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Bosons-Bosons Oscillator Dynamics in a Quantum Control System

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Abstract: In this work, we developed a study of a set of Cooper pairs in a superconducting system and the control dynamics of the corresponding Quantum Control System. This paper specifically addresses the interaction between a bosonic field, represented by the Cooper pairs, and another bosonic field, which is the System of vibrational modes of phonons in the crystal, in the context of applications related to Quantum Controllers. We consider an interaction of the boson-boson type, which refers explicitly to the possible coupling between the phonon field and the bosonic field constituted by Cooper pairs. We develop the Hamiltonian of the entire system, which comprises the Hamiltonian of the Cooper Pairs, the Hamiltonian of the interacting phonon system, and the interaction Hamiltonian. In our study, we define, for the Quantum Control System relative to the bosons-bosons interaction, a Quantum Kalman Filter. We obtain, as a first result of this work, the expression that makes explicit the Quantum Kushner-Stratonovich equation in our case of interaction between Cooper pairs and phonons of the crystal lattice of a superconductor. It expresses the density of states ρ of the Cooper pairs as a function of time in the non-Markovian statistical system described by us. Furthermore, we obtain a second fundamental result of this work, expressing the Hamiltonian of the overall open system as a function of the Pauli matrix operator. This operator allows us to represent the same Hamiltonian of the open system described by us through the logic gates of a quantum controller. Ultimately, we obtain the control law governing the temporal evolution of the bosonic system and demonstrate its representability through the logic gates of a quantum control system.

Keywords: Quantum Controller, Bosons-Bosons Interaction, Quantum Kalman Filter, Quantum Kushner-Stratonovich Equation, Pauli Matrices.

I. INTRODUCTION

The peculiarity of our study is given by considering the electronic field as constituted by a set of vibrational elements constituted by Cooper pairs in a superconducting system. This work, in general, deals with the interaction in a crystal between a bosonic field, given by Cooper pairs, and another bosonic field, provided by the System of vibrational modes of phonons in the crystal, concerning applications related to quantum computers. Several works have been dedicated to studying interactions in crystals and dissipative cavities with quantum dots.

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Therefore, based on the qubit/photon system. In the paper [1], they developed a general framework for modeling the control of flying qubits based on the quantum stochastic differential equation (QSDE) that describes the input-output process actuated by a standing quantum system. A power of a qubit under Markovian and non-Markovian noise was developed in the paper [2]. In work [3], the properties of emission spectra of dissipative cavities coupled with quantum dots were studied. The quantum Kalman filter is presented in the paper [4] for quantum linear systems, which is the quantum analogy of the Kalman filter for classical (namely, non-quantum-mechanical) linear systems. The paper [5] presents a scheme for enhancing nonlinear quantum effects via the recently developed coherent feedback techniques. In paper [6] the authors define a study of an asymptotically optimal statistical inference concerning the unknown state of N identical quantum systems. Last, in the work [7] the authors studied the dynamics of a quantum coherent feedback network, where an N-level atom is coupled with a cavity, and the cavity is also associated with single or multiple parallel waveguides.

In the recent past, we have found an accurate description of the interaction between an electron field, defined as a two-level system, and the vibrational field constituted by the phonons of a crystal [8]. In general, the study of quantum controllers has so far led to the analysis of either purely fermionic systems or fermionic systems (electronics) interacting with bosonic systems such as the phonons of a crystal [9]. The latter line appears to be the most promising about situations in which the system is superconducting [10]. In this context, the electronic components of the system are represented with qubits, while the phonon modes are defined with their vibrational states. In these works, the dynamics of electron-phonon systems are described using real quantum hardware.

In the quantum domain, we consider a significant departure from previous works. We consider a boson-boson interaction, which refers explicitly to the possible coupling between the phonon field and the bosonic field constituted by Cooper pairs. Cooper pairs are formed by two electrons, with opposite wave vectors and opposite spins. The energy involved in this coupling is minimal, of the order of milli-electronvolts (or a few tens of milli-electronvolts in superconductors that operate on liquid nitrogen). These fermion pairs behave quite similarly to bosons and form the supercurrent.

The boson-boson interaction in superconductors appears to be a promising approach to defining quantum controllers with properties distinct from those obtained so far.

Considering open quantum system control, it is typically assumed that the environmental evolution timescale is considerably shorter than the timescale of the atomic system. Therefore, we can take as static [11]. As a

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consequence, it can model the interaction between the quantum system and the environment with a static decaying rate, which is a Markovian approximation method. However, often, and in our case, this Markovian approximation is inadequate for accurately analysing the dynamics of an open quantum system [11]. The interaction between the quantum system and its environment, considered non-Markovian, can be modelled by the stochastic Schrödinger equation.

Furthermore, in this work, we finish focusing our considerations on a quantum controller. Quantum controllers are being increasingly used in practice. Besides, they are easy systems to insert into design problems. In the quantum domain, we consider a hybrid classical-quantum description. In the quantum world, information cannot be copied. Therefore, the jump points, as in classical block diagrams, cannot be used for quantum information. The method used in the present paper is based on the works on Quantum Kalman Filtering and controllers defined in some recent works. The paper [12] provides an introduction to quantum filtering theory. In paper [13] the authors define an introduction to noncommutative (quantum) filtering theory. Last, in the paper [14] authors define a focus on the SLH framework, a robust modeling framework for networked quantum systems. Our method uses the notions of a controlled quantum stochastic evolution defined in these fundamental works by Bouten and van Handel [12] and [13], and by Combes et al. [14]. We also consider the previous work of the author of this paper [15].

The advantageous novelty of this method lies in the fact that the interaction between two bosonic fields in a superconducting crystal is considered for the first time. This enables us to explore new potential developments related to quantum controllers in superconductors.

We organised this work as follows. In Section II, the BCS model, consisting of Cooper pairs formed by two electrons and interacting phonons, is represented. Hamiltonians expressed by quantum harmonic oscillators represent both the dynamics of the Cooper pairs and that of the phonons. In Section III, the Hamiltonian in second quantization is obtained in all its fundamental parts. The Belavkin equation is obtained in Section IV by explicating the L function in the first approximation. In section V, we get the Quantum Kushner-Stratonovich equation and the representation of the Quantum Kalman Filter for the stochastic process of time evolution of our System. In section VI, the fundamental results of our work are presented. In particular, the equation describing the time evolution of the density of states matrix and the Hamiltonian of the open System in terms of Pauli matrices.

II. BOSONS MODEL WITH INTERACTIONS

The interaction between bosonic vibrational elements gives the model described in this work. In particular, it is assumed that superconductivity is present. In the superconductor, Cooper pairs are formed, which are formed by two electrons (mainly) of the opposite wave vector and opposite spin.

The energy involved in this coupling is minimal, of the order of milli-electronvolts (or a few tens of milli-electronvolts in superconductors that operate on liquid nitrogen). These pairs of fermions behave quite similarly to bosons and form the supercurrent. All these pairs form a condensate, which can be considered qualitatively similar to a Bose-Einstein condensate of bosons, in which all the pairs participate cooperatively. An energy gap is created at the Fermi level, and the system exhibits powerful magnetic properties.

1. The electronic part of the Hamiltonian, describing the system of bosons described by the Cooper pairs, is given by:

$$H_{SYS(Cc)} = \sum_{i=1}^{N_i} \left\{ \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 q_i^2 \right) \right\} + \sum_{i=1}^{N_i} \sum_{j=1}^{N_j} \frac{\gamma_{ij}}{\sqrt{\pi\sigma^2}} \exp\left\{ -\frac{1}{2} \left(\frac{q_i - q_j}{\sigma} \right)^2 \right\} \quad \dots (1)$$

Where p_i is the momentum, m_i the mass, ω_i the angular frequency, q_i the lagrangian coordinate of the single Cooper pair i, q_j the lagrangian coordinate of the pair j, Υ_{ij} is the interaction strength between the Cooper pairs i and j. σ is the variance of the normal distribution of the position coordinates of the Cooper pairs. The first term of the Hamiltonian accounts for the vibrational energy of the Cooper pairs considered, in a harmonic approximation, as a set of independent quantum harmonic oscillators (QHOs). The second term, instead, takes into account the effective interaction energy between individual Cooper pairs, considering this interaction energy as distributed according to a Gaussian function based on the distance between the individual pairs.

2. The Phononic part of the Hamiltonian describing the environment of bosons, described by the phonons in the crystal lattice, is given by:

$$H_{ENV(Ph)} = \sum_{k=1}^{N_k} \left\{ \frac{p_k^2}{2m_k} + \frac{1}{2} m_k \omega_k^2 q_k^2 \right\} \quad \dots (2)$$

Where p_k is the momentum, m_k is the mass, ω_k is the angular frequency, and qk is the Lagrangian coordinate of the single phonon k. Like Cooper pairs, phonons are also considered, in harmonic approximation, as a set of independent quantum harmonic oscillators (QHOs).

3. The interaction term between the two bosonic fields described in points 1 and 2 is given by:

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$$H_{INT(cp)} = \sum_{i=1}^{N_i}\sum_{k=1}^{N_k}\frac{\beta_{ik}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_i-q_k}{\sigma_{cp}}\right)^2\right\} \quad \dots \quad (3)$$

Where q_i is the lagrangian coordinate of the Cooper pair i, q_k is the lagrangian coordinate of the single phonon k, β_{ik} is the interaction strength between the Cooper pair i and the bosonic system k, σ_{cp} is the variance of the normal distribution of the relative position coordinates between the Cooper pairs and the phonons.

III. HAMILTONIAN IN SECOND QUANTIZATION

We will now use the formalism of the second quantization, and then we will use the creation and destruction operators $\hat{a}^\dagger$ and $\hat{a}$ Which, respectively, increase or decrease the number of particles in the quantum state by one.

1. The electronic part of the Hamiltonian, describing the system of bosons described by the Cooper pairs, is given by:

$$H_{SYS(Cc)} = \sum_{i=1}^{N_i}\sum_{j=1}^{N_j}\hbar\omega_{ij}\left(\hat{a}_{ij}^\dagger\hat{a}_{ij}+{}^1/_2\right) + \sum_{i=1}^{N_i}\sum_{j=1}^{N_j}V_{ij}\left(|i\rangle\langle j| + |j\rangle\langle i|\right) \quad \dots \quad (4)$$

with

$$V_{ij} = \frac{\gamma_{ij}}{\sqrt{\pi\sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{q_i-q_j}{\sigma}\right)^2\right\} \quad \dots \quad (5)$$

where $|i\rangle\langle j|$ is the inner product of the states i and j of the Cooper pairs, and V_{ij} Is the interaction energy between the two Cooper pairs, i.e., j. The expression of interaction energy V_{ij} It is assumed to be a Gaussian based on the relative position of the two Cooper pairs i and j. γ_{ij} Represents the strength of the interaction between the two Cooper pairs.

2. The Phonon part of the Hamiltonian describing the environment of bosons, described by the phonons in the crystal lattice, is given by:

$$H_{ENV(Ph)} = \sum_{k=1}^{N_k}\sum_{l=1}^{N_l}\hbar\omega_{kl}\left(\hat{a}_{kl}^\dagger\hat{a}_{kl}+{}^1/_2\right) \quad \dots \quad (6)$$

3. The interaction term between the two bosonic fields described in points 1 and 2 is given by:

$$H_{INT(cp)} = \sum_{i=1}^{N_i}\sum_{k=1}^{N_k}\frac{\Theta_{ik}}{\sqrt{\pi\sigma_{cp}}^{2}}\left(|i\rangle\langle k| + |k\rangle\langle i|\right) \quad \dots \quad (7)$$

Where Θ_{ik} is the interaction strength between the Cooper pairs i and the bosonic system k in second quantization.

The advantage of expressing the system in second quantization is that it, as we will see in the following paragraph, allows us to define a Circuit implementation for the expressions found in a Quantum control system.

IV. BELAVKIN EQUATION

Concerning [13] and [14], we can write:

$$d\psi = -\left(\tfrac{1}{2}L^*L + \tfrac{i}{\hbar}H\right)\psi dt + L\psi dy \quad \dots \quad (8)$$

Contrary to the Schrödinger equation, which describes the deterministic evolution of the wave-function of a closed system (without interaction), the Belavkin equation represents the stochastic evolution of a random wave-function ψ of an open quantum system interacting with an observer.

L is a self-adjoint operator of the system coupled to the external field, and y(t) is a stochastic process representing the measurement noise. It is a martingale, and this noise has dependent increments concerning the output probability measure. This measure represents the output of the innovation process (the observation). For L=0, this equation becomes the basic Schrödinger equation.

The stochastic process y(t) can be of two kinds: the Poisson type, which corresponds to a counting observation. And y(t)=w(t), being w(t) a Brownian (or diffusion) movement that corresponds to a Wiener process with continuous observation.

The Hamilton operator is, in our case, the following:

$$H = \sum_{i=1}^{N_i}\left\{\left(\frac{\hat{p}_i^2}{2m_i}+\frac{1}{2}m_i\omega_i^2q_i^2\right)\right\} + \sum_{i=1}^{N_i}\sum_{j=1}^{N_j}\frac{\gamma_{ij}}{\sqrt{\pi\sigma^2}}\exp\left\{-\frac{1}{2}\left(\frac{\hat{q}_{ij}^2}{\sigma}\right)\right\} \quad \dots \quad (9)$$

The L function, in our case, is established as a first approximation by:

$$L \approx H_{INT} + \sum_{k=1}^{N_k}\left\{\frac{p_{k(p)}^2}{2m_{k(p)}} + \frac{1}{2}m_{k(p)}\omega_{k(p)}^2q_{k(p)}^2\right\} \quad \dots \quad (10)$$

That is, as a first approximation, we consider the operator L given by H_{INT} + the term related to the field, which is determined by the surrounding phonon environment.

V. QUANTUM KALMAN FILTER

Concerning [12] and [13], starting from eq. (8), it is possible to arrive at the quantum version of the conditional probability density form of the Kushner-Stratonovich equation. The stochastic process has forward increments. Therefore, it becomes a standard Wiener process concerning the input probability measure. Substituting w(t) for y(t), we obtain the linear Belavkin equation for the unnormalized random wavefunction undergoing continuous observation. The

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output Wiener process becomes a diffusion innovation process with increments:

$$d\tilde{w}(t,\omega) = dw(t,\omega) - 2\mathrm{Re}\langle\varphi(t,\omega)|L\varphi(t,\omega)\rangle dt \quad \dots \quad (11)$$

Normalizing the function $\psi(t,\omega)$:

$$\varphi(t,\omega) = \psi(t,\omega)/\|\psi(t,\omega)\| \quad \dots \quad (12)$$

We consider the normalized random posterior density operator:

$$\rho(t,\omega) = \psi(t,\omega)\psi^*(t,\omega) \quad \dots \quad (13)$$

Therefore, if we start with the Belavkin equation (8), we can obtain the Quantum Kushner-Stratonovich equation:

$$\begin{aligned} d\rho &= -[K\rho + \rho K^* - L\rho L^*]dt \\ &+[L\rho + \rho L^* - \rho Tr\{(L+L^*)\rho\}]d\tilde{w} \end{aligned} \quad \dots \quad (14)$$

where

$$K = \frac{1}{2}L^*L + \frac{i}{\hbar}H \quad \dots \quad (15)$$

Taking the average of Eq. (14), overall ω, leads to the Lindblad equation:

$$\frac{d\rho}{dt} = -[K\rho + \rho K^*] + L\rho L^* \quad \dots \quad (16)$$

Considering our system, and neglecting the higher order terms, we can write in first approximation, from (16):

$$\begin{aligned}\frac{d\rho}{dt} &= -\left[\frac{i}{\hbar}H_{SYS}\rho - \rho\frac{i}{\hbar}H^*_{SYS}\right]dt \\ &+[L\rho + \rho L^*]d\tilde{w}\end{aligned} \quad \dots \quad (17)$$

Therefore, considering (1) and (10), we obtain the fundamental result:

$$\begin{aligned}\frac{d\rho}{dt} = &-\left\{\begin{array}{l}\frac{i}{\hbar}\left[\sum_{i=1}^{N_i}\left(\frac{p^2_{i(Cc)}}{2m_{i(Cc)}} + \frac{1}{2}m_i\omega^2_{i(Cc)}q^2_{i(Cc)}\right)\right]\rho \\ -\rho\frac{i}{\hbar}\left[\sum_{i=1}^{N_i}\left(\frac{\left(p^*_{i(Cc)}\right)^2}{2m_{i(Cc)}} + \frac{1}{2}m_{i(Cc)}\omega^2_{i(Cc)}\left(q^*_{i(Cc)}\right)^2\right)\right]\end{array}\right\}dt \\ &+\left\{\begin{array}{l}\left[\begin{array}{l}\sum_{i=1}^{N_i}\sum_{k=1}^{N_k}\frac{\beta_{ik(cp)}}{\sqrt{\pi\sigma_{cp}^{2}}}\exp\left\{-\frac{1}{2}\left(\frac{q_{i(c)}-q_{k(p)}}{\sigma_{cp}}\right)^2\right\} + \\ \sum_{k=1}^{N_k}\left(\frac{p^2_{k(p)}}{2m_{k(p)}} + \frac{1}{2}m_k\omega^2_{k(p)}q^2_{k(p)}\right)\end{array}\right]\rho \\ -\rho\left[\begin{array}{l}\sum_{i=1}^{N_i}\sum_{k=1}^{N_k}\frac{\beta^*_{ik(cp)}}{\sqrt{\pi\sigma_{cp}^{2}}}\exp\left\{-\frac{1}{2}\left(\frac{q^*_{i(c)}-q^*_{k(p)}}{\sigma_{cp}}\right)^2\right\} + \\ \sum_{k=1}^{N_k}\left(\frac{\left(p^*_{k(p)}\right)^2}{2m_{k(p)}} + \frac{1}{2}m_k\omega^2_{k(p)}\left(q^*_{k(p)}\right)^2\right)\end{array}\right]\end{array}\right\}d\tilde{w}\end{aligned} \quad \dots \quad (18)$$

VI. RESULT AND DISCUSSION

A. Quantum Control System

The first result we obtain in this work is the development of the expression for Eq explicitly. (18), considering all the components of the evolution matrix of the states ρ. We will then express the density matrix of the evolved states ρ in all its components. We will thus obtain the following expression that makes explicit the Quantum Kushner-Stratonovich equation of our case of interaction between Cooper pairs and phonons of the crystal lattice in a superconductor:

$$\begin{pmatrix}\frac{d\rho_{1,1}}{dt} & \cdots & \frac{d\rho_{1,N_p}}{dt} \\ \vdots & \ddots & \vdots \\ \frac{d\rho_{N_{Cc},1}}{dt} & \cdots & \frac{d\rho_{N_{Cc},N_p}}{dt}\end{pmatrix} = \left\{\begin{array}{l}\frac{i}{\hbar}\begin{pmatrix}\left(\frac{p^2_{1(Cc)}}{2m_{1(Cc)}} + \frac{1}{2}m_1\omega^2_{1(Cc)}q^2_{1(Cc)}\right) \\ \vdots \\ \left(\frac{p^2_{N_{Cc}(Cc)}}{2m_{N_{Cc}(Cc)}} + \frac{1}{2}m_{N_{Cc}}\omega^2_{N_{Cc}(Cc)}q^2_{N_{Cc}(Cc)}\right)\end{pmatrix}\left(\rho_1 \cdots \rho_{N_p}\right) \\ -\frac{i}{\hbar}\left[\begin{pmatrix}\rho_1 \\ \vdots \\ \rho_{N_p}\end{pmatrix}\begin{pmatrix}\left(\frac{\left(p^*_{1(Cc)}\right)^2}{2m_{1(Cc)}} + \frac{1}{2}m_1\omega^2_{1(Cc)}\left(q^*_{1(Cc)}\right)^2\right) \\ \vdots \\ \left(\frac{\left(p^*_{N_{Cc}(Cc)}\right)^2}{2m_{N_{Cc}(Cc)}} + \frac{1}{2}m_{N_{Cc}}\omega^2_{N_{Cc}(Cc)}\left(q^*_{N_{Cc}(Cc)}\right)^2\right)\end{pmatrix}\right]^T\end{array}\right\}dt +$$

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$$+\left\{\begin{pmatrix} \frac{\beta_{11(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(c)}-q_{1(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{1(p)}^{2}}{2m_{1(p)}}+\frac{1}{2}m_{1}\omega_{1(p)}^{2}q_{1(p)}^{2}\right) & \cdots & \frac{\beta_{1N_p(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(c)}-q_{N_p(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{N_p(p)}^{2}}{2m_{N_p(p)}}+\frac{1}{2}m_{N_p}\omega_{N_p(p)}^{2}q_{N_p(p)}^{2}\right) \\ \vdots & \ddots & \vdots \\ \frac{\beta_{N_p,1(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(c)}-q_{N_p(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{N_p(p)}^{2}}{2m_{N_p(p)}}+\frac{1}{2}m_{N_p}\omega_{N_p(p)}^{2}q_{N_p(p)}^{2}\right) & \cdots & \frac{\beta_{N_{Cc}N_p(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{N_{Cc}(c)}-q_{N_p(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{N_p(p)}^{2}}{2m_{N_p(p)}}+\frac{1}{2}m_{N_p}\omega_{N_p(p)}^{2}q_{N_p(p)}^{2}\right) \end{pmatrix}\begin{pmatrix}\rho_{1,1} & \cdots & \rho_{1,N_p} \\ \vdots & \ddots & \vdots \\ \rho_{N_{Cc},1} & \cdots & \rho_{N_{Cc},N_p}\end{pmatrix} - \left[\begin{pmatrix}\rho_{1,1} & \cdots & \rho_{1,N_{Cc}} \\ \vdots & \ddots & \vdots \\ \rho_{N_p,1} & \cdots & \rho_{N_p,N_{Cc}}\end{pmatrix}\begin{pmatrix} \frac{\beta_{11(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(c)}-q_{1(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{1(p)}^{2}}{2m_{1(p)}}+\frac{1}{2}m_{1}\omega_{1(p)}^{2}q_{1(p)}^{2}\right) & \cdots & \frac{\beta_{1N_{Cc}(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(c)}-q_{N_{Cc}(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{N_{Cc}(p)}^{2}}{2m_{N_{Cc}(p)}}+\frac{1}{2}m_{N_{Cc}}\omega_{N_{Cc}(p)}^{2}q_{N_{Cc}(p)}^{2}\right) \\ \vdots & \ddots & \vdots \\ \frac{\beta_{N_p,1(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(c)}-q_{N_p(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{N_p(p)}^{2}}{2m_{N_p(p)}}+\frac{1}{2}m_{N_p}\omega_{N_p(p)}^{2}q_{N_p(p)}^{2}\right) & \cdots & \frac{\beta_{N_{Cc}N_p(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{N_{Cc}(c)}-q_{N_p(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{N_p(p)}^{2}}{2m_{N_p(p)}}+\frac{1}{2}m_{N_p}\omega_{N_p(p)}^{2}q_{N_p(p)}^{2}\right) \end{pmatrix}\right]^{T}\right\} d\tilde{w} \quad (19)$$

From equation (19), we derive a simplified expression of the Quantum Kushner-Stratonovich equation, considering only the components of the state evolution density matrix related to the Cooper pairs, as these are the ones we are interested in. For this purpose, we consider the maximum number of phonons corresponding to the number of Cooper pairs. With the final equation (20), we have derived the expression for the state control of our Quantum control system as a function of time. This is the first of the goals we wanted to achieve with our work.

$$\begin{pmatrix}\frac{d\rho_{1,1}}{dt} \\ \vdots \\ \frac{d\rho_{N_{Cc},N_{pMax}}}{dt}\end{pmatrix} = \left\{\frac{i}{\hbar}\begin{pmatrix}\left(\frac{p_{1(Cc)}^{2}}{2m_{1(Cc)}}+\frac{1}{2}m_{1}\omega_{1(Cc)}^{2}q_{1(Cc)}^{2}\right) \\ \vdots \\ \left(\frac{p_{N_{Cc}(Cc)}^{2}}{2m_{N_{Cc}(Cc)}}+\frac{1}{2}m_{N_{Cc}}\omega_{N_{Cc}(Cc)}^{2}q_{N_{Cc}(Cc)}^{2}\right)\end{pmatrix}\left(\rho_{N_{pMAX}}\right) - \frac{i}{\hbar}\left(\rho_{N_{pMAX}}\right)\begin{pmatrix}\left(\frac{\left(p_{1(Cc)}^{*}\right)^{2}}{2m_{1(Cc)}}+\frac{1}{2}m_{1}\omega_{1(Cc)}^{2}\left(q_{1(Cc)}^{*}\right)^{2}\right) \\ \vdots \\ \left(\frac{\left(p_{N_{Cc}(Cc)}^{*}\right)^{2}}{2m_{N_{Cc}(Cc)}}+\frac{1}{2}m_{N_{Cc}}\omega_{N_{Cc}(Cc)}^{2}\left(q_{N_{Cc}(Cc)}^{*}\right)^{2}\right)\end{pmatrix}\right\} dt \; +$$

$$+\left\{\begin{pmatrix} \frac{\beta_{11(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(c)}-q_{1(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{1(p)}^{2}}{2m_{1(p)}}+\frac{1}{2}m_{1(p)}\omega_{1(p)}^{2}q_{1(p)}^{2}\right) & \cdots & \frac{\beta_{1N_p(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(Cc)}-q_{N_p(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{N_p(p)}^{2}}{2m_{N_p(p)}}+\frac{1}{2}m_{N_p(p)}\omega_{N_p(p)}^{2}q_{N_p(p)}^{2}\right) \\ \vdots & \ddots & \vdots \\ \frac{\beta_{N_{pMAX},1(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{N_{Cc}(c)}-q_{N_{pMAX}(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{N_{pMAX}(p)}^{2}}{2m_{N_{pMAX}(p)}}+\frac{1}{2}m_{N_{pMAX}}\omega_{N_{pMAX}(p)}^{2}q_{N_{pMAX}(p)}^{2}\right) & \cdots & \frac{\beta_{N_{Cc}N_{pMAX}(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{N_{Cc}(c)}-q_{N_{pMAX}(p)}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{p_{N_{pMAX}(p)}^{2}}{2m_{N_{pMAX}(p)}}+\frac{1}{2}m_{N_{pMAX}}\omega_{N_{pMAX}(p)}^{2}q_{N_{pMAX}(p)}^{2}\right) \end{pmatrix}\begin{pmatrix}\rho_{1,1} \\ \vdots \\ \rho_{N_{Cc},N_{pMAX}}\end{pmatrix} - \left[\left(\rho_{1,1}\cdots\rho_{N_{Cc},N_{pMAX}}\right)\begin{pmatrix} \frac{\beta_{11(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(c)}^{*}-q_{1(p)}^{*}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{\left(p_{1(p)}^{*}\right)^{2}}{2m_{1(p)}}+\frac{1}{2}m_{1(p)}\left(\omega_{1(p)}^{*}\right)^{2}\left(q_{1(p)}^{*}\right)^{2}\right) & \cdots & \frac{\beta_{1N_p(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{1(Cc)}^{*}-q_{N_p(p)}^{*}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{\left(p_{N_p(p)}^{*}\right)^{2}}{2m_{N_p(p)}}+\frac{1}{2}m_{N_p}\left(\omega_{N_p(p)}^{*}\right)^{2}\left(q_{N_p(p)}^{*}\right)^{2}\right) \\ \vdots & \ddots & \vdots \\ \frac{\beta_{N_{pMAX},1(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{N_{Cc}(c)}^{*}-q_{N_{pMAX}(p)}^{*}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{\left(p_{N_{pMAX}(p)}^{*}\right)^{2}}{2m_{N_{pMAX}(p)}}+\frac{1}{2}m_{N_{pMAX}}\left(\omega_{N_{pMAX}(p)}^{*}\right)^{2}\left(q_{N_{pMAX}(p)}^{*}\right)^{2}\right) & \cdots & \frac{\beta_{N_{Cc}N_{pMAX}(cp)}}{\sqrt{\pi\sigma_{cp}}^{2}}\exp\left\{-\frac{1}{2}\left(\frac{q_{N_{Cc}(c)}^{*}-q_{N_{pMAX}(p)}^{*}}{\sigma_{cp}}\right)^{2}\right\}+\left(\frac{\left(p_{N_{pMAX}(p)}^{*}\right)^{2}}{2m_{N_{pMAX}(p)}}+\frac{1}{2}m_{N_{pMAX}}\left(\omega_{N_{pMAX}(p)}^{*}\right)^{2}\left(q_{N_{pMAX}(p)}^{*}\right)^{2}\right) \end{pmatrix}\right]^{T}\right\} d\tilde{w} \quad \ldots. \quad (20)$$

B. HAMILTONIAN IN TERMS OF PAULI MATRICES

Starting from the formulation expressed in various texts, e.g. [8], we can consider the expression of the time evolution Operator:

$$\hat{U}(t) = \exp\left\{-{}^{i\hat{H}t}/_{\hbar}\right\} \quad \dots \quad (21)$$

starting from some initial state $|\varphi_0\rangle$.

Let us consider the expression of the total time evolution Operator, given in [11]:

$$\hat{U}_{TOT}(t) = \left(e^{-{}^{i\hat{H}_{SYS}t}/_{\hbar\gamma}} e^{-{}^{i\hat{H}_{ENV}t}/_{\hbar\gamma}} e^{-{}^{i\hat{H}_{INT}t}/_{\hbar\gamma}}\right)^{\gamma} \quad \dots \quad (22)$$

Via a Suzuki-Tropper decomposition. It can be implemented in a quantum computer using logic gates.
The considered error is $O(\delta^2)$being $\delta = {}^{t}/_{\gamma}$.

$$\hat{a}_{ij}^{\dagger}\hat{a}_{ij} = \sum_{k=0}^{N_k} 2^k\, \hat{\sigma}_{k+1}^{z} = 2^0\hat{\sigma}_1^z + 2^1\hat{\sigma}_2^z + \dots \quad (23)$$

$$\exp\left(-i\vartheta\hat{a}_{ij}^{\dagger}\hat{a}_{ij}\right) = \prod_{k=0} \exp\left(-i\,{}^{\vartheta}/_{2}\, 2^{k+1}\hat{\sigma}_{k+1}^{z}\right) = \prod_{k=0} \hat{R}_z^k(-i\vartheta 2^k) \quad \dots \quad (24)$$

with

$$\vartheta = \frac{\hbar\omega_{ij}t}{\gamma} \quad \dots \quad (25)$$

$$\hat{a}_{ij}^{\dagger} + \hat{a}_{ij} \Rightarrow \otimes_{i=1} \hat{\sigma}_i^{j} \Rightarrow \hat{\sigma}_i^{z} \otimes \hat{P}_{ij}$$
$$\Rightarrow \exp\left(-i\hat{H}_{INT}\right) = \prod_i \prod_j \exp\left(-i\vartheta\hat{\sigma}_i^{z}\hat{P}_{ij}\right) \quad \dots \quad (26)$$

$$e^{-{}^{i\hat{H}_{SYS}t}/_{\hbar\gamma}} = \left\{\frac{1}{2}\left(-i\frac{\vartheta_{SYS}}{2}2\hat{\sigma}_1^z\right) + \frac{1}{2}\left(-i\frac{\vartheta_{SYS}}{2}2^2\hat{\sigma}_1^z\right)^2 + \dots\right\} + \sum_{i=1}^{N_i}\sum_{j=1}^{N_j} \frac{1}{\sqrt{2\pi\beta^2}} \exp\left\{-\frac{1}{2}\left(\frac{q_{ij}}{\beta}\right)^2\right\} \quad \dots \quad (27)$$

where

$$q_{ij} \triangleq q_i - q_j \quad \dots \quad (28)$$

Is the coupling coordinate between the Cooper pairs?
The second term is expressible as:

$$q_{ij} = \hat{\sigma}_i^x\hat{\sigma}_j^x - \hat{\sigma}_i^y\hat{\sigma}_j^y \quad \dots \quad (29)$$

We thus obtain the second fundamental result of our work. We have expressed the three parts of the Hamiltonian as a function of the Pauli matrix operator. This operator allows us to represent the Hamiltonian itself through the logic gates of a quantum control.

Finally, we obtain the control law governing the temporal evolution of the bosonic system and demonstrate the possibility of representing it through the logic gates of a quantum controller. This last topic will be given in a subsequent work.

VII. CONCLUSION

In this work, we developed a study of a set of Cooper pairs in a superconducting system.

The BCS model, consisting of Cooper pairs formed by two electrons and interacting phonons, is represented. Hamiltonians expressed by quantum harmonic oscillators represent both the dynamics of the Cooper pairs and that of the phonons. The Hamiltonian in second quantisation is obtained by describing it for both the Cooper pairs and the phonons, as well as the interaction between them. Furthermore, the Belavkin equation is obtained by making explicit the L function in the first approximation. From it, we derive the Quantum Kushner-Stratonovich equation and the representation of the Quantum Kalman Filter for the stochastic process of time evolution of our system. Finally, we obtain the fundamental results of our work, namely, the equation describing the time evolution of the density matrix of states of our system and the Hamiltonian of the open system itself in terms of Pauli matrices.

In the future, the development of this work seems possible on two fronts. On the one hand, define the algorithm that describes the temporal evolution of the System in terms of the circuit diagram given by the logic gates whose expression has been defined in this work. Then it will be possible to arrive at a simulation of the Population of the Cooper pairs. On the other hand, explain, in terms of statistical temporal evolution, the control law of the density of states of the Cooper pairs and any control feedback.

DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

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