frac{\partial\theta(x,0)}{\partial F_o} = 0, \qquad (51)$$

boundary condition

$$\theta(1,F_o) = \theta_w + A\cos(\omega F_o), \qquad (52)$$

and symmetric condition

$$\frac{\partial\theta(0,F_o)}{\partial x} = 0 \qquad (53)$$

The solution of (50) is solved by using the method of Laplace transform and then its inverse Laplace transform under the periodic boundary condition (51) and initial condition (52). The solution becomes:

$$\theta(x,F_o) = \theta_w\left(\frac{\cosh(P_f x)}{\cosh(P_f)} + \sum_{n=1}^{\infty}\frac{2e^{R_{n1}F_o}\cosh(\beta_{R_{n1}}x)\beta_{R_{n1}}}{E_{n1}\sinh(\beta_{R_{n1}})} + \right.$$

n=1∞2eRn2FocoshβRn2xβRn2En2sinhβRn2+AeiωFocosh(βiωx)cosh(βiω)+e−iωFocosh(β−iωx)cosh(β−iω)+An=1∞2eRn1FoRn1βRn1cosh(βRn1x)(Rn12+ω2)En1sinh(βRn1)+n=1∞2eRn2FoRn2βRn2cosh(βRn2x)(Rn22+ω2)En2sinh(βRn2)−Ncosh(Pfx)Pf2cosh(Pf)+n=1∞eR3Focosh(βR3x)R3(R4−R3)+n=1∞eR4Focosh(βR4x)R4(R3−R4)+Nn=1∞2eRn1FocoshβRn1xβRn1Rn1(1+Rn1Foq)βRn12En1sinhβRn1−n=1∞2eRn2FocoshβRn2xβRn2Rn2(1+Rn2Foq)βRn22En2sinhβRn2+N

$$\left(\frac{1}{P_f^2} + \frac{e^{R_3F_o}}{R_3\left[F_{oq}R_3^2 + \left(1 + P_f^2F_{oq}\right)R_3 + P_f^2\right]} + \right.$$

eR4FoR4FoqR42+1+Pf2FoqR4+Pf2, (54)

$$N = P_f^2\theta_b + P_m,$$

$$b_n = 1 + P_f^2F_{oq} + (2n-1)^2\frac{\pi^2}{4}F_{oT},$$

$$E_{n1} = -\left(F_{oT}F_{oq}R_{n1}^2 + 2F_{oq}R_{n1} + 1\right),$$

$$E_{n2} = -\left(F_{oT}F_{oq}R_{n2}^2 + 2F_{oq}R_{n2} + 1\right),$$

$$R_{n1} = \frac{-b_n + \sqrt{b_n^2 - 4F_{oq}\left(P_f^2 + (2n-1)^2\frac{\pi^2}{4}\right)}}{2F_{oq}},$$

$$R_{n2} = \frac{-b_n - \sqrt{b_n^2 - 4F_{oq}\left(P_f^2 + (2n-1)^2\frac{\pi^2}{4}\right)}}{2F_{oq}},$$

$$\beta_{R_{n1}} = \sqrt{\frac{F_{oq}R_{n1}^2 + R_{n1}\left(1 + P_f^2F_{oq}\right) + P_f^2}{1 + F_{oT}R_{n1}}},$$

$$\beta_{R_{n2}} = \sqrt{\frac{F_{oq}R_{n2}^2 + R_{n2}\left(1 + P_f^2F_{oq}\right) + P_f^2}{1 + F_{oT}R_{n2}}},$$

$$R_3 = \frac{\left[-\left(1 + P_f^2F_{oq}\right) + \sqrt{(1 + P_f^2F_{oq})^2 - 4P_f^2F_{oq}}\right]}{2F_{oq}},$$

$$R_4 = \frac{\left[-\left(1 + P_f^2F_{oq}\right) - \sqrt{(1 + P_f^2F_{oq})^2 - 4P_f^2F_{oq}}\right]}{2F_{oq}}$$

VI. RESULTS AND DISCUSSION

We considered the temperature dependent blood perfusion and three types of metabolic heat generation as constant, linear and exponent variation in non-linear DPLBHT model. These two functions of temperature are validated experimentally. The comparison of DPLBHT model with three cases and exact solution is made in this paper. The parameters are also explained which are different from reference value. For the evaluation of dimensionless temperature in living tissue in finite domain. The selected reference values are taken from [4],[17] to calculate other parameters. $\rho = 1000kg/m^3, k = 0.5W/m℃$, L=0.05m, $\tau_T = 20s$, $\tau_q = 28s$, $c_b = 3.340 \times 10^3 J/kg℃$, $T_o = 37℃$, $Q_{mo} = 0.17W/m^3$, $T_b = 37℃, c = 4 \times 10^3\ J/kg℃$, $w_{bo} = 3.075 \times 10^{-3}s^{-1}$, $\beta = 1℃, P = 41.4397W$, $r_p = 0.005m, a = -127m^{-1}, S = 12.5\ kg^{-1}$.

In this paper, we have computed exact solution for DPLBHT model with constant, linear and exponential metabolic heat source and accuracy of numerical method during periodic boundary condition is evaluated. Effect of different parameters on temperature distribution for four cases namely, constant, linear and exponential temperature metabolism studied in detail for different values.

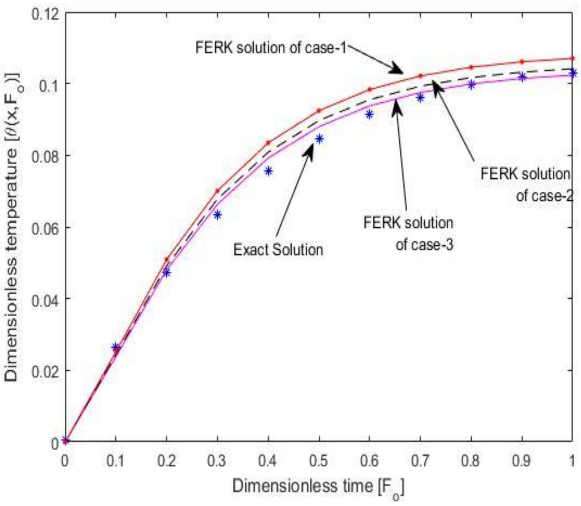

Fig.2. Comparison of Exact and FERK (4, 5) solution with different blood perfusion rate w_{bo}.

In Fig.2, we show the comparison of exact solution and analytical solution (FERK (4, 5)) with three types of metabolic heat source. From that fig it is observed that FERK (4, 5) method with exponential metabolic source gives good results with exact solution.

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The effect of space coordinate x with non-dimensional temperature and time is shown in Fig.3. The effect of x with three cases is also shown in that fig. It is observed that temperature profile decreases as increasing the values of x and from three cases temperature is highest in case-1 (for constant metabolism) and lowest for case-3 (for exponential metabolism).

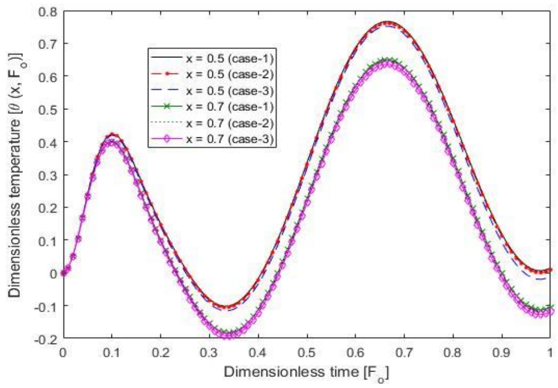

Fig.3. Plot of different values of space coordinate with dimensionless temperature and time for three cases.

The observation of P_m with non-dimensional temperature and time is shown in Fig.4. It is shown that the temperature profile is highest for case-3 which is exponential metabolic heat generation and the temperature is increases as increasing the value of P_m. Metabolic heat source and blood perfusion rate plays an important role in living tissues. The effect of perfusion of blood P_f with non-dimensional temperature and time is shown in Fig.5. It is observed that there is slightly difference in temperature profile for three cases. But temperature rises as increasing the value to blood perfusion rate P_f.

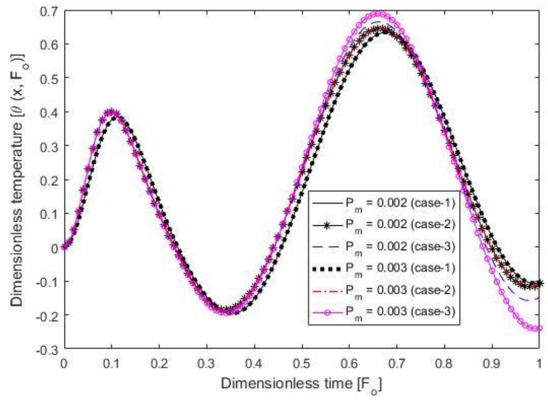

Fig.4. Plot of different values of metabolic heat source with dimensionless temperature and time for three cases.

The effect of external heat source P_r is shown in Fig.6 with non-dimensional temperature and time. It is concluded that temperature profile is approximately same for three cases but decreasing as decreasing the P_r.

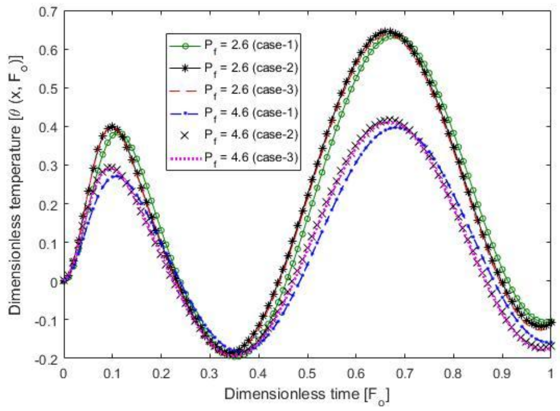

Fig.5. Plot of different values of blood perfusion rate with dimensionless temperature and time for three cases.

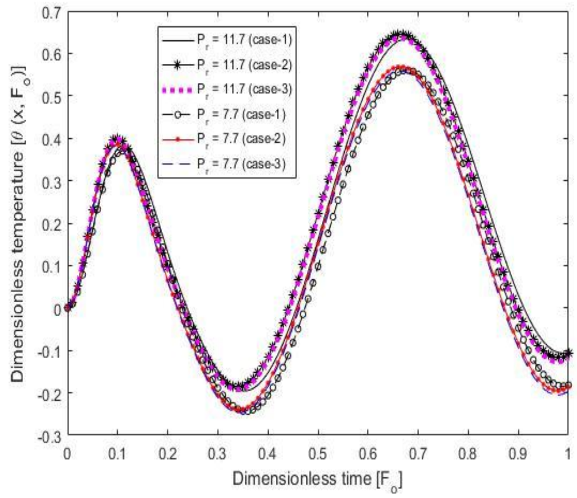

Fig.6. Plot of different values of external heat source with dimensionless temperature and time for three cases.

Metabolic heat constant α effect is observed by Fig.7 which shows that temperature is approximately same for case-2 and case-3 but decreases for case-1. It is also observed that temperature value falls when metabolic constant rises. In Fig.8, there is the effect of phase lag time because of heat flux is shown with non-dimensional temperature and time. It is observed that temperature decreases as increases the value of F_{oq} and temperature is highest for case-1 and lowest for case-3. The plot of dimensionless phase lag time which is because of temperature gradient F_{oT} with temperature and time is shown in Fig.9. We concluded from fig that temperature decreases as increasing the value of phase lag F_{oT} and temperature decreases form case-1 to case-3. The effect of periodicity ω is given in Fig.10. The periodicity is same for all three cases which is constant, linear and exponential metabolic heat source. This fig also shows that thermal wave highest for $\omega = 10$ and decreases as ω increases.

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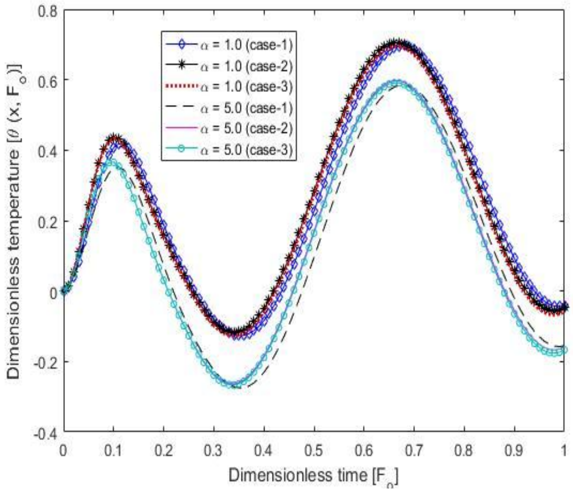

Fig.7. Plot of different values of associated metabolism constant with dimensionless temperature and time for three cases.

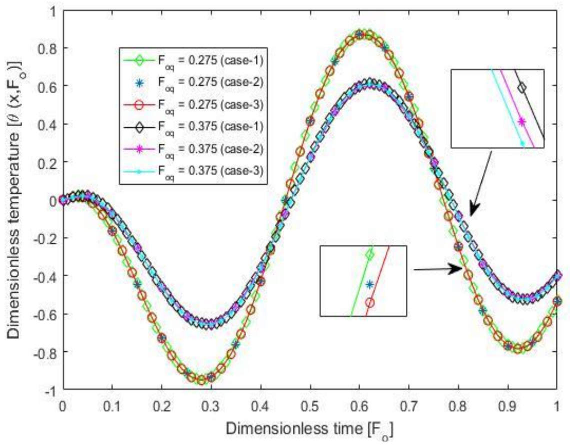

Fig.8. Plot of different values of phase lag due to heat flux with dimensionless temperature and time for three cases.

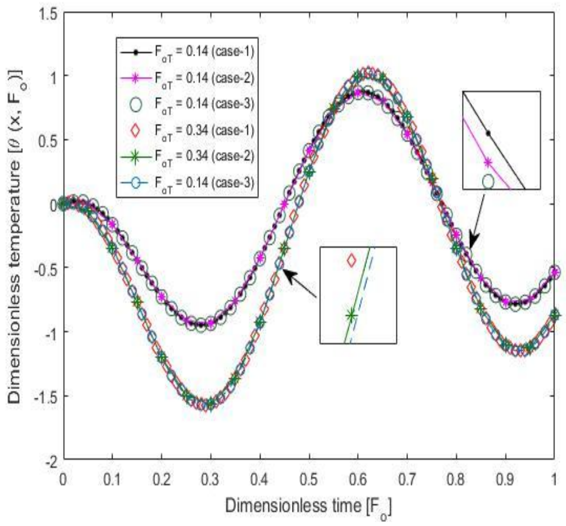

Fig.9. Plot of different values of dimensionless phase lag due to temperature gradient with dimensionless temperature and time for three cases.

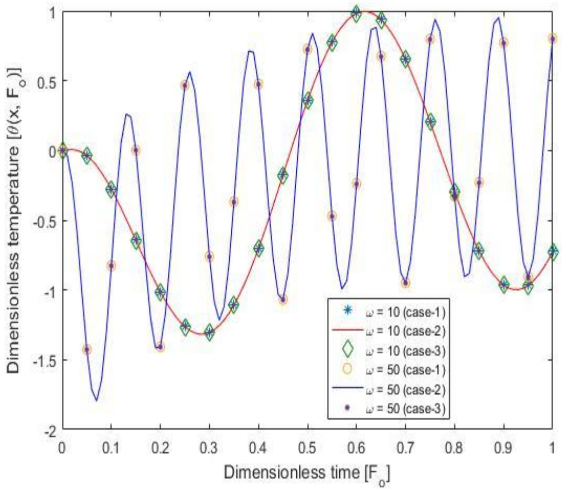

Fig.10. Plot of different values of periodicity with dimensionless temperature and time for three cases.

The amplitude of temperature profile is highest when $A = 1$ for all three cases and increases as the value of A increases for non-dimensional temperature and time which is shown in Fig.11.

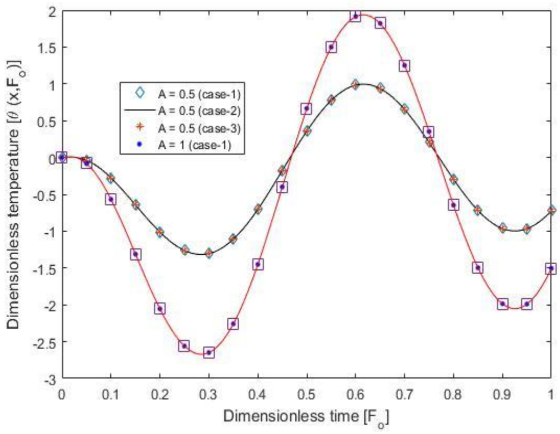

Fig.11. Plot of different values of amplitude with dimensionless temperature and time for three cases.

VII. CONCLUSION

The thermal behavior of DPLBHT model is studied in this paper in which we take the three different cases of metabolic heat generation. The metabolism has very important properties in biological system. In this paper the problem is converted into non-dimensional parameters.

The present problem is solved by using two methods one is finite-difference method and second is Runge-Kutta (4, 5) method. Comparison of exact solution obtained in particular case with DPLBHT method for three cases has been presented. Two types of conclusion have been found from each figure, one is for different parameters and other is for three given cases in the problem. We get some results by using hybrid method which is as follows:

- The DPLBHT model with exponential metabolic heat source gives good results with exact solution as from other two cases.

- The thermal wave of temperature profile decreases as the values of space coordinate x and metabolic constant α increases.
- The temperature profile rises as increasing the value of perfusion of blood P_f and metabolic heat source P_m and decreases as decreasing the value of external heat source P_r.
- The phase lag time because of heat flux F_{oq} and temperature gradient F_{oT} increases then temperature profile decreases.
- On increasing the periodicity ω and amplitude$A$, the temperature distribution decreases for ω and increases for A.
- It is also concluded from above figures that exponential metabolic heat source gives good results with each parameter.

On the basis of above observations, we concluded metabolic heat source gives important role in living tissue.

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