On Secondary Normal and Unitary Polynomial Matrices

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***Abstract:** In this paper we introduce polynomial s-normal and polynomial s-unitary matrices and study some of its properties.*

***Keywords:** s-normal polynomial matrix, s-unitary polynomial matrix.*

I. INTRODUCTION

The concept of symmetric and normal matrices have significant importance in spectral analysis . The idea of secondary symmetric and secondary orthogonal matrices was introduced by Anna Lee [1]. As a generalization of Krishnamoorthy and Vijakumar [3] have studied some equivalent conditions on s-normal matrices and Krishnamoorthy and Govindarasu [2] have given some results on secondary unitary matrices. Recently, Ramesh , Sudha and Gajalakshmi [4] have introduced and investigated some properties on normal and unitary polynomial matrices. For any matrix $A = [a_{ij}] \in C^{n\times n}$ (set of all complex matrices of order n x n) , the secondary transpose of A is denoted by A^s and $A^s = [a_{n-j+1,n-i+1}]$ i , j = 1 to n. A matrix $A \in C^{n\times n}$ is said to be s-symmetric if $A = A^s$ and is said to be s-hermitian if $A = A^\theta$, where A^θ denotes conjugate secondary transpose of A. A matrix $A \in C^{n\times n}$ is said to be secondary normal (s-normal if $AA^\theta = A^\theta A$ and is said to be s-unitary, if $AA^\theta = A^\theta A = I_n$.Here, we have introduced the definition of secondary normal (s-normal) polynomial and secondary unitary (s-unitary) polynomial matrices and some of its algebraic properties are studied.

II. POLYNOMIAL S-NORMAL MATRICES

In this section we extend the results studied for normal polynomial matrices [4] to s-normal polynomial matrices.

Definition:2.1

A polynomial s-normal matrices is a polynomial matrix whose coefficient matrices are s-normal matrices.

Example :2.2

$$A(x)=\begin{bmatrix} ix^2+ix+x & xi+i \\ ix+i & ix^2+ix+x \end{bmatrix} = x^2\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} + x\begin{bmatrix} i+1 & i \\ i & i+1 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$= A_2 x^2 + A_1 x + A_0$,where A_0 , A_1 and A_2 are s-normal matrices.

Theorem :2.3

If $A(\lambda)$ and $B(\lambda)$ are polynomial s-normal matrices and $A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$, then $A(\lambda)B(\lambda)$ is a polynomial s-normal matrices**.**

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Proof :

Let $A(\lambda) = A_0 + A_1\lambda + \ldots + A_n\lambda^n$ and $B(\lambda) = B_0 + B_1\lambda + \ldots + B_n\lambda^n$ be polynomial s-normal matrices, A_0 , $A_1 \ldots A_n$ and B_0 , $B_1 \ldots B_n$ are s-normal matrices. Also given , $A(\lambda)B(\lambda) = B(\lambda)A(\lambda)$

$A(\lambda)B(\lambda) = A_0 B_0 + (A_0B_1+A_1B_0)\lambda+\ldots+(A_0B_n+A_1B_{n-1}+\ldots+A_nB_0)\lambda^n$

$B(\lambda)A(\lambda) = B_0A_0 + (B_0A_1+B_1A_0)\lambda+\ldots+(B_0A_n+B_1A_{n-1}+\ldots+B_nA_0)\lambda^n$

Here each coefficient of λ and constants terms are equal.

(i.e) $A_0B_0 = B_0A_0$

$A_0B_1 + A_1B_0 = B_0A_1+B_1A_0$

$\Rightarrow A_0B_1 = B_0A_1$ and $A_1B_0 = B_1A_0$ ....

$A_0B_n+A_1B_{n-1}+\ldots+A_nB_0 = B_0A_n+B_1A_{n-1}+\ldots+B_nA_0$

$\Rightarrow A_nB_0 = B_0A_n$, $A_1B_{n-1} = B_1A_{n-1}$,..., $A_0B_n = B_nA_0$

Now we have to prove $A(\lambda)B(\lambda)$ is s-normal**.**

$A(\lambda)B(\lambda)\ [A(\lambda)B(\lambda)]^\theta = A(\lambda)B(\lambda)\ A(\lambda)^\theta B(\lambda)^\theta = A(\lambda)A(\lambda)^\theta B(\lambda)B(\lambda)^\theta = A(\lambda)^\theta A(\lambda)B(\lambda)^\theta B(\lambda)$

$= [B(\lambda)A(\lambda)]^\theta[A(\lambda)\ B(\lambda)] = [A(\lambda)\ B(\lambda)]^\theta[A(\lambda)\ B(\lambda)]$

Hence $A(\lambda) B(\lambda)$ is s-normal.

Example:2.4

$$\text{Let } A(\lambda)=\begin{bmatrix} i\lambda^2+i\lambda+\lambda & \lambda i+i \\ i\lambda+i & i\lambda^2+i\lambda+\lambda \end{bmatrix} = \lambda^2\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} + \lambda\begin{bmatrix} i+1 & i \\ i & i+1 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$= A_2\lambda^2 + A_1\lambda + A_0$ be polynomial s- normal matrix, where A_0, A_1, A_2 are s-normal matrices and let

$$B(\lambda)=\begin{bmatrix} 0 & i\lambda^2+(1+i)\lambda-i \\ i\lambda^2+(1+i)\lambda-i & 0 \end{bmatrix} = \lambda^2\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + \lambda\begin{bmatrix} 0 & i+1 \\ i+1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$= B_2\lambda^2 + B_1\lambda + B_0$ be polynomial s- normal matrix, where B_0, B_1, B_2 are s-normal matrices. Now we have to prove $A(\lambda)B(\lambda)$ is s-normal.

That is, $[A(\lambda)B(\lambda)]\ [A(\lambda)B(\lambda)]^\theta = [A(\lambda)\ B(\lambda)]^\theta[A(\lambda)\ B(\lambda)] =$

$$\begin{bmatrix} \lambda^8+4\lambda^7+9\lambda^6+6\lambda^5+2\lambda^4-6\lambda^3+3\lambda^2+1 & \lambda^7+2\lambda^6-2\lambda^5-3\lambda^4+\lambda+i(\lambda^6-4\lambda^3-\lambda^2+\lambda) \\ \lambda^7+2\lambda^6-2\lambda^5-3\lambda^4+\lambda+i(\lambda^6-4\lambda^3-\lambda^2+\lambda) & \lambda^8+4\lambda^7+9\lambda^6+6\lambda^5+2\lambda^4-6\lambda^3+3\lambda^2+1 \end{bmatrix}$$

Hence $A(\lambda)B(\lambda)$ is s-normal.

Theorem :2.5

If $A(\lambda)$ is a polynomial s-normal matrix if and only if every polynomial s-unitarily
equivalent to $A(\lambda)$ is polynomial s-normal matrix.

Proof :

Suppose $A(\lambda)$ is polynomial s-normal matrix and $B(\lambda) = U(\lambda)^\theta A(\lambda)U(\lambda)$, where $U(\lambda)$ is polynomial s-unitary matrix.

Now we have to show $B(\lambda)$ is polynomial s-normal matrix.

$B(\lambda)^\theta B(\lambda) = U(\lambda)^\theta A(\lambda)^\theta U(\lambda)U(\lambda)^\theta A(\lambda)U(\lambda)$ (since $U(\lambda)$ is s-unitary matrix)

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$= U(\lambda)^\theta A(\lambda)^\theta A(\lambda)U(\lambda) = U(\lambda)^\theta A(\lambda)A(\lambda)^\theta U(\lambda)$
$=[U(\lambda)^\theta A(\lambda)U(\lambda)][U(\lambda)^\theta A(\lambda)^\theta U(\lambda)]= B(\lambda)B(\lambda)^\theta$.

Hence $B(\lambda)$ is polynomial s-normal matrix.

Conversely , assume $B(\lambda)$ is polynomial s-normal matrix. We shall show that $A(\lambda)$ is polynomial s-normal matrix.

$B(\lambda) = U(\lambda)^\theta A(\lambda)U(\lambda)$ is s-normal matrix.

$B(\lambda)B(\lambda)^\theta = B(\lambda)^\theta B(\lambda) \Rightarrow [U(\lambda)^\theta A(\lambda)U(\lambda)][U(\lambda)^\theta A(\lambda)^\theta U(\lambda)] = [U(\lambda)^\theta A(\lambda)^\theta U(\lambda)][U(\lambda)^\theta A(\lambda)U(\lambda)]$

$\Rightarrow U(\lambda)^\theta A(\lambda)[U(\lambda)U(\lambda)^\theta]A(\lambda)^\theta U(\lambda) = U(\lambda)^\theta A(\lambda)^\theta[U(\lambda)U(\lambda)^\theta]A(\lambda)U(\lambda)$

$\Rightarrow U(\lambda)^\theta A(\lambda)A(\lambda)^\theta U(\lambda) = U(\lambda)^\theta A(\lambda)^\theta A(\lambda)U(\lambda)$

Pre and post multiply by $U(\lambda)$ and $U(\lambda)^\theta$, we get $A(\lambda)A(\lambda)^\theta = A(\lambda)^\theta A(\lambda)$.

Hence $A(\lambda)$ is a polynomial s-normal matrix.

Theorem :2.6

Let $A(\lambda) = A_0 + A_1\lambda+\ldots+ A_m\lambda^m$ and $B(\lambda) = B_0 + B_1\lambda+\ldots+ B_m\lambda^m$ be polynomial matrix , where A_i's B_i's $\in C^{n\times n}$, i = 0,1,2,…m are s-normal matrices and $A_0B_0 = A_1B_1 = \ldots A_mB_m = 0$. Then $A_0^\theta B_0 = A_1^\theta B_1=\ldots = A_m^\theta B_m = 0$.

Proof :

Let $A(\lambda) = A_0A_1\lambda + A_2\lambda^2 +\ldots+A_m\lambda^m$ is a polynomial s-normal matrix. Where $A_{0,}\ A_{1,\ldots}\ A_m$ are s-normal matrices and $B(\lambda) = B_0B_1\lambda + B_2\lambda^2 +\ldots+B_m\lambda^m$ is a polynomial matrix and also given $A_0B_0 = A_1B_1 = \ldots =A_mB_m = 0$. We know that if M, N are s-normal and MN = 0 then $M^\theta N = 0$. Thus $A_0^\theta B_0 = A_1^\theta B_1 = \ldots = A_m^\theta B_m = 0$.

III. POLYNOMIAL S-UNITARY MATRICES

Here we define polynomial s-unitary matrices and establish a relation between s-hermition and s-normal polynomial matrices.

Definition:3.1

A polynomial s-hermitian matrix is a polynomial matrix whose coefficients are s-hermitian matrices.

Definition :3.2

A polynomial s-unitary matrix is a polynomial matrix whose coefficient matrices are s-unitary matrices.

Theorem :3.3

If $U(\lambda)$ is a polynomial s-unitary matrix if and only if $U(\lambda)^\theta$ is polynomial s-unitary matrix.

Proof :

Let $U(\lambda) = A_0 + A_1\lambda + A_2\lambda^2+\ldots+ A_n\lambda^n$ be a polynomial s-unitary matrix , A_i's are s-unitary matrices and $A_0A_0^\theta = I$, $A_1A_1^\theta = I$ …, $A_nA_n^\theta = I$. We shall show that $U(\lambda)^\theta$ is a polynomial s-unitary matrix.

Now , $[U(\lambda)]^\theta = [A_0 + A_1\lambda + A_2\lambda^2+\ldots+ A_n\lambda^n]^\theta = A_0^\theta + A_1^\theta\lambda + A_2^\theta\lambda^2 +\ldots+ A_n^\theta\lambda^n$, and A_i's are s-unitary matrices. Hence $U(\lambda)^\theta$ is a polynomial s-unitary matrix.

Conversely ,Assume that , $U(\lambda)^\theta$ is a polynomial s-unitary matrix.

We shall show that $U(\lambda)$ is a polynomial s-unitary matrices.

$[U(\lambda)]^\theta = [A_0 + A_1\lambda + A_2\lambda^2+\ldots+ A_n\lambda^n]^\theta$,Taking secondary conjugate transpose on both sides ,

$U(\lambda) = A_0 + A_1\lambda + A_2\lambda^2+\ldots+ A_n\lambda^n$. Hence $U(\lambda)$ is a polynomial s-unitary matrices.

Remark:3.4

Since the coefficient matrices of a polynomial s-unitary is s-unitary their determinant value is one**.**

Theorem :3.5

If $U(\lambda) = A_0 + A_1\lambda + A_2\lambda^2+\ldots+ A_n\lambda^n$ be polynomial s-unitary matrix ,

Where $A_i \in C^{n\times n}$ and $A_0, A_1, A_2,\ldots A_n$ are s-unitary matrices then A_i's are diagonalizable.

Proof:

Given $U(\lambda) = A_0 + A_1\lambda + A_2\lambda^2+\ldots + A_n\lambda^n$ is a polynomial s-unitary matrix. By definition $A_0, A_1, A_2,\ldots A_n$ are s-unitary matrices are diagonalizable and hence are diagonalizable.

Remark:3.6

If $A(\lambda)$ is a polynomial s-unitary matrix then the absolute value of all the eigen values of the coefficient matrices are unity. Polynomial s-unitary matrices are polynomial s-normal matrices.

Theorem :3.7

Polynomial s-hermitian matrix is polynomial s-normal matrix and the coefficient matrices satisfy $AA^\theta = A^2 = A^\theta A$.

Proof:

We know that s-hermitian matrix with $AA^\theta = A^2 = A^\theta A$ is s-normal.

In polynomial s-hermitian matrix we have all the coefficient matrices are s-hermitian.

By hypothesis , we have coefficient matrices satisfy the condition $AA^\theta = A^2 = A^\theta A$.

Hence we get all coefficient matrices are s-normal**.**

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