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Mobius Function Graph $M_n(G)$

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Abstract - The study of graphs on positive integer n as its vertex set from 1 to n, the adjacency of vertices defined using tools of number theoretic functions is interesting and may focus new light on structure of the integers. This paper is concerned on some of the structural properties of the Möbius function graphs from the number theoretic Möbius function. Further we have discussed some basic observations, results concerning |E|, subgraph, perfect matching, completeness, independence number and chromatic number of Möbius function graphs along with new induced proper coloring method.

I. INTRODUCTION

The theory of numbers is a field of mathematics which concern with the properties of whole and rational numbers. Analytic number theory is one of its branches, which includes study of arithmetical functions, their characteristics and the relations that exist among these kind of functions. The function was introduced by Möbius in 1832, and the notation μ(n) was first used by Mertens in 1874. The perception of Number theory were introduced in Graph theory to layout a graph is imported in 1980 by Nathanson **[1]**. He used the theory of congruences for characterizing the adjacency between two nodes of a graph. Later in 1994, Vasumathi [2] carried a new class of graph namely Möbius graph by assigning the Möbius function on the adjacency of 2 nodes in a graph and she examined the independence number relations of Möbius graphs in [3]. Srimitra, Sajana and Bharathi [4] analyzed this graph and examined the adjacency, valency of the nodes of this graphs and they noted the values of some parameters like radius, diameter and girth of the graph in [5]. In this paper, there are few theorems which concerns the basic observations from the graphs, number of edges of $M_n(G)$, completeness, matching and on induced subgraphs of $M_n(G)$.

II PRELIMINARIES

Definition 2.1:

The Möbius function is defined by μ(1) = 1 and if n > 1, then write $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$

Now, $\mu(n) = \begin{cases} -1^r, & \text{if } a_1 = a_2 = \cdots = a_r = 1 \\ 0, & \text{otherwise} \end{cases}$

Definition 2.2: Alternate Definition of Möbius function

- μ(n) = 1, if n is a square free positive integer with an even number of prime factors.
- μ(n) = -1, if n is a square free positive integer with an odd number of prime factors.
- μ(n) = 0, if n has a squared prime factor.

Definition 2.1

[5] A Shell graph is defined as a cycle C_n with n − 3 chords sharing a common end point called the apex. Shell graph is denoted by C(n, n − 3). A shell S_n is also called fan f_{n-1}.

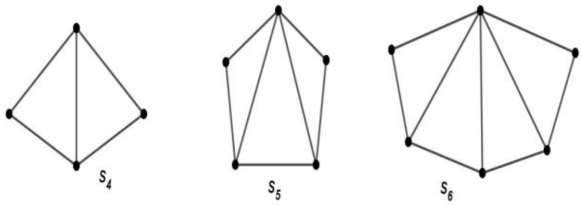

Figure 1: Examples for Shell Graph

III MÖBIUS FUNCTION GRAPH Mn(G)

Definition 3.1

Let n > 1 be any integer. Consider all the positive integers from 1, 2, . . . , n. Now we define a simple graph G = (V, E) with vertex set V = {1, 2, . . . , n} placed in clockwise or anticlockwise order successively along with their Möbius function values and edge set E = {uv | μ(uv) = μ(u)μ(v), where u, v ∈ V } is called a Möbius Function Graph.

The following graphs are some of the examples of Möbius Function Graph.

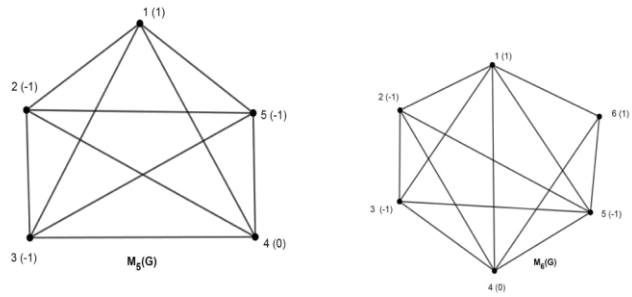

Figure 2: Möbius Function Graphs for n = 5 and 6.

IV OBSERVATIONS FROM Mn(G)

1. $M_n(G)$ is complete for n = 2, 3, 4, 5.

2. Möbius value of a square free semi-prime and sphenic number is 1 and −1 respectively.

3. There exists an Hamiltonian circuit in every $M_n(G)$, n ≥

4. $M_n(G)^c$ is always disconnected.

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5. $M_n(G)$ is planar for n = 2, 3, 4.

6. Removal of vertices 1 and n from $M_n(G)$ gives the complete graphs for n = 3, 4, 5.

7. Removal of all non-adjacent pairs of vertices from the graph leads to complete graph.

8. d(1) = n − 1 and the degree of any arbitrary vertex u in $M_n(G)$ is at most n − 1.

9. Every induced subgraph of $M_n(G)$ preserves the construction of $M_n(G)$.

V. RESULTS ON Mn(G)

Theorem 5.1

$M_n(G)$ is always connected and $|E(M_n(G))| = \binom{n}{2} - |\mu'|$ where μ' is the set of all non co-prime pairs with Möbius function value both 1 and −1 (or) either −1 or 1.

Proof:

Let n > 1 be an integer. Consider all positive integers from 1 to n as vertices. Now label the vertices from 1, 2, . . . , n along with their Möbius function value. Since the pair 1 and n is always relatively prime to each other, from the definition of $M_n(G)$, we must have $\mu(1 * a) = \mu(1)\mu(a)$, for all $a \in V(M_n(G))$. Thus, there exist always an edge between 1 and other (n −1) vertices of $M_n(G)$. Hence any vertex of $M_n(G)$ can be reachable from any other vertex. Therefore the Möbius Function Graph $M_n(G)$ is always connected. We know that, for any graph G with n vertices, it must have at most $\frac{(n-k)(n-k+1)}{2}$ edges, where k=number of components of G. Since the graph is already connected then we have k = 1. So, $\frac{(n-k)(n-k+1)}{2} \rightarrow \frac{n(n-1)}{2}$.

Suppose there exists a non co-prime pair (a, b) with $\mu(a) = \mu(b) = 1/-1$ or $\mu(a) = 1$ and $\mu(b) = -1$, we must have $\mu(a * b) \neq \mu(a) * \mu(b)$, contradicts the definition of $M_n(G)$. So collect all such pairs (a, b) and denote the set by μ'. There cannot be an edge between such pairs. Further, we can subtract $|\mu'|$ from $\frac{n(n-1)}{2}$ to obtain the total number of edges of $M_n(G)$.

Hence $|E(M_n(G))| = \binom{n}{2} - |\mu'|$.

Result 5.1

Shell graph is a subgraph of $M_n(G)$, $\forall n \geq 4$.

Proof:

We know that, the vertex 1 is always adjacent with all the remaining (n−1) vertices of $M_n(G)$, and also there is an edge between successive vertices (Since $\mu(ab) = \mu(a)\mu(b)$, if a and b are co-prime) in $M_n(G)$ together forms a C_n. Now, there are n edges on C_n and the number of chords from the vertex 1(common vertex adjacent to all) is always n − 3.

∴ The cycle with n vertices and (n − 3) chords from the common vertex is a shell graph.

Theorem 5.2

Möbius Function Graph $M_n(G)$ has a perfect matching if and only if n is even.

Proof:

Let $M_n(G)$ be a Möbius Function Graph with n vertices, n > 1. Suppose $M_n(G)$ has a perfect matching S, then S saturates all the n vertices of $M_n(G)$. It is clear that n is even because each edge in S contributes two vertices and all the vertices must appear exactly once in S since it admits perfect matching.

Suppose $M_n(G)$ has even number of vertices. We have to show that $M_n(G)$ admits perfect matching .

Since 1 is adjacent with all other (n − 1) vertices of $M_n(G)$, initially there exist (n − 1) odd number of edges which are incident with the vertex 1. Also there exits edges consecutively between the remaining (n − 1) vertices starting from the vertex 2. (i.e.) we have $e_{23}, e_{34}, \ldots, e_{(n-1)n}$ in total n − 2 edges appears in the outer cycle. Along with e_{12} and e_{n1}, we have n edges in the outer cycle.

Now choose $\frac{n}{2}$ edges from the n outer edges alternating. we may have $e_{12}, e_{34}, \ldots, e_{(n-1)n}$ (or) $e_{23}, e_{45}, \ldots, e_{n1}$ which are independent edges as well as saturates all the vertices of $M_n(G)$. Hence $M_n(G)$ has a perfect matching always if n is even.

Result 5.2

An induced subgraph of $M_n(G)$, n > 2 formed by prime numbers less than or equal to n is always complete.

Proof:

Take any arbitrary n > 2, by the definition of Möbius function, the Möbius value for any prime number is −1.

Now by the definition of $M_n(G)$, we must have an edge if and only if $\mu(a * b) = \mu(a)\mu(b)$, $\forall a, b \in V(M_n(G))$, already the chosen numbers are prime, so that $\mu(p) = -1$.

case (i):

Product of any two odd prime numbers = odd.

By the definition of Möbius Function Graph,

$\mu(a)\mu(b) = (-1) * (-1)$, if a and b are primes.

Now, $\mu(a*b) = 1$ since the product $a * b$ has even number of prime factors, we have $\mu(a)\mu(b) = \mu(ab)$, $\forall a, b \in V$. So there exists an edge between all such pairs. Hence the resultant graph is complete.

case (ii):

2*(any other prime number) = even

If a = 2 and b may be any other prime both have the Möbius function value −1, so $\mu(a)\mu(b) = (-1) * (-1) = 1$ and also $\mu(a * b) = 1$ because it is same as the earlier case that the product has even number of prime factors.

Result 5.3

An induced subgraph of $M_n(G)$, n > 2 formed by odd numbers less than equal to n will be complete upto n = 14.

Proof:

Since the odd integers from 3 to 13 have the Möbius function value −1 except 9(non-prime). We know that by the definition of Möbius function graph, 1 is connected with all the remaining vertices in $M_n(G)$. So except 9 remaining all are prime from 3 to 13. Hence we must have edges between

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all such pairs(by previous result).

Now, $\mu(9 * p) = \mu(9) * \mu(p) = 0 * (-1) = 0$.

Also, $\mu(9 * p) = 0, \forall\, p \in [3, 13]$.

Since 9 * p has a squared prime factor. Therefore we have edges between 9 and all the remaining primes from 3 to 13, and hence the graph is complete.

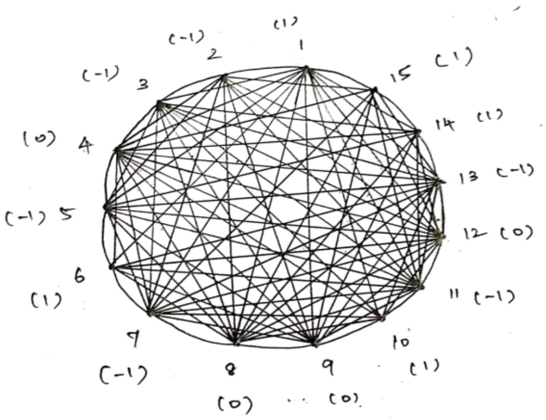
Example 5.3

(Counter Example for the above result)

For n = 15, the odd integers ≤ 15 are 1, 3, 5, 7, 9, 11, 13, 15.

$M_n(G)$ formed by the above set of integers is not complete because $\mu(3 * 15) = \mu(45) = 0 6= \mu(3) * \mu(15)$.

$\mu' = \{(2, 6), (2, 10), (2, 14), (3, 6), (3, 15), (5, 10), (5, 15), (6, 10), (6, 14), (6, 15), (7, 14), (10, 14), (10, 15)\}$.



VI RESULTS ON INDEPENDENCE NUMBER OF Mn(G)

Proposition 6.1

$\alpha(M_n(G)) = 2$, for n = 6 to 9.

Proof: In $M_n(G)$ for n = 6 to 9, the vertices 2, 6 and 3, 6 are non-adjacent, but 2 and 3 are adjacent to each other. So we have 2, 6 or 3, 6 in the independent set.

∴ $\alpha(M_n(G)) = 2$.

Proposition 6.2

$\alpha(M_n(G)) = 3$, for n = 10 to 14.

Proof:

In $M_{10}(G)$, we have 5 non-adjacent pairs {(2, 6), (2, 10), (3, 6), (5, 10), (6, 10)}. So we have {2, 6, 10} as the independent set and hence $\alpha(M_{10}(G)) = 3$[Since we can't add 3, 5 to the independent set].

For n = 11, W.K.T. $\mu(11) = -1$, we can't have 11 in the independent set, because 11 is adjacent to 2, 6, 10. For n = 12, W.K.T. $\mu(12) = 0$, it is clear that 12 is adjacent to the vertices 2, 6, 10. Similarly, 13($\mu(13) = -1$) and 14 ($\mu(14) = 1$) are adjacent to 2, 6, 10. Hence for n = 10 to 14, we have {2, 6, 10} as the independent set.

∴ $\alpha(M_n(G)) = 3$, for n = 10 to 14.

Theorem 6.3

$\alpha(M_n(G))$ = Number of even numbers upto n with Möbius function value 1 and −1.

Proof:

For n = 6 to 9, we have $\alpha(M_n(G)) = 2$. The corresponding independent set S is {2, 6} or {3, 6}. If we consider n = 10 ($\mu(10) = 1$), then there is no edge between (2, 10) and (6, 10). Since $\mu(2 * 10) = 0$, but $\mu(2) * \mu(10) = -1$ $\mu(6 * 10) = 0$, but $\mu(6) * \mu(10) = -1$.

Suppose if we consider 3 in our set S, we can't have 10, since $\mu(3 * 10) = -1 = \mu(3)*\mu(10)$. So excluding this 3, we may fix our set S as {2, 6, 10}. Further the next vertex is 11.

Case (i.)

We need not to add any prime further to the independent set S, because $\mu(p) = -1$. So $\mu(a * p) = \mu(a) * \mu(p)$, for all $a \in S$.

Case (ii.)

We need not to add any other composite number to the set S with Möbius value zero, because any vertex with Möbius value zero is adjacent to all the vertices of $M_n(G)$.

Case (iii.)

We need not to add any odd composite number with Möbius value 1 to the set S because if some vertex a with even number of prime factors exists then it must be adjacent to any one of the vertices of the set S. Since the first odd composite number greater than 6 with Möbius value 1 is 15, W.K.T. $\mu(15) = 1$. So we have $\mu(2 * 15) = -1 = \mu(2) * \mu(15)$. Any odd composite with Möbius value 1 must be adjacent to 2. Hence we can only add an even numbers with Möbius value either 1 or −1 to the set S because these would not have edges between the vertices that already we have S (by theorem (5.1). Since all are even and hence $\mu(a * b)$ will have squared prime factor, so $\mu(a * b) = 0$ but $\mu(a) * \mu(b) = -1$ or 1, $\forall$ a, b ∈ S.

By proceeding the above theorem 6.3, we obtain the following tabulated numerical results:

Table 1: Independence Number of $M_n(G)$

Number Range	$\alpha(M_n(G))$
6 − 9	2
10 − 13	3
14 − 21	4
22 − 25	5
26 − 29	6
30 − 33	7
34 − 37	8
38 − 41	9
42 − 45	10

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VII. RESULTS ON COLORABILITY OF Mn(G)

Definition 7.1

An assignment of colors to the nodes such that no two nearer nodes share same colors. Let f : V → {C_1, C_2, (C_1, C_2), C_3, (C_1, C_3), ...} such that if (a, b) ∈ E then f(a) ≠ f(b). Here C_1 = White. This type of coloring is called as ***induced proper coloring.***

1. $\chi_i(C_n) = 2$.
2. $\chi_i(T) = 2$.
3. $\chi_i(G) = 2 \Longleftrightarrow$ G is a bi-partite graph.

Proposition 7.2

[6] For any graph G with n vertices, $\frac{n}{\alpha} \leq \chi(G) \leq n-\alpha+1$.

Result 7.1

$\chi_i(M_n(G)) = 3$ for n = 6.

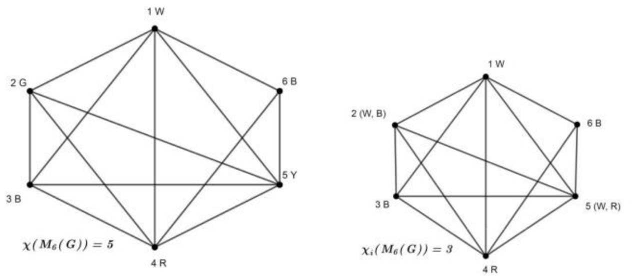

Figure 3: Chromatic number and induced chromatic number of $M_6(G)$

Proposition 7.3

$\chi(M_n(G)) = n - \alpha + 1$.

Proof:

To color any $M_n(G)$, we require at most n colors, so we can subtract the independence number from n. Hence, we require $(n - \alpha)$ colors and also we require one more color for the non-adjacent vertices.

Hence $\chi(M_n(G)) = n - \alpha + 1$.

Theorem 7.4

Let $M_n(G)$ be a Möbius Function Graph. Then

$\chi_i(M_n(G)) = \frac{n-\alpha}{2} + 1$; if $n - \alpha$ is even and

$\chi_i(M_n(G)) = \left\lceil \frac{n-\alpha}{2} \right\rceil + 1$, otherwise.

Proof:

Case (i.)

If $n - \alpha$ is even.

It is clear that α number of vertices need only one color, since they are non-adjacent and hence the remaining $n - \alpha$ vertices are all adjacent to each other. So let we start from the first vertex 1, we can assign C_1(White) and assign C_2 to the vertices in the independent set S.

Since $n - \alpha$ = even, we require exactly half of the $n - \alpha$ colors needed to color our graph properly.

Hence, $\chi_i(M_n(G)) = \frac{n-\alpha}{2} + 1$; if $n - \alpha$ is even.

Case (ii.)

If $n - \alpha$ is odd, then we require $\left\lceil \frac{n-\alpha}{2} \right\rceil + 1$ colors are needed to color those $n - \alpha$ nodes properly.

Thus, $\chi_i(M_n(G)) = \left\lceil \frac{n-\alpha}{2} \right\rceil + 1$, otherwise.

Table 2: Induced Proper Chromatic Number of $M_n(G)$

Number of Vertices	*$\chi_i(M_n(G))$*
3	***2***
4	***3***
5	***3***
6	***3***
7	***4***
8	***4***
9	***5***
10	***5***

VIII. CONCLUSION AND FUTURE WORK

While this paper provides a good introduction to the study of Möbius Function Graphs. Results concerning the cardinality of edge set, subgraphs, completeness, independence number of $M_n(G)$ and coloring are given. Further, we have introduced a new type of coloring namely induced proper coloring and we proved that its chromatic number is always less than the usual chromatic number of any graphs.

REFERENCE

1. Nathanson Melvyn. B, "Connected components of Arithmetic Graphs, Monatshefte fur Mathematik, 89, 219-222, 1980.
2. Vasumathi, N, Graphs on Numbers, Ph.D. Thesis, Sri Venkateswara University, Tirupati, India, 1994.
3. N. Vasumathi and S. Vangipuram, Mobius Graphs, Journal of Informatics and Mathematical Sciences Volume 1 (2009), Number 1, pp. 5560
4. Srimitra, K. K., Shaik Sajana, Bharathi, D., Adjacency and Degree of Graph of Mobius function, i Managers Journal on Mathematics, 5(1), 3138, 2016.
5. S. Meena M. Renugha M. Sivasakthi, CORDIAL LABELING FOR DIFFERENT TYPES OF SHELL GRAPH, International Journal of Scientific & Engineering Research, Volume 6, Issue 9, September-2015.
6. R. Balakrishnan and G. Ranganathan, A textbook of Graph Theory, Springer-Verlog, NewYork, 2000.

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