New Oscillation and Nonoscillation Criteria for a Class of Linear Delay Differential Equation

P. Sowmiya, G.K. Revathi, M. Sakthipriya And V. Ramya

***Abstract:** In this article the authors established sufficient condition for the first order delay differential equation in the form*

$$x'(t) + p(t)x(\tau(t)) = 0, t \geq t_0 \qquad (*)$$

where $\tau(t) < t$, $lim_{t\to\infty} \tau(t) = \infty$ and $p(t)$ is a non negative piecewise continuous function. Some interesting examples are provided to illustrate the results.

***Keywords:** Oscillation, delay differential equation and bounded. AMS Subject Classification 2010: 39A10 and 39A12.*

I. INTRODUCTION

The derivative of unknown function at a certain time are represented in terms of values of those functions at previous time are referred as Delay differential equation.

Time delays are the natural components in dynamic processes of physiology, epidemiology, ecology, biology and mechanics which includes time delays.

Infact time delays represents the resource regeneration times in biological models.

Delay differential equations includes much more complicated dynamics than ordinary differential equation. Time delay may cause stable equilibrium to unstable and may leads to populations fluctuation which has been discussed in H. Smith [8].

The idea of nonoscillation criteria of linear delay differential equation has appeared as early in the works of O. Arino [1]. Now a days, peoples' concentration has been directed towards the oscillation criteria for linear delay differential equation. Among numerous papers dealing with the subject, we refer [2, 3, 4, 5, 6, 7, 9, 10] cited there in.

The above observations are motivated our interest in the study of new oscillation and nonoscillation linear delay differential equation which can be extended to neutral delay differential equation.

Our aim in this paper is to obtain the new oscillatory behavior of the solutions of the first order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, t \geq t_0 \qquad (1)$$

where $(t) < t$, $\lim_{t\to\infty} \tau(t) = \infty$.

We assume the following conditions:

(i) Let $p(t)$ be a nonnegative piecewise continuous function in $t \geq t_0$.

(ii) $\tau \in C([t_0,\infty), R), \tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

(iii) $\tau(t)$ is non decreasing, where there exists $\{t_n\}_{n-1}^{\infty}$ be a real sequence such that $\tau(t_{n+1}) = t_n$, n = 1,2,…. Also we assume $\tau(t_1) = t_0$.

(iv) Let τ^{-k} (t) be defined on $[t_0,\infty)$ by $\tau^{-(k+1)}$ (t) = $\tau^{-1}(\tau^{-k}(t))$, $k = 1,2,\cdots$ and let $t_k = \tau^{-k}(t_0)$, $k = 1,2,\cdots$ where $t_k \to \infty$ as k → ∞.

Define $\{p_n(t)\}$ as follows:

$$p_1(t) = \int_{\tau(t)}^{t} p(s)ds, \qquad t \geq t_1,$$

$$p_{k+1}(t) = \int_{\tau(t)}^{t} p(s)p_k(s)ds,$$

$$t \geq t_{k+1}, \quad k = 1,2,\cdots$$

For any positive integer n, define a sequence $\{q_k\}$ as

$q_0 = \min_{t_0 \leq t \leq t_n}\{p_n(t)\}$

$q_k = \min_{t_{kn} \leq t \leq t_{(k+1)n}}\{p_n(t)\}$,

$$k = 1,2,\ldots \qquad (2)$$

We always assume that

$$p_n(t) \geq \frac{1}{e^n}, t \geq t_n \qquad (3)$$

Also the best results on the oscillation of (1) is obtained when $\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e}$

and $\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$.

II. SOME BASIC LEMMAS

Lemma: 1

Consider $x(t)$ as a solution of (1), and $\bar{t}$ be a positive number with $x(t) > 0$ on $[\tau^{n+2}(\bar{t}), \bar{t}]$ where $\tau^{n+2}(\bar{t}) \geq \bar{t}$. Let

$$N = \min_{\tau(t) \leq t \leq \bar{t}} \frac{x(\tau(t))}{x(t)}.$$

Then $N < 4e^{2n}$.

Proof:

It is true from (1) that $x'(t) \leq 0$ for $t \in [\tau^{n+2}(\bar{t}), \bar{t}]$.

Integrating equation (1),

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^{t} p(s)x(\tau(t))ds = 0, \qquad t \geq t1,$$

therefore for $t \in [\tau^{n-1}(\bar{t}), \bar{t}]$, we obtain

$$x(\tau(t)) > \int_{\tau(t)}^{t} p(s)x(\tau(s))ds$$
$$\geq x(\tau(t))p_1(t)$$

Thus for $t \in [\tau^{n-2}(\bar{t}), \bar{t}]$, we have

$$x(\tau(t)) > \int_{\tau(t)}^{t} p(s)x(\tau(s))ds$$
$$\geq \int_{\tau(t)}^{t} p(s)p_1(s)x(\tau(s))ds$$
$$\geq x(\tau(t))p_2(t)$$

It is true that for $t \in [\tau(\bar{t}), \bar{t}]$,

$$x(\tau(t)) > x(\tau(t))p_{n-1}(t) \qquad (4)$$

Based on (3), there exists $t^* \in [\tau(\bar{t}), \bar{t}]$ such that $\int_{\tau(\bar{t})}^{t^*} p(s)p_{n-1}(s)ds \geq \frac{1}{2e^n}$

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P. Sowmiya, Assistant Professor, Shri Sakthikailassh Women's college Salem-636 003. Tamilnadu, India .

G.K.Revathi, Assistant Professor , Division of Mathematics, School of advanced sciences, Vellore Institute of Technology,Chennai 127 Tamilnadu, India .

M. Sakthipriya, Research Scholar. Shri Sakthikailassh Women's college Salem-636 003. Tamilnadu, India.

V. Ramya, Assistant Professor, Shri Sakthi kailassh Women's college, Salem-636 003. Tamilnadu, India.

And

$$\int_{t^*}^{\bar{t}} p(s)p_{n-1}(s)ds \geq \frac{1}{2en} \quad (5)$$

By integrating equation (1), we have

$$x(t^*) - x(\tau(\bar{t})) = -\int_{\tau(\bar{t})}^{t^*} p(s)x(\tau(s))ds$$

and

$$x(\bar{t}) - x(t^*) = -\int_{t^*}^{\bar{t}} p(s)x(\tau(s))ds,$$

$$x(t^*) \geq \frac{1}{2e^n} x(\tau(\bar{t})) \geq \frac{1}{4e^{2n}} x(\tau(t^*)).$$

Thus

$$N \leq \frac{x(\tau(t^*))}{x(t^*)} < 4e^{2n}.$$

The proof is complete.

Lemma: 2

Assume that (2) and (3) holds true. Suppose there exist two positive integers k > m with a solution $x(t)$ in (1) such that $x(t) > 0$ for $t \in [t_{(m-1)n}, t_{kn}]$, consider

$$H(t) = \frac{x(\tau(t))}{x(t)},$$

$h(t) = \min\{ H(s) : \tau^n(t) \leq s \leq t \}$.

Then h(t) is non decreasing for $t \in [t_{mn}, t_{kn}]$.

Proof:

From (1) ,

$$\frac{x'(t)}{x(t)} = -p(t)\frac{x(\tau(t))}{x(t)}, \; t \geq t_{(m-1)n}.$$

Integrating form $\tau(t)$ to t, for $t \in [t_{mn}, t_{kn}]$,

$$H(t) = \exp\left\{\int_{\tau(t)}^{t} p(s)\frac{x(\tau(s))}{x(s)}ds\right\}$$

$$= \exp\left\{\int_{\tau(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{s} p(u)\frac{x(\tau(u))}{x(u)}du\right)ds\right\} = \exp\left\{e\int_{\tau(t)}^{t} p(s_1)\int_{\tau(s_1)}^{s_1} p(s_2)\frac{x(\tau(s_2))}{x(s_2)}ds_2\, ds_1\right\}$$

$$\vdots$$

$$\geq \exp\left\{e^{n-1}\int_{\tau(t)}^{t} p(s_1)\int_{\tau(s_1)}^{s_1} p(s_2)\ldots \int_{\tau(s_{n-1})}^{s_{n-1}} p(s_n)\frac{x(\tau(s_n))}{x(s_n)}ds_n\, ds_1\right\}$$

Where $s_n \in [\tau^n(t), t]$, for any $t \in [t_{mn}, t_{kn}]$,

$$H(t) \geq \exp(e^{n-1}h(t)p_n(t)).$$

In $[t_{mn}, t_{kn}]$, $h(t)$ is not increasing, then there exists t and u with u > t such that $h(t) >$ H(u). Let us choose c ≠ e with $h(t) >$ c > H(u), where H is continuous on $[t_{mn}, t_{kn}]$, S = min { s : H(s) = c, u ≥ s ≥ t } exists. Hence t < S < u and H(S) = $h(S)$ = c.

Therefore

c = H(S) ≥ exp $(e^{n-1}h(S)p_n(s))$ ≥ exp $\left(\frac{c}{e}\right)$ > c a refutation.

Therefore $h(t) \leq$H(v) for all v ≥ t and t, v ∈ $[t_{mn}, t_{kn}]$.

Thus $h(t) \leq h(v)$ for t ≤ v and t , v ∈ $[t_{mn}, t_{kn}]$.

The proof is complete.

Lemma: 3

Assume that $x(t)$ is a solution of (1) for some positive integer m, and such that $x(t) > 0$ on $[t_{(m-2)n-1}, t_{(m+1)n}]$ and let M, N be defined by

$$M = \min_{t_{(m-1)n} \leq t \leq t_{mn}} \frac{x(\tau(t))}{x(t)},$$

$$N = \min_{t_{mn} \leq t \leq t_{(m+1)n}} \frac{x(\tau(t))}{x(t)},$$

Then M > 1, and

$$N \geq \exp(e^{n-1}Mq_m) \geq \exp\left(\frac{M}{e}\right) \geq M,$$

where q_m is defined as in (2).

Proof:

From (1), $x'(t) \leq 0$ on $[t_{(m-1)n-1}, t_{(m+1)n}]$.

Hence M > 1. Dividing (1) by $x(t)$ and integrating for $t \in [t_{(m-1)n-1}, t_{mn+1}]$, we have

$$\frac{x(\tau(t))}{x(t)} = \exp\left(\int_{\tau(t)}^{t} p(s)\frac{x(\tau(s))}{x(s)}ds\right). \quad (6)$$

Let

$\bar{p}_k = \min_{t_{mn+k-1} \leq t \leq t_{mn+k}} \{p_n(t)\}$,

k = 1,2,…n,

$N_k = \min_{t_{mn+k-1} \leq t \leq t_{mn+k}} \frac{x(\tau(t))}{x(t)}$,

k = 1,2,…n,

$M_{\bar{k}} = \min_{t_{(m-1)n+\bar{k}} \leq t \leq t_{mn+\bar{k}}} \frac{x(\tau(t))}{x(t)}$,

$\bar{k} = 1,2,\ldots n-1$,

Then, by the $q_m \leq \bar{p}_k$, k = 1,2,…n, and N = $\min_{1 \leq i \leq n}\{N_i\}$.

Now,

$$\frac{x(\tau(t))}{x(t)} = \exp\left\{\int_{\tau(t)}^{t} p(s)\left(\exp\left(\int_{\tau(s)}^{s} p(u)\frac{x(\tau(u))}{x(u)}du\right)\right)ds\right\}$$

$$\geq \exp\left\{e\int_{\tau(t)}^{t} p(s)\int_{\tau(s_1)}^{s} p(u)\frac{x(\tau(u))}{x(u)}du\; ds\right\}$$

$$\vdots$$

$$\geq \exp\left\{e^{n-1}\int_{\tau(t)}^{t} p(s_1)\int_{\tau(s_1)}^{s_1} p(s_2)\ldots \int_{\tau(s_{n-1})}^{s_{n-1}} p(s_n)\frac{x(\tau(s_n))}{x(s_n)}ds_n\, ds_1\right\}$$

where $s_n \in [t_{(m-1)n}, t_{mn+1}]$.

Hence, N ≥ M, and

$N_1 \geq \exp(e^{n-1}\min(M, N_1)p_1)$

$\geq \exp(e^{n-1}\min(M, N)q_m)$

$\geq \exp\left(\frac{m}{e}\right) \geq$ M.

Here, $\min(M, N_1)$ = M.

$M_1 = \min_{t_{(m-1)n+1} \leq t \leq t_{mn+1}} \frac{x(\tau(t))}{x(t)}$

$\geq \min(M, N_1)$ = M.

By using the same methodology, for $t \in [t_{mn+1}, t_{mn+2}]$

$N_2 \geq \exp\left(\frac{\min(M,N_2)}{e}\right) \geq \min(M_1, N_1)$

$\geq \min(M, N_2)$.

Therefore $\min(M, N_2) = M$ and

$N_2 \geq \exp(e^{n-1}Mq_m) \geq \exp\left(\frac{M}{e}\right) \geq$ M.

$M_2 \geq \min(M_1, N_1)$

$\geq \min(M, N_1, N_2) = M.$

$N_K \geq \exp(e^{n-1}Mq_m) \geq \exp\left(\frac{M}{e}\right) \geq$ M,

k = 1,2,…

which implies that

$$N = \min_{1 \leq k \leq n}\{N_K\} \geq \exp(e^{n-1}Mq_m)$$

$$\geq \exp\left(\frac{M}{e}\right) \geq M.$$

Note:

consider the sequence $\{r\}_{i=0}^{\infty}$ which is defined by

$r_0 = 1$, $r_{i+1} = \exp\left(\frac{r_i}{e}\right)$

for i = 0,1,2,… (7)

Lemma: 4

The following relations hold for the sequence $\{r_i\}$ given in (7) the following relations hold
(a) $r_i < r_{i+1}$;
(b) $r_i <$ e ;
(c) $\lim_{i\to\infty} r_i$ = e;
(d) $r_i >$ e $- \frac{2e}{i+2}$, for i= 0,1,……

Proof:
By using induction method the first two relations can be proved. A consequence of (a) and (b) $\lim_{i\to\infty} r_i$ = r exists and it is finite. Then by (7) we have r = $e^{\frac{r}{e}}$. It is easy to verify that

$e^{\frac{x}{e}} >$ x for x ≠ e (8)

The above inequality shows that the limit r equals e.
For i = 0 and i = 1, it is immediate for the proof of (d). For i ≥ 1 the proof goes by induction method, so we have $r_{i+1} = e^{\frac{r_i}{e}} > e^{\frac{1-2}{(i+2)}}$,
And it is sufficient to show

$e^{\frac{1-2}{(i+2)}} >$ e $-\frac{2e}{i+3}$,

or

$f(x) = e^{\frac{-2}{x}} + \frac{2}{x+1} > 1$ for x = i + 2.

Since

$f'(x) = \frac{2}{x^2}\left(e^{\frac{-1}{x}} + \frac{x}{x+1}\right)\left(e^{\frac{-1}{x}} - \frac{x}{x+1}\right)$

And

$e^{\frac{1}{x}} > 1 + \frac{1}{x} = \frac{x+1}{x}$,

We have $f'(x) < 0$ and $f(x) > \lim_{x\to\infty} f(x) = 1$, which has been proved.
The proof of above lemma is complete.

III. OSCILLATION AND NONOSCILLATION RESULTS

Theorem: 5

Suppose that (2) and (3) holds. If $\sum_{i=1}^{\infty}\left(q_i - \frac{1}{e^n}\right) = \infty$, then the solution in (1) oscillates. (9)

Proof:

On the contrary let us assume, that there exists $x(t)$ such that $x(t) > 0$ for $t \ge t_{(k-1)n-2}$ where k is a sufficiently large integer.
Let us consider the sequence $\{N_i\}$ by

$N_i = \min_{t_{(k+i-1)n} \le t \le t_{(k+i)+n}} \frac{x(\tau(t))}{x(t)}$, i = 0,1,⋯

By lemma (2), we have $N_i > 1$, i = 0,1,….. and

$N_{i+1} \ge \exp(e^{n-1} N_i q_{k+i})$

$= \exp\left(\frac{N_i}{e}\right)\exp\left(N_i\left(e^{n-1} q_{k+i} - \frac{1}{e}\right)\right)$

$> \exp\left(\frac{N_i}{e}\right) \ge N_i$, (10)

That is $\{N_i\}$ is increasing and bounded.
Hence $\lim_{i\to\infty} N_i$ = N exists. From (10) N ≥ exp $\left(\frac{N_i}{e}\right)$.
It is clear that for $x \neq e$, $\exp\left(\frac{x}{e}\right) > x$,.
Based on the above result, N = e and 1 < $N_0 < N_1 <$…. <e.
Also,

$N_{i+1} \ge N_i\left(1 + e^{n-1} N_i (q_{k+i} - e^{-n})\right)$, i = 0,1,⋯

Hence

$N_{i+1} - N_i \ge e^{n-1} N_i^2 (q_{k+i} - e^{-n})$,

And

$N_{i+2} - N_{i+1} \ge e^{n-1} N_{i+1}^2 (q_{k+i+1} - e^{-n})$,
$> e^{n-1} N_i^2 (q_{k+i+1} - e^{-n}) \cdots$

By collating all the known results,

e$-N_i > e^{n-1} N_i^2 \sum_{j=1}^{\infty}(q_{k+j} - e^{-n})$ (11)

for some sufficiently large k, which leads to a contradiction to (9).
Hence the proof is complete.

Corollary: 1

Consider that condition (2) holds. If there is a positive integer n such that $\lim_{t\to\infty}\inf p(t) > \frac{1}{e^n}$, then all solution of (1) oscillates.

Example: 1

Consider the equation of delay

$x'(t) + \frac{1}{2e}(1 + \cos t)\, x(t-\pi) = 0$, t ≥ 0. (12)

We have

$\lim_{t\to\infty}\inf\int_{t-\pi}^{t} p(s)ds = \frac{\pi-2}{2e} < \frac{1}{e}$

which does not satisfy $\lim_{t\to\infty}\inf\int_{t-\pi}^{t} p(s)ds > \frac{1}{e}$

$p_4(t) = \frac{1}{16e^4}(\pi^4 - 4\pi^2 + 2(\pi^3 - 6\pi)\sin t -$ $4\pi2-4\cos t$,

and

$\lim_{t\to\infty}\inf p_4(t) = \frac{1}{16e^4}\left(\pi^4 - 4\pi^2 -\right.$ $2\pi3-6\pi2+4\pi2-4 2>1e4$

By using corollary (1), every solution of (12), oscillates.

Theorem: 6

Consider (2) and (3), and assume that either

$\limsup_{k\to\infty} k\sum_{i=k}^{\infty}\left(q_i - \frac{1}{e^n}\right) > \frac{2}{e^n}$ (13)

or

$\liminf_{k\to\infty} k\sum_{i=k}^{\infty}\left(q_i - \frac{1}{e^n}\right) > \frac{1}{2e^n}$ (14)

Then all solution of (1) oscillates.

Proof:

By following the method of proof of Theorem (5), the sequences N_i and $N_{i+1} \ge \exp\left(\frac{N_i}{e}\right)$. Additionally, (7) and the induction methods were implemented to obtain
$N_0 > r_0 = 1$, $N_i > r_i$, for i = 1,2,…..
Then by using lemma (4) (d),

e$-N_i <$ e$-r_i < \frac{2e}{i+2}$ (15)

By using simple calculation, (k+i) $\frac{2e}{i+2} > e^{n-1} N_i^2$ (k + i) $\sum_{j=k+1}^{\infty}((q_i - e^{-n}))$.
In the above inequality take the limit as k → ∞ , then

2e ≥ $e^{n+1}\limsup_{k\to\infty} k\sum_{j=k}^{\infty}(q_j - e^{-n})$,

A violation to (13). Define A as
A = $\liminf_{k\to\infty} k\sum_{j=k}^{\infty}(q_j - e^{-n})$.
By (13), every solution of equation (1) oscillates if A =∞. In case $A \in (0,\infty)$, for any $\varepsilon > 0$ there exists $\bar{k}_\varepsilon$ such that $\bar{A} = A - \varepsilon > 0$ and

$\sum_{j=k}^{\infty}(q_j - e^{-n}) > \frac{\bar{A}}{k}$ for k > $\bar{k}_\varepsilon$.

Using the inequality

exp $\left(\frac{x}{e}\right) > x + \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(\frac{1-x}{e}\right)^2$

for $\xi < x <$ e.

in (10), we obtain for $N_i>\xi$ and k + i >$\bar{k}_\varepsilon$

$$N_{i+1} \geq \exp\left(\frac{N_i}{e}\right)\exp\left(e^{n-1} N_i(q_{k+i} - e^{-n})\right)$$
$$> \left[N_i + \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(\frac{1-N_i}{e}\right)^2\right]$$
$$[1 + e^{n-1} N_i(q_{k+i} - e^{-n})]$$

Consequently, we obtain

$$N_{i+1} - N_i > \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(\frac{1-N_i}{e}\right)^2 + e^{n-1}\xi^2(q_{k+i} - e^{-n}).$$

Summing the above inequality from i to ∞, we obtain

$$e - N_i > \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(\frac{1-N_j}{e}\right)^2 + e^{n-1}\xi^2\sum_{j=k+i}^{\infty}(q_j - e-n.$$

$$\mathrm{e} - N_i > \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\sum_{j=k}^{\infty}\left(\frac{1-N_j}{e}\right)^2 + \frac{e^{n-1}\xi^2\bar{A}}{k+i}. \quad (16)$$

Hence we obtain

$\mathrm{e} - N_i > \frac{U_0}{k+i}$ and $U_0 = e^{n-1}\xi^2\bar{A}.$

For improvisation of the above inequality iteration was done.

$$\mathrm{e} - N_i > \frac{U_m}{k+i} \quad m=0,1,\ldots \quad (17)$$

where

$U_{m+1} = \frac{U^2 m}{2e^2}\exp\left(\frac{\xi}{e}\right) + e^{n-1}\xi^2\bar{A}$

m = 0,1,…

For any n, if (17) holds Then by (16)

$$\mathrm{e} - N_i > \frac{1}{k+i}\left[\frac{U^2 n}{2e^2}\exp\left(\frac{\xi}{e}\right) + e^{n-1}\xi^2\bar{A}\right] = \frac{U_{n+1}}{k+i}$$

Since $U_1>U_0$, for any m

$U_{m+1} - U_m = \frac{1}{2e^2}\exp\left(\frac{\xi}{e}\right)(U_m + U_{m-1})$

$(U_m - U_{m-1}) > 0,$

It is true that $U_m \leq 2\mathrm{e}$. Then

$\lim_{m\to\infty} U_m = U$ exits, and

$$\mathrm{U} = \frac{U^2}{2e^2}\exp\left(\frac{\xi}{e}\right) + e^{n-1}\xi^2\bar{A}$$

Now, $1 - 2\xi^2\bar{A}\exp\left(\frac{\xi}{e} + n - 3\right) \geq 0.$

Let $\varepsilon \to 0$ and $\xi \to e$. Then$A \leq \frac{1}{2e^n}$,

a contradiction.

THE BEST RESULTS ON THE OSCILLATION OF (1) IS OBTAINED WHEN $\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e}$ AND $\lim_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$ WHICH IS CLEAR FROM THE PROCEEDING LEMMAS.

Definition: 1

The piecewise continuous function (t) : $[t_0, \infty)$ →$[0, \infty)$ belongs to $\mathcal{A}_\lambda$ if

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e}, \quad t \geq t_1.$$

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} + \lambda_k\left(\int_{t_k}^{t_{k+1}} p(s)ds - \frac{1}{e}\right).$$

for $t_k < t \leq t_{k+1}$, k = 1,2,…For some $\lambda_k \geq 0$ and if $\liminf_{k\to\infty} \lambda_k = \lambda > 0.$

Lemma: 7

In equation (1), let x(t) be positive solution on $[t_{k-2}, t_{k+1}]$ for some k ≥ 2.

Let N be defined by

$$\mathrm{N} = \min_{t_k\leq t\leq t_{k+1}} \frac{x(\tau(t))}{x(t)}.$$

Then N <$(2e)^2$.

Proof:

Let us put n =1 and $\bar{t} = t_{k+1}$ in lemma (1)

Then $\tau^{n+2}(\bar{t}) = \tau^{n+2}(t_{k+1})$ and

$\tau^3(\bar{t}) = \tau^3(t_{k+1})$
$= \tau\tau^2(t_{k+1})$
$= \tau^2(t_k)$
$= \tau(\tau(t_k))$
$= \tau(t_{k-1})$

Therefore,

$$\tau^3(t_{k+1}) = t_{k-2}$$

Therefore,

$$x(t) > 0 \text{ on } [\tau^3(\bar{t}), \bar{t}]$$

shows that

$x(t) > 0$ on $[t_{k-2}, t_{k+1}]$.

If k = 2, we get

$$\tau^3(t_3) = t_{3-3} = t_0.$$

Therefore

$$\mathrm{N} = \min_{t_k\leq t\leq t_{k+1}} \frac{x(\tau(t))}{x(t)}.$$

And by lemma (1),

N < $4e^2 = (2e)^2$.

Lemma: 8

For some k ≥ 3, let us assume that $x(t)$ is a positive solution of (1) on $[t_{k-3}, t_{k+1}]$.

And $p(t) \in A_\lambda$. Let M, N be defined by

$$\mathrm{M} = \min_{t_{k-1}\leq t\leq t_k} \frac{x(\tau(t))}{x(t)},$$

$$\mathrm{N} = \min_{t_k\leq t\leq t_{k+1}} \frac{x(\tau(t))}{x(t)}.$$

Then take M > 1 and N ≥ $\exp\left(M\left[\frac{1}{e} + \lambda_k\left(\int_{t_k}^{t_{k+1}} p(s)ds - 1e \geq M.\right.\right.\right.$

Proof:

From Lemma (3), M > 1

For, since $\mathrm{M} = \min_{t_{k-1}\leq t\leq t_k} \frac{x(\tau(t))}{x(t)}$

Put m = k, n = 1 in lemma (3)

Therefore we get,

$$\mathrm{M} = \min_{t_{(m-1)n}\leq t\leq t_{mn}} \frac{x(\tau(t))}{x(t)}$$
$$= \min_{t_{(k-1)}\leq t\leq t_k} \frac{x(\tau(t))}{x(t)}$$

And

$$\mathrm{N} = \min_{t_{mn}\leq t\leq t_{(m+1)n}} \frac{x(\tau(t))}{x(t)}$$
$$= \min_{t_k\leq t\leq t_{k+1}} \frac{x(\tau(t))}{x(t)}.$$

Therefore by lemma (3),$M > 1$.

Now,

$N \geq \exp(e^{n-1}Mq_m)$

$= \exp(Mq_k)$, where $n = 1, m = k$

$$= \exp\left\{M \min_{t_{kn}\leq t\leq t_{(k+1)n}} \{p_n(t)\}\right\},$$
$$k = 1,2,\cdots$$

If $k \geq 3$

$x(t) > 0$ on $\left[t_{(m-2)n-1}, t_{(m+1)n}\right]$

gives

$x(t) > 0$ on $[t_{k-3}, t_{k+1}]$

when $n = 1, m = k$.

Therefore

$$N \geq \exp(e^{n-1}Mq_m)$$

gives

$$N \geq \exp\left(M\left[\frac{1}{e}+\lambda_k\left(\int_{t_k}^{t_{k+1}} p(s)ds-\frac{1}{e}\right)\right]\right). \quad (18)$$

Since we can assume $p(t) \in A_1$ and $\lim_{k\to\infty}\inf \lambda_k = \lambda = 1$, we may have $\lambda_k = 1$ and (18) holds.

Hence $N \geq M$, by lemma(3).

IV. MAIN RESULTS

Theorem: 9

Assume that the function $p(t)$ in (1) belongs to A_λ for some $\lambda \in (0,1]$ where

$$\sum_{i=1}^{\infty}\left(\int_{t_{i-1}}^{t_i} p(s)ds-\frac{1}{e}\right)=\infty \qquad (19)$$

Then equation (1) oscillates.

Proof:

On the contrary, let us assume that there exists a solution $x(t)$ such that $x(t) > 0$ for $t \geq t_{k-3}$ for $k \geq 3$.

Consider the sequence $\{N_i\}_{i=0}^{\infty}$ and be defined by

$$N_i = \min_{t_{k+i-1}\leq t\leq t_{k+i}} \frac{x(\tau(t))}{x(t)}$$

By using lemma (8) we have $N_0 > 1$ and

$$N_{i+1} \geq \exp\left(\frac{N_i}{e}\right)\exp\left(N_i\lambda_{k+i}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds-\frac{1}{e}\right)\right) \geq N_i \qquad (20)$$

Therefore the sequences $\{N_i\}_{i=0}^{\infty}$ is nondecreasing and by lemma (7), it is bounded and also the sequence converges.

Let $\lim_{i\to\infty} N_i = N$. Then (20) implies $N \geq \exp\left(\frac{N}{e}\right)$

Hence by (8) we have $N = e$ and

$$1 < N_0 < N_1 < \cdots < e \qquad (21)$$

From (20), in view of (8)

$$N_{i+1} \geq N_i\left(1+N_i\lambda_{k+i}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds-\frac{1}{e}\right)\right)$$

Thus

$$N_{i+1}-N_i > N_i^2\lambda_{k+i}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds-\frac{1}{e}\right) \qquad (22)$$

By the definition of $\mathcal{A}_\lambda$, $\lambda = \lim_{k\to\infty}\inf \lambda_k > 0$, for any sufficiently small $\varepsilon > 0$ there is a value k_ε such that $\lambda_{k+i} > \lambda-\varepsilon$ for $k+i > k_\varepsilon$.

Hence for such i's from (21) and (22), we have

$$N_{i+1}-N_i > N_i^2(\lambda-\varepsilon)\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds-\frac{1}{e}\right)$$

$$N_{i+2}-N_{i+1} > N_{i+1}^2(\lambda-\varepsilon)\left(\int_{t_{k+i+1}}^{t_{k+i+2}} p(s)ds-\frac{1}{e}\right)$$

$$> N_i^2(\lambda-\varepsilon)\left(\int_{t_{k+i+1}}^{t_{k+i+2}} p(s)ds-\frac{1}{e}\right)$$

$$\vdots$$

Collating all the above inequalities,

$$e-N_i > N_i^2(\lambda-\varepsilon)\sum_{j=1}^{\infty}\left(\int_{t_{k+j}}^{t_{k+j+1}} p(s)ds-\frac{1}{e}\right) \text{ for } k+i > k_\varepsilon$$

which leads to a contradiction to (19). Hence the proof is complete.

Theorem: 10

Assume that $p(t) \in \mathcal{A}_\lambda$, for some $0 < \lambda \leq 1$ and either

$$\lambda \lim_{k\to\infty}\sup k\sum_{i=k}^{\infty}\left(\int_{t_{i-1}}^{t_i} p(s)ds-\frac{1}{e}\right) > \frac{2}{e} \qquad (23)$$

or

$$\lambda \lim_{k\to\infty}\inf k\sum_{i=k}^{\infty}\left(\int_{t_{i-1}}^{t_i} p(s)ds-\frac{1}{e}\right) > \frac{1}{2e}. \qquad (24)$$

Hence every solution of (1) oscillates.

Proof:

Put $n = 1$ in Theorem (6). We get

$$\lim_{k\to\infty}\sup k\sum_{i=k}^{\infty}(q_j-e^{-1}) > \frac{1}{e},$$

Using the definition of q_k,

$$\lim_{k\to\infty}\sup k\sum_{i=k}^{\infty}\left(\int_{t_{i-1}}^{t_i} p(s)ds-\frac{1}{e}\right) > \frac{2}{e} > \frac{2}{e\lambda}$$

where $0 < \lambda \leq 1$

Therefore $\lambda \lim_{k\to\infty}\sup k\sum_{i=k}^{\infty}\left(\int_{t_{i-1}}^{t_i} p(s)ds-\frac{1}{e}\right) > \frac{2}{e}$, which is equivalent to (23) in Theorem (10).

In a similar way we get

$$\lambda \lim_{k\to\infty}\inf k\sum_{i=k}^{\infty}\left(\int_{t_{i-1}}^{t_i} p(s)ds-\frac{1}{e}\right) > \frac{1}{2e},$$

which is equivalent to (24) in Theorem (10). Therefore every solution of (1) oscillates.

A Criterion For Nonoscillation Is Given In The Following Theorem.

Theorem: 11

Let $\tau(t) = t-1$, $p(t) = \frac{1}{e}+a(t)$ and $t_0 = 1$ in (1) then

$$x'(t)+\left[\frac{1}{e}+a(t)\right]x(t-1) = 0, t \geq 1. \quad (25)$$

Assume that

$$a(t) \leq \frac{1}{8et^2}$$

Then (25) has a solution $x(t) \geq \sqrt{t}e^{-t}$.

Proof:

Let the functions $A(t), B(t), C(t)$ be defined as

$$A(t) = \frac{1}{e}+a(t)$$

$$B(t) = \frac{1}{e}+\frac{1}{8et^2}$$

$$C(t) = \frac{1}{e}\frac{1-\frac{1}{2t}}{\sqrt{1-\frac{1}{t}}}, \qquad t > 1$$

By the assumption we have $A(t) \leq B(t)$.

Now, we show that the inequality $B(t) < C(t)$ also holds.

Namely, for $\theta = \frac{1}{2t} \in (0, 1/2)$, we have

$$C(t)-B(t)=\frac{\theta^3\left(\frac{1}{2}\theta^2-\frac{1}{4}\theta+2\right)}{e\sqrt{1-2\theta}\left[1-\theta+\left(1+\frac{1}{2}\theta^2\right)\sqrt{1-2\theta}\right]} > 0$$

Let us compare the differential equations,

$$x'(t)+A(t)x(t-1)=0$$
$$z'(t)+B(t)z(t-1)=0$$
$$u'(t)+C(t)u(t-1)=0$$

Let us observe that function $u(t)=\sqrt{t}e^{-t}$ is a solution of the last differential equation. Let the initial functions $\varphi(t)$ is the function $\sqrt{t}e^{-t}$ on [0, 1] and consider $x(t)$ and $z(t)$ be solutions of the first and second order differential equations, associated with this initial function $\varphi(t)$. Then by known comparison theorems in Elbert [5], we take $x(t)\geq z(t) > u(t)=\sqrt{t}e^{-t}$ for $t>1$

The proof is complete.

V. FINDINGS:

By taking $p(t)$ as a nonnegative piecewise continuous function and $\tau(t)$ as a non decreasing real sequence in the oscillatory behavior of solutions of first order linear delay differential equation (1) have been evaluated. Also the criterion for nonoscillation solutions are discussed.

VI. CONCLUSION:

In a delay differential equation the time derivatives at the current time which depend on the solution and possibly its derivatives at pervious times. In general the first order differential equations posses oscillatory solutions by using variable coefficients. But in this article the authors have found the oscillatory solutions to delay differential equation in the first order by taking $p(t)$as a piecewise continuous function with delay as a nondecreasing real sequence. These findings can be extended further to the study of neutral delay differential equations.

BIBLIOGRAPHY:

1. O. Arino et al, Delay Differential Equations and Applications, Springer (2006), 477-517.
2. Chuan Jun Tian and B.G Zhang, New Oscillation Criteria for a Class of Linear Delay Differential Equations, **Bull London Math. Soc. 31**(1999), 196-206.
3. S.G. Deo, V. Lakshmikantham and V. Ragavendra. Text Book of Ordinary Differential Equations, **Tata Mc Graw Hill Pub. Co. New Delhi,** 1997.
4. A. Elbert and I.P. Stavroulakis, Oscillation and Non Oscillation Criteria for Delay Differential Equations, **Proc. Ameri. Math. Soc. 123**(1995), 1503-1510.
5. A. Elbert, Comparison Theorems for First Order Nonlinear Differential with Delay, **Studia Sci. Math. Hungan. 11** (1976), 259-267.
6. G.S. Ladde, V.Lakshmikantham, B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Dekker, New York, 1987.
7. B. Li, Oscillation of First Order Delay Differential Equations, **Proc. Amer. Math. Soc. 124**(1996), 3729-3737.
8. H. Smith, An introduction to delay Differential Equations with Applications to the Life Sciences, Springer 2011.
9. X.H Tang and J.S. Yu, Oscillation of First Order Delay Differential Equation**J. Math. Anal. Appl. 248**(2000), 247-259.
10. Xianhua Tang Jianhua Shan, Oscillations of Delay Differential Equations with Variable Coefficients, **J. Math. Anal. Appl. 217**(1998), 43-71.

AUTHORS PROFILE

P. Sowmiya, Assistant Professor, Shri Sakthikailassh Women's college Salem-636 003. Tamilnadu, India . psowmiyababu@gmail.com

G.K.Revathi, Assistant Professor , Division of Mathematics, School of advanced sciences, Vellore Institute of Technology,Chennai 127 Tamilnadu, India . gk_revathi@yahoo.co.in

M. Sakthipriya, Research Scholar. Shri Sakthikailassh Women's college Salem-636 003. Tamilnadu, India.

sakthipriyavel159@gmail.com

V. Ramya, Assistant Professor, Shri Sakthi kailassh Women's college, Salem-636 003. Tamilnadu, India.

ramya24797@gmail.com