

Implementation of Gaussian- Elimination

Awatif M.A. Elsiddieg

Abstract: Gaussian elimination is an algorithm for solving systems of linear equations, can also use to find the rank of any matrix ,we use Gaussian Jordan elimination to find the inverse of a non singular square matrix. This work gives basic concepts in section (1) , show what is pivoting , and implementation of Gaussian elimination to solve a system of linear equations. Section (2) we find the rank of any matrix. Section (3) we use Gaussian elimination to find the inverse of a non singular square matrix. We compare the method by Gauss Jordan method. In section (4) practical implementation of the method we inherit the computation features of Gaussian elimination we use programs in Matlab software.

Keywords: Gaussian elimination, algorithm Gauss, Jordan, method, computation, features, programs in Matlab, software.

I. INTRODUCTION

Section (1) Basic Concepts:

A. Echelon form:

The steps of replacing the argument matrix into reduced row-echelon form. The goal when solving a system of linear equations is to replace the augmented matrix into reduced row-echelon form, if possible. There are three elementary row operations that you may use to accomplish placing a matrix into reduced row-echelon form. Each of the requirements of a reduced row-echelon matrix can satisfied using the elementary row operations.

- If there is a row of all zeros, then it is at the bottom of the matrix. Interchange two rows of a matrix to move the row of all zeros to the bottom.
- The first non-zero element of any row is one. That element is called the leading one. Multiply (divide) the row by a non-zero constant to make the first non-zero element into a one.
- The leading one of any row is to the right of the leading one of the previous row. Multiply a row by a non-zero constant and add it to another row, replacing that row. The point of this elementary row operation is to make numbers into zeros. By making the numbers under the leading ones into zero, it forces the first non-zero element of any row to be to the right of the leading one of the previous row.
- All elements above and below a leading one are zero. Multiply a row by a non-zero constant and add it to another row, replacing that row. The point of this elementary row operation is to make numbers into zero. The difference here is that you're clearing (making zero) the elements above the leading one instead of just below the leading one.

Manuscript published on 30 April 2016.

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II. PIVOTING

The objective of pivoting is to make an element above or below a leading one into a zero.

The "pivot" or "pivot element" is an element on the left hand side of a matrix that you want the elements above and below to be zero. Normally, this element is a one. If you can find a book that mentions pivoting, they will usually tell you that you must pivot on a one. If you restrict yourself to the three elementary row operations, then this is a true statement. However, if you are willing to combine the second and third elementary row operations, you come up with another row operation (not elementary, but still valid).

- You can multiply a row by a non-zero constant and add it to a non-zero multiple of another row, replacing that row.

So what? If you are required to pivot on a one, then you must sometimes use the second elementary row operation and divide a row through by the leading element to make it into a one. Division leads to fractions. While fractions are your friends, you're less likely to make a mistake if you don't use them.

What's the catch? If you don't pivot on a one, you are likely to encounter larger numbers. Most people are willing to work with the larger numbers to avoid the fractions.

The Pivot Process:

Pivoting works because a common multiple (not necessarily the least common multiple) of two numbers can always be found by multiplying the two numbers together. Let's take the example we had before, and clear the first column.

$$\begin{array}{ccc|c} x & y & z & \text{rhs} \\ 3 & 2 & -4 & 3 \\ 2 & 3 & 3 & 15 \\ 5 & -3 & 1 & 14 \end{array}$$

Helpful Hints:

- Although you do not have to pivot on a one, it is highly desirable. Pivoting on a one means that you're multiplying by 1 (which is easy to do).
- It is nice to pivot on the main diagonal, but not absolutely necessary. Some people like to start in the upper left and work their way down to the lower right.
- As long as you pivot only once per row and column, the columns which have been cleared will remain cleared.
- Since the point of pivoting is to clear the pivot column, picking a column which already has zeros in it saves time because you don't have to change the row that contains the zero.



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Selecting a Pivot:

- Pick the column with the most zeros in it.
- Use a row or column only once
- Pivot on a one if possible
- Pivot on the main diagonal
- Never pivot on a zero
- Never pivot on the right hand side

Since there is no one in the first row, we have two options: Either we divide the first row by three and work with fractions, or we pivot on the three and get large numbers. That is the option I'm going to use. I'll pivot on the three in R_1C_1 . Go ahead and circle that as the pivot element. Depending on your browser, you may see the pivot elements circled in red or just with a * in front of it.

$$\begin{array}{c|ccc} x & y & z & \text{rhs} \\ \hline *3 & 2 & -4 & 3 \\ 2 & 3 & 3 & 15 \\ 5 & -3 & 1 & 14 \end{array}$$

The idea is to make the boxed (yellow) numbers into zero. Using the combined row operation (this is not an elementary operation), that could be done by:

$$3R_2 - 2R_1 \rightarrow R_2 \text{ and } 3R_3 - 5R_1 \rightarrow R_3.$$

The only row not being changed is the row containing the pivot element (the 3). The whole point of the pivot process is to make the boxed values into zero. Go ahead and rewrite the pivot row and clear (make zero) the pivot column.

$$\begin{array}{c|ccc} x & y & z & \text{rhs} \\ \hline *3 & 2 & -4 & 3 \\ 0 & & & \\ 0 & & & \end{array}$$

To replace the values in row 2, each new element is obtained by multiplying the element being replaced in the second row by 3 and subtracting 2 times the element in the first row from the same column as the element being replaced. To perform the pivot, place one finger on the pivot (circled number), and one finger on the element being replaced. Multiply these two numbers together. Now, place one finger on the boxed number in the same row as the element you're replacing and the other finger in the pivot row and the same column as the number your replacing. Multiply these two numbers together. Take the product with the pivot and subtract the product without the pivot.

$$\begin{array}{c|ccc} x & y & z & \text{rhs} \\ \hline *3 & 2 & -4 & 3 \\ 2 & 3 & 3 & 15 \\ 5 & -3 & 1 & 14 \end{array}$$

To replace the 3 in R_2C_2 , you would take $3(3) - 2(2) = 9 - 4 = 5$.

To replace the 3 in R_2C_3 , you would take $3(3) - 2(-4) = 9 + 8 = 17$.

To replace the 15 in R_2C_4 , you would take $3(15) - 2(3) = 45 - 6 = 39$.

To replace the -3 in R_3C_2 , you would take $3(-3) - 5(2) = -9 - 10 = -19$.

To replace the 1 in R_3C_3 , you would take $3(1) - 5(-4) = 3 + 20 = 23$

To replace the 14 in R_3C_4 , you would take $3(14) - 5(3) = 42 - 15 = 27$.

Here's how the process looks.

$$\begin{array}{c|ccc} x & y & z & \text{rhs} \\ \hline \text{pivot row, copy} & \text{pivot row, copy} & \text{pivot row, copy} & \text{pivot row, copy} \\ 3 & 2 & -4 & 3 \\ \hline \text{pivot column, clear} & 3(3) - 2(2) & 3(3) - 2(-4) & 3(15) - 2(3) \\ 0 & 5 & 17 & 39 \\ \hline \text{pivot column, clear} & 3(-3) - 5(2) & 3(1) - 5(-4) & 3(14) - 5(3) \\ 0 & -19 & 23 & 27 \end{array}$$

Or, if you remove the comments, the matrix after the first pivot looks like

This:

$$\begin{array}{c|ccc} x & y & z & \text{rhs} \\ \hline 3 & 2 & -4 & 3 \\ 0 & 5 & 17 & 39 \\ 0 & -19 & 23 & 27 \end{array}$$

It is now time to repeat the entire process. We go through and pick another place to pivot. We would like it to be on the main diagonal, a one, or have zeros in the column. Unfortunately, we can't have any of those. But since we have to multiply all the other numbers by the pivot, we want it to be small, so we'll pivot on the 5 in R_2C_2 and clear out the 2 and -19.

$$\begin{array}{c|ccc} x & y & z & \text{rhs} \\ \hline 3 & 2 & -4 & 3 \\ 0 & *5 & 17 & 39 \\ 0 & -19 & 23 & 27 \end{array}$$

Begin by copying down the pivot row (2^{nd} row) and clearing the pivot column (2^{nd} column). Previously cleared columns will remain cleared.

$$\begin{array}{c|ccc} x & y & z & \text{rhs} \\ \hline & 0 & & \\ 0 & *5 & 17 & 39 \\ 0 & 0 & & \end{array}$$

Here are the calculations to find the next iteration. By careful attention to the 3^{rd} row where we're subtracting -19 times a value. Since we're subtracting a negative, I went ahead and wrote it as plus 19.

$$\begin{array}{c|c|c|c} x & y & z & \text{rhs} \\ \hline 5(3) - 2(0) & \text{pivot column, clear} & 5(-4) - 2(17) & 5(3) - 2(39) \\ 15 & 0 & -54 & -63 \\ \hline \text{pivot row, copy} & \text{pivot row, copy} & \text{pivot row, copy} & \text{pivot row, copy} \\ 0 & 5 & 17 & 39 \\ \hline \text{previously cleared} & \text{pivot column, clear} & 5(23) + 19(17) & 5(27) + 19(39) \\ 0 & 0 & 438 & 876 \end{array}$$

And the resulting matrix:

$$\begin{array}{c|c|c|c} x & y & z & \text{rhs} \\ \hline 15 & 0 & -54 & -63 \\ 0 & 5 & 17 & 39 \\ 0 & 0 & 438 & 876 \end{array}$$

Notice that all the elements in the first row are multiples of 3 and all the elements in the last row are multiples of 438. We'll divide to reduce the rows.

$$\begin{array}{c|c|c|c} x & y & z & \text{rhs} \\ \hline 5 & 0 & -18 & -21 \\ 0 & 5 & 17 & 39 \\ 0 & 0 & 1 & 2 \end{array}$$

That had the added benefit of giving us a 1, exactly where we want it to be to pivot. So, we'll pivot on the 1 in R_3C_3 and clear out the -18 and 17. Circle your pivot and box the other numbers in that column to clear.

$$\begin{array}{c|c|c|c} x & y & z & \text{rhs} \\ \hline 5 & 0 & -18 & -21 \\ 0 & 5 & 17 & 39 \\ 0 & 0 & 1 & 2 \end{array}$$

Copy down the pivot row and clear the pivot column. Previously cleared columns will remain cleared as long as you don't pivot in a row or column twice.

$$\begin{array}{c|c|c|c} x & y & z & \text{rhs} \\ \hline & 0 & 0 & \\ 0 & & 0 & \\ 0 & 0 & 1 & 2 \end{array}$$

Notice that each time, there are fewer calculations to perform. Here are the calculations for this pivot. Again, since the value in the pivot column in the first row is -18 and we're subtracting, I wrote it as +18.

$$\begin{array}{c|c|c|c} x & y & z & \text{rhs} \\ \hline 1(5) + 18(0) & \text{previously cleared} & \text{pivot column, clear} & 1(-21) + 18(2) \\ 5 & 0 & 0 & 15 \\ \hline \text{previously cleared} & 1(5) - 17(0) & \text{pivot column, clear} & 1(39) - 17(2) \\ 0 & 5 & 0 & 5 \end{array}$$

$$\begin{array}{c|c|c|c} & & 0 & \\ \hline \text{pivot row, copy} & \text{pivot row, copy} & \text{pivot row, copy} & \text{pivot row, copy} \\ 0 & 0 & 1 & 2 \end{array}$$

And the resulting matrix.

$$\begin{array}{c|c|c|c} x & y & z & \text{rhs} \\ \hline 5 & 0 & 0 & 15 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}$$

Notice that the first and second rows are multiples of 5, so we can reduce those rows.

$$\begin{array}{c|c|c|c} x & y & z & \text{rhs} \\ \hline 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}$$

And the final answer is $x = 3$, $y = 1$, and $z = 2$. You can also write that as an ordered triplet $\{(3,1,2)\}$.

Gauss Elimination Method:

Definition (1):

Gaussian elimination is a method of solving a linear system $Ax = b$ (Consisting of m equations in n unknowns) by bringing the augmented matrix:

$$[A \ b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

to an upper triangular form

$$\left[\begin{array}{cccc|c} c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\ 0 & c_{22} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{mn} & d_m \end{array} \right]$$

This elimination process is also called the forward elimination method.

The following examples illustrate the Gauss elimination procedure.

Example (1):

Solve the linear system by Gauss elimination method.

$$\begin{array}{rcl} y + z & = & 2 \\ 2x + 3z & = & 5 \\ x + y + z & = & 3 \end{array}$$

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Solution:

In this case, the augmented matrix

is $\begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}$. The method proceeds along the following steps.

1. Interchange 1st and 2nd equation (or R_{12}).

$$\begin{array}{rcl} 2x + 3z & = & 5 \\ y + z & = & 2 \\ x + y + z & = & 3 \end{array} \quad \begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

2. Divide the 1st equation by 2 (or $R_1(1/2)$).

$$\begin{array}{rcl} x + \frac{3}{2}z & = & \frac{5}{2} \\ y + z & = & 2 \\ x + y + z & = & 3 \end{array} \quad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

3. Add -1 times the 1st equation to the 3rd equation (or $R_{31}(-1)$).

$$\begin{array}{rcl} x + \frac{3}{2}z & = & \frac{5}{2} \\ y + z & = & 2 \\ y - \frac{1}{2}z & = & \frac{1}{2} \end{array} \quad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

4. Add -1 times the 2nd equation to the 3rd equation (or $R_{32}(-1)$).

$$\begin{array}{rcl} x + \frac{3}{2}z & = & \frac{5}{2} \\ y + z & = & 2 \\ -\frac{3}{2}z & = & -\frac{3}{2} \end{array} \quad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}.$$

5. Multiply the 3rd equation by $-\frac{2}{3}$ (or $R_3(\frac{-2}{3})$).

$$\begin{array}{rcl} x + \frac{3}{2}z & = & \frac{5}{2} \\ y + z & = & 2 \\ z & = & 1 \end{array} \quad \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The last equation gives $z = 1$ the second equation now gives $y = 1$ Finally the first equation gives $x = 1$. Hence the set of solutions is $(x, y, z)^t = (1, 1, 1)^t$ is a unique solution.

Example (2):

Solve the linear system by Gauss elimination method.

$$x + y + z = 3$$

$$x + 2y + 2z = 5$$

$$3x + 4y + 4z = 11$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 11 \end{bmatrix}$$

In this case, the augmented matrix is and the method proceeds as follows:

1. Add -1 times the first equation to the second equation.

$$\begin{array}{rcl} x + y + z & = & 3 \\ y + z & = & 2 \\ 3x + 4y + 4z & = & 11 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 3 & 4 & 4 & 11 \end{bmatrix}.$$

2. Add -3 times the first equation to the third equation.

$$\begin{array}{rcl} x + y + z & = & 3 \\ y + z & = & 2 \\ y + z & = & 2 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

3. Add -1 times the second equation to the third equation

$$\begin{array}{rcl} x + y + z & = & 3 \\ y + z & = & 2 \\ y + z & = & 2 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the set of solutions is $(x, y, z)^t = (1, 2 - z, z)^t = (1, 2, 0)^t + z(0, 1, -1)^t$ with z arbitrary. In other words, the system has an infinite number of solutions.

Example (3):

Solve the linear system by Gauss elimination method.

$$x + y + z = 3$$

$$x + 2y + 2z = 5$$

$$3x + 4y + 4z = 12$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 12 \end{bmatrix}$$

In this case, the augmented matrix is and the method proceeds as follows:

1. Add -1 times the first equation to the second equation.

$$x + y + z = 3$$

$$y + z = 2$$

$$3x + 4y + 4z = 12$$

2. Add -3 times the first equation to the third equation.

$$\begin{array}{rcl} x + y + z & = & 3 \\ y + z & = & 2 \\ y + z & = & 3 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}.$$

3. Add -1 times the second equation to the third equation

$$\begin{array}{rcl} x + y + z & = & 3 \\ y + z & = & 2 \\ 0 & = & 1 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The third equation in the last step is:

$$0x + 0y + 0z = 1$$

This can never hold for any value of x, y, z . Hence, the system has no solution.

Remark:

Note that to solve a linear system, $Ax = b$ one needs to apply only the elementary row operations to the augmented matrix $[A \ b]$.

Algorithm for Gauss Elimination:

Let $T = (t_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be an $m \times n$ matrix. Then T is a matrix in an upper row echelon form, if $r \in \{0, 1, \dots, n\}$ and indices $j_1, j_2, \dots, j_r \in \{1, \dots, n\}$ exist with:

1. $j_1 < j_2 < \dots < j_r$.
2. For each $i \in \{1, \dots, r\}$: $t_{i,1} = t_{i,2} = \dots = t_{i,j_i-1} = 0$.
3. For each $i \in \{r+1, \dots, m\}$: $t_{i,j} = 0$ for each $j \in \{1, \dots, n\}$.

The indices j_1, j_2, \dots, j_r are the characteristic column indices of the matrix T .

Section (2)

In this section, we describe a method for finding the rank of any matrix. This method assumes familiarity with echelon matrices and echelon transformations.

The maximum number of linearly independent vectors in a matrix is equal to the number of non-zero rows in its row echelon matrix. Therefore, to find the rank of a matrix, we simply transform the matrix to its row echelon form and count the number of non-zero rows.

Definition (1):

A linear system is called **consistent** if it admits a solution and is called **inconsistent** if it admits no solution.

The question arises, as to whether there are conditions under which the linear system $Ax = b$ is consistent. The answer to this question is in the affirmative. To proceed further, we need a few definitions and remarks.

Let

Recall that the row reduced echelon form of a matrix is unique and therefore, the number of non-zero rows is a unique number. Also, note that the number of non-zero rows in either the row reduced form or the row reduced echelon form of a matrix are same.

The Rank of a Matrix:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The i^{th} row of A is

$$row_i(A) = [a_{i1} \ a_{i2} \ \dots \ a_{in}], \ i = 1, 2, \dots, m,$$

and the j^{th} column of A is

$$col_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \ j = 1, 2, \dots, n.$$

Row space and column space:

$$span\{row_1(A), row_2(A), \dots, row_m(A)\}$$

which is a vector space under standard matrix addition and scalar multiplication, is referred to as the row space.

Similarly,

$$span\{col_1(A), col_2(A), \dots, col_n(A)\}$$

which is also a vector space under standard matrix addition and scalar multiplication, is referred to as the column space.

Definition (2): (row equivalence)

A matrix B is row equivalent to a matrix A if B result from A via elementary row operations.

Example(1):

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B_1 = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Since

$$A = \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{(1) \leftrightarrow (2)} B_1 = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A = \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{(1)=2*(1)} B_2 = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A = \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{(2)=(2)-(1)} B_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

are all row equivalent to A, B_1, B_2, B_3

Important Result:

If A and B are two $m \times n$ row equivalent matrices, then the row spaces of A and B are equal.

How to find the bases of the row and column spaces:

Suppose A is a $m \times n$ matrix. Then, the basis of the row and column spaces can be found via the following steps.

Step 1:

Transform the matrix A to the matrix in reduced row echelon form.

Step 2:

- The nonzero rows of the matrix in reduced row echelon form a basis of the row space of A.
- The columns corresponding to the ones containing the leading 1's form a basis. For example, if $n = 6$ and the reduced row echelon matrix is :

$$A = \begin{bmatrix} 1 & \times & \times & \times & \times & \times \\ 0 & 0 & 1 & \times & \times & \times \\ 0 & 0 & 0 & 1 & \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

then the 1st, the 3rd, and the 4th columns contain a leading 1 and

thus $col_1(A), col_3(A), col_4(A)$ form a basis of the column space of A.

Note:

To find the basis of the column space is to find to basis for the vector space

$$\text{span}\{col_1(A), col_2(A), \dots, col_n(A)\}$$

Two methods introduced in the previous section can also be used. The method used in this section is equivalent to the second method in the previous section.

Example(2):

Let

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \xrightarrow{\text{in reduced row echelon form}} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2:

- The basis for the row space is :

$$\left\{ [1 \ 0 \ 2 \ 0 \ 1], [0 \ 1 \ 1 \ 0 \ 1], [0 \ 0 \ 0 \ 1 \ -1] \right\}$$

- The columns corresponding to the ones containing the leading 1's are the 1st, the 2nd, and the 4th columns. Thus,

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix} \right\}$$

form a basis of the column space.

Definition (3): (row rank and column rank)

The dimension of the row space of A is called the row rank of A and the dimension of the column space of A is called the column rank of A.

Example(3):

Since the basis of the row space of A is:

$$\left\{ [1 \ 0 \ 2 \ 0 \ 1], [0 \ 1 \ 1 \ 0 \ 1], [0 \ 0 \ 0 \ 1 \ -1] \right\}$$

,
the dimension of the row space is 3 and the row rank of A is 3. Similarly,

Find the basis of the row and column spaces of A.

Solution:

Step 1:

Transform the matrix A to the matrix in reduced row echelon form,

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix} \right\}$$

is the basis of the column space of A. Thus, the dimension of the column space is 3 and the column rank of A is 3.

Definition (4) :

The number of non-zero rows in the row reduced form of a matrix is called the row-rank of the matrix.

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By the definition, it is clear that row-equivalent matrices have the same row-rank. For a matrix A we write 'row-rank(A)' to denote the row-rank of A .

Remark (1) :

Let $Ax = b$ be a linear system with m equations and n unknowns. Then the row-reduced echelon form of A agrees with the first n columns of

$[A \mid b]$ and hence

$$\text{rank} - \text{row}(A) \leq \text{row} - \text{rank} ([A \mid b])$$

The reader is advised to supply a proof.

Remark (2):

Consider a matrix A . After application of a finite number of elementary column operations to the matrix A we can have a matrix, say B which has the following properties:

1. The first nonzero entry in each column is 1.
2. A column containing only 0's comes after all columns with at least one non-zero entry.
3. The first non-zero entry (the leading term) in each non-zero column moves down in successive columns.

Therefore, we can define **column-rank** of A as the number of non-zero columns in B . It will be:

$$\text{Rank} - \text{row} (A) = \text{column} - \text{rank} (A).$$

Remark(3):

Let A be an $m \times n$ matrix and consider the linear system $Ax = b$. Then we see that the linear system $Ax = b$ is consistent if and only if

$$\text{rank}(A) = \text{rank}([A|b]).$$

The following corollary is a very important result about the homogeneous linear system $Ax = 0$.

Corollary (1) :

1- Let A be an $n \times n$ matrix.

- A is nonsingular if and only if $\text{rank}(A) = n$.
- $\text{rank}(A) = n \Leftrightarrow A$ is nonsingular $\Leftrightarrow \det(A) \neq 0$
- $\Leftrightarrow Ax = 0$ has a unique solution.
- $\text{rank}(A) < n \Leftrightarrow Ax = 0$ has a nontrivial solution.

Corollary (2) :

Let A be an $n \times n$ matrix of rank $r < n$. Then the system of equations $Ax = b$ has infinite number of solutions.

The row rank and column rank of the $m \times n$ matrix A are equal.

Since the row rank and the column rank of a $m \times n$ matrix A are equal, we only refer to the rank of A and write

$$\text{rank}(A).$$

2- If A is a $m \times n$ matrix, then

Let A be an $m \times n$ matrix. Then the homogeneous system $Ax = 0$ has a non-trivial solution if and only if $\text{rank}(A) < n$.

Proof:

Suppose the system $Ax = 0$ has a non-trivial solution, x_0 . That is, $Ax_0 = 0$ and $x_0 \neq 0$. Under this assumption, we need to show that $\text{rank}(A) < n$. So, $n = \text{rank}(A) = \text{rank}([A|0])$.

Theorem(1) :

Let A be a matrix of rank r . Then there exist elementary matrices E_1, E_2, \dots, E_k and F_1, F_2, \dots, F_ℓ such that

$$E_1 E_2 \dots E_k A F_1 F_2 \dots F_\ell = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof:

Let C be the row reduced echelon matrix obtained by applying elementary row operations to the given matrix A . As $\text{rank}(A) = r$ the matrix C will have the first r rows as the non-zero rows. So by

Remark (2) C will have r leading columns, say i_1, i_2, \dots, i_r . Note that, for $1 \leq s \leq r$ the i^{th} column will have 1 in the s^{th} row and zero elsewhere.

We now apply column operations to the matrix C . Let D be the matrix obtained from C by successively interchanging the s^{th} and i^{th} column of C for $1 \leq s \leq r$. Then the matrix

D can be written in the form $\begin{bmatrix} I_r & B \\ 0 & 0 \end{bmatrix}$ where B is a matrix of appropriate size. D is an identity matrix, can be made the zero matrix by application of column operations to D . This gives the required result.

Important Results:

$$\begin{aligned} & \text{rank}(A) + \text{nullity}(A) \\ &= \text{the dimension of the column space} + \text{the dimension of the null space} \\ &= n \end{aligned}$$

Example(4) :

Determine the row-rank of:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Solution:

To determine the row-rank of A we proceed as follows.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_{21}(-2), R_{31}(-1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2(-1), R_{32}(1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3(1/2), R_{12}(-2)} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_{23}(-1), R_{13}(1)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The last matrix is the row reduced form of A which has 3 non-zero rows. Thus, rank -row (A) = 3. This result can also be easily deduced from the last matrix.

Example (5):

Determine the row-rank of :

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution:

Here we have

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_{21}(-2), R_{31}(-1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_2(-1), R_{32}(1)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the last matrix, we deduce row-rank (A)=2

Example(6):

Let:

$$\text{and } n = 5. A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis of column space and thus $\text{rank}(A) = 3$. The solutions of $Ax = 0$ are:

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = s_1, x_5 = s_2, s_1, s_2 \in \mathbb{R}.$$

Thus, the solution space (the null space) is:

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$$s_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Then, } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ are the basis of the null space. and}$$

$$\text{nullity}(A) = 2.$$

Therefore,

$$\text{rank}(A) + \text{nullity}(A) = 3 + 2 = 5 = n$$

Let A be an $m \times n$ matrix. Then,

$$\text{has a solution } Ax = b$$

$$\text{rank}(A) = \text{rank}[A | b]$$

You can think of an $r \times c$ matrix as a set of r row vectors, each having c elements; or you can think of it as a set of c column vectors, each having r elements.

The rank of a matrix is defined as (a) the maximum number of linearly independent column vectors in the matrix or (b) the maximum number of linearly independent row vectors in the matrix. Both definitions are equivalent.

Remark (3):

For an $r \times c$ matrix,

- If r is less than c , then the maximum rank of the matrix is r .
- If r is greater than c , then the maximum rank of the matrix is c .

The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its minimum rank would be one.

How to find matrix rank:

Consider the matrix A and its row echelon matrix, A_{ref} . Previously, we showed how to find the row echelon form for matrix A.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$A \qquad A_{\text{ref}}$

Because the row echelon form A_{ref} has two non-zero rows, we know that matrix A has two independent row vectors; and we know that the rank of matrix A is 2.

Section(3)

In this section we give details for finding the inverse of a non singular square matrix by Gauss-Jordan Method.

Definition (1):

A matrix A is invertible if there exists a matrix A^{-1} such that $AA^{-1} = I$. Not all square matrices have an inverses.

Notes:

- * The inverse exist if and only if elimination produces n pivots (row changes are allowed). Elimination solves $Ax = b$ without explicitly using the matrix A^{-1} .
- * The matrix A cannot have two different inverses.
- * If A is invertible, the one and only one solution to $Ax = b$ is $x = A^{-1}b$.

* Suppose there is a non zero vector x such that $Ax = 0$. Then A cannot have an inverse.

* The inverse of a matrix does not always exist. If the determinant of the matrix is zero, if the inverse does not exist, then the matrix is singular

Corollary (1) :

Let A be an invertible $n \times n$ matrix. Suppose that a sequence of elementary row-operations reduces A to the identity matrix. Then the same sequence of elementary row-operations when applied to the identity matrix yields A^{-1} .

Proof:

Let A be a square matrix of order n . Also, let $E_1, E_2, E_3, \dots, E_k$ be a sequence of elementary row operations such $E_1 E_2 \dots E_k A = I_n$ that Then $E_1 E_2 \dots E_k I_n = A^{-1}$ This implies $A^{-1} = E_1 E_2 \dots E_k$.

Calculating A^{-1} by Gauss – Jordan Elimination:

Let A be an $n \times n$ matrix. Apply the Gauss-Jordan method to the matrix $[A | I_n]$ Suppose the row reduced echelon form of the matrix $[A | I_n]$ is $[B | C]$ If $B = I_n$, then $A^{-1} = C$ or else A is not invertible.

Say we have matrix A, and a sequence of row elementary row operations $E_1, E_2, E_3, \dots, E_k$, which will reduce A to I_n . It turns out that the same sequence of row operations will reduce I_n to A^{-1} .

An elementary row operation on an $n \times n$ matrix can be represented by an elementary matrix and performed with matrix multiplication.

For example, the operation " $R_1 \leftrightarrow R_2$ " can be performed by left multiplying

by the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The operation " $2R_1 + R_3 =$

R_3 " is represented by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$. So the sequence of row operations E_1, E_2, \dots, E_k , on the matrix A can be written:

$E_1 E_2 \dots E_k A = I$ If A^{-1} exists, then we can right-multiply both sides by A^{-1} :

$$E_1 E_2 \dots E_k A A^{-1} = I A^{-1}$$

$$E_1 E_2 \dots E_k I = A^{-1}$$

We can use this fact to develop a method to find the inverse of a matrix. To find the inverse of $n \times n$ matrix A , we augment with the Identity to form an $n \times n$ matrix $[A | I]$. We perform Gauss-Jordan reduction on the matrix and the result is $[I | A^{-1}]$. If we cannot reduce A to I using row operations, then A has no inverse.

Example (1):

Find the inverse of A matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ using Gauss Jordan law?

Solution:

First you put it in the form $[A | I]$

$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix}$, by using $-2R_1 + R_2 \rightarrow R_2 \rightarrow R_1 = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}$

Now we want to change -1 in the second row to 1 to put it in the form

$[I | A]$ so we use $-1R_2 \rightarrow R_2$, to be $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}$.

Now we want to change 3 in the first row to 0 to put it in the form $[I | A]$ so we use $-3R_2 + R_1 \rightarrow R_1$, to be $\begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{bmatrix}$

$$\therefore A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

Example (2) :

Find the inverse of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ using the Gauss-Jordan method.

Solution:

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

Consider the matrix A sequence of steps in the Gauss-Jordan method are:

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1(1/2)} \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

1.

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\begin{matrix} R_{21}(-1) \\ R_{31}(-1) \end{matrix}} \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{bmatrix}$$

2.

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{bmatrix} \xrightarrow{R_2(2/3)} \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{bmatrix}$$

3.

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 0 & 1 \end{bmatrix} \xrightarrow{R_{32}(-1/2)} \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 4/3 & -1/3 & -1/3 & 1 \end{bmatrix}$$

4.

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 4/3 & -1/3 & -1/3 & 1 \end{bmatrix} \xrightarrow{R_3(3/4)} \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

5.

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix} \xrightarrow{\begin{matrix} R_{23}(-1/3) \\ R_{13}(-1/2) \end{matrix}} \begin{bmatrix} 1 & 1/2 & 0 & 5/8 & 1/8 & -3/8 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

6.

$$\begin{bmatrix} 1 & 1/2 & 0 & 5/8 & 1/8 & -3/8 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix} \xrightarrow{R_{12}(-1/2)} \begin{bmatrix} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{bmatrix}$$

Thus, the inverse of the given matrix is :

$$\begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

Also $A0 = 0$ implies that 0 is a solution of the linear system

$Ax = 0$ Hence, by the uniqueness of the solution under the condition

$r = r_a = n$ we get $x_0 = 0$ A contradiction to the fact that x_0 was a given non-trivial solution.

Now, let us assume that $\text{rank}(A) < n$ Then

$$r_a = \text{rank}([A \ 0]) = \text{rank}(A) < n.$$

So, the solution set of the linear system $Ax = 0$ has infinite number of vectors x satisfying $Ax = 0$ From this infinite set, we can choose any vector x_0 that is different from 0 Thus, we have a solution $x_0 \neq 0$ That is, we have obtained a non-trivial solution x_0 .

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Proposition (1):

Consider the linear system $Ax = b$. Then the two statements given below cannot hold together.

1. The system $Ax = b$ has a unique solution for every b .
2. The system $Ax = 0$ has a non-trivial solution.

Remark (1):

1. Suppose x_1, x_2 are two solutions of $Ax = 0$. Then $k_1x_1 + k_2x_2$ is also a solution of $Ax = 0$ for any $k_1, k_2 \in \mathbb{R}$.
2. If u, v are two solutions of $Ax = b$ then $u - v$ is a solution of the system $Ax = 0$. That is, $u - v = x_h$ for some solution x_h of

$Ax = 0$. That is, any two solutions of $Ax = 0$ differ by a solution of the associated homogeneous system $Ax = 0$.

In conclusion, for $b \neq 0$ the set of solutions of the system $Ax = b$ is of the form, $\{x_0 + x_h\}$ where x_0 is a particular solution of $Ax = b$ and x_h is a solution $Ax = 0$.

Section (4)

In this section we write a Matlab programs and apply them to solve chosen problems for systems of linear equations, for finding the rank of any matrix, finding the inverse of a non singular matrices as an implementation of Gauss Elimination our main work.

Remarks:

- If A is a square matrix, $A \setminus b$ is roughly equal to $\text{inv}(A)*b$, but MATLAB processes $A \setminus b$ differently and more robustly.
- If the rank of A is less than the number of columns in A , then $x = A \setminus b$ is not necessarily the minimum norm solution. The more computationally expensive.
 $x = \text{pinv}(A)*b$ computes the minimum norm least-squares solution.
- For full singular inputs, you can compute the least-squares solution using the function `linsolve`.

Description:

$x = A \setminus b$ solves the system of linear equations $A*x = b$. The matrices A and b must have the same number of rows.

MATLAB displays a warning message if A is badly scaled or nearly singular, but performs the calculation regardless.

- If A is a scalar, then $A \setminus b$ is equivalent to $A \setminus b$.
- If A is a square n -by- n matrix and b is a matrix with n rows, then
 $x = A \setminus b$ is a solution to the equation $A*x = b$, if it exists.
- If A is a rectangular m -by- n matrix with $m \approx n$, and b is a matrix with m rows, then $A \setminus b$ returns a least-squares solution to the system of equations $A*x = b$. $x = \text{modified}(A, b)$ is an alternative way to execute $x = A \setminus b$, but is rarely used. It enables operator overloading for classes.

Example (1):

Solve a system of linear equations, $A*x = b$.

$A = [1 \ 2 \ 0; 0 \ 4 \ 3];$

$b = [8; 18];$

$x = A \setminus b$

$x =$

4.0000

0.6667

Example (2):

Solve a simple system of linear equations, $A*x = b$.

$A = \text{magic}(3);$

$b = [15; 15; 15];$

$x = A \setminus b$

$x =$

1.0000

1.0000

1.0000

Linear system with singular matrix:

Example(3):

Solve a linear system of equations $A*x = b$ involving a singular matrix, A .

$A = \text{magic}(4);$

$b = [34; 34; 34; 34];$

$x = A \setminus b$

Warning:

Matrix is close to singular or badly scaled. Results may be inaccurate.

$\text{RCOND} = 1.306145\text{e-}17.$

$x =$

1.5000

2.5000

-0.5000

0.5000

Least -squares solution of undetermined system:

Example(4):

Solve a system of linear equations, $A*x = b$.

$A = [1 \ 2 \ 0; 0 \ 4 \ 3];$

$b = [8; 18];$

$x = A \setminus b$

$x =$

4.0000

0.6667

Linear system with sparse matrix:

Example(5):

Solve a simple system of linear equations using sparse matrices.

Consider the matrix equation $A*x = b$.

$A = \text{sparse}([0 \ 2 \ 0 \ 1 \ 0; 4 \ -1 \ -1 \ 0 \ 0; 0 \ 0 \ 0 \ 3 \ -6; -2 \ 0 \ 0 \ 0 \ 2; 0 \ 0 \ 4 \ 2 \ 0]);$

$b = \text{sparse}([8; -1; -18; 8; 20]);$

$x = A \setminus b$

$x =$

(1,1) 1.0000

(2,1) 2.0000

(3,1) 3.0000

(4,1) 4.0000

(5,1) 5.0000

The inverses of matrices:

Example

```
>> a = [3 5 -1 -4 ; 1 4 -0.7 -3 ; 0 -2 0 1 ; -2 6 0 0.3];
```

```
a =
3.0000 5.0000 -1.0000 -4.0000
1.0000 4.0000 -0.7000 -3.0000
0 -2.0000 0 1.0000
-2.0000 6.0000 0 0.3000
```

```
>> inv(a)
```

```
ans =
0.6544 -0.9348 -0.1912 0.0142
0.1983 -0.2833 -0.1034 0.1558
0.3683 -1.9547 -4.2635 -0.4249
0.3966 -0.5666 0.7932 0.3116
```

Example(2):

```
a=
1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1
```

Then, our function shows its result:

```
(inv(a)) =
0.6544 -0.9348 -0.1912 0.0142
0.1983 -0.2833 -0.1034 0.1558
0.3683 -1.9547 -4.2635 -0.4249
0.3966 -0.5666 0.7932 0.3116
```

```
>> rank(a)
```

```
ans = 4
```

Example (3):

```
a = [ 1 2 3; 2 3 4; 1 2 5];
```

```
a =
```

```
1 2 3
2 3 4
1 2 5
```

```
inv(a)
```

```
ans =
-3.5000 2.0000 0.5000
3.0000 -1.0000 -1.0000
-0.5000 0 0.5000
```

Example(4):

```
>>> A= rand(7, 5);
```

```
>>> rank(A)
```

```
ans =
```

```
5
```

Example (5):

We have the linear independent equations :

```
x1 + x2 = -ln(m1)
x3 + x4 = -ln(m2)
x1 + .5x2 + .5x3 + x4 = -ln(m3)
x1 + x3 = -ln(m4)
```

```
>> a = [ 1 1 0 0
```

```
0 0 1 1
```

```
1 .5 .5 1
```

```
1 0 1 0];
```

```
>> rank(a)
```

```
ans =
```

```
(1): >> inv(a)
ans =
-0.5000 -1.0000 1.0000 0.5000
1.5000 1.0000 -1.0000 -0.5000
0.5000 1.0000 -1.0000 0.5000
-0.5000 0 1.0000 -0.5000
```

```
>> b= [-log(m1)
```

```
-log(m2)
```

```
-log(m3)
```

```
-log(m4)];
```

```
>> m1 =.3679
```

```
m1 =
```

```
0.3679
```

```
>> m2 =.0907
```

```
m2 =
```

```
0.0907
```

```
>> m3 =.0954
```

```
m3 =
```

```
0.0954
```

```
>> m4 =.1827
```

```
m4 =
```

```
0.1827
```

```
>> b= [-log(m1)
```

```
-log(m2)
```

```
-log(m3)
```

```
-log(m4)];
```

```
>> x=inv(a)*b
```

```
x =
```

```
0.2995
```

```
0.7005
```

```
1.4004
```

```
0.9997
```

Example(6):(Solving the square system of linear equations)

$$2x_1 + x_2 - 3x_3 - x_4 = 1$$

$$-x_1 - 2x_2 + 4x_3 + 5x_4 = 6$$

$$7x_3 - 6x_4 = 7$$

$$-3x_1 - 8x_2 - 9x_3 + 3x_4 = 9$$

```
>> a= [2 1 -3 -1; -1 -2 4 5; 7 0 0 -6; -3 -8 -9 3]
```

```
a =
```

```
2 1 -3 -1
```

```
-1 -2 4 5
```

```
7 0 0 -6
```

```
-3 -8 -9 3
```

```
>> b=[1 6 7 9]'
```

```
b =
```

```
1
```

```
6
```

```
7
```

```
9
```

```
>> x= a\b
```

```
x =
```

```
1.7843
```

```
-1.5062
```

```
0.0491
```

```
0.9151
```

```
>> inv(a)
```

```
ans =
```

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```
0.2498  0.1649  0.0908 -0.0100
0.2273 -0.0300 -0.1074 -0.0891
-0.1882  0.0358  0.0450 -0.0325
0.2914  0.1923 -0.0608 -0.0117
```

```
>> rank(a)
```

```
ans =
```

```
4
```

REFERENCES

1. D. Eugene, Schaum's Outline of Theory and Problems of Mathematica, McGraw-Hill, NY, (2001).
2. E .B. Magrab and others , An Engineer's Guide to Matlab ,Prentice Hall, Upper Saddle River ,NJ, (2000).
3. Eivind Eriksen B/ Norwegian School of Management Department of Economics (2010).
4. Jim Hefferon Mathematics ,Saint Michael College Colcheser Verno USA (2014).
5. M. Golubitsky and M. Dellnitz , Linear Algebra and Differential Equations Using Matlab, Books/Cole Publishing Co., NY, (1999).
6. Stephen Boyd Convex Optimization . Department of Electrical Engineering Stanford University.
7. Stephen G. Nash Linear and non linear programming (1996).

