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Riccati Techniques, Discrete Oscillation and Conjugacy Criteria for Fourth Order Nonlinear Difference Equations

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Abstract*: *We discuss the discrete Oscillatory properties of Fourth order non linear difference equation.

$$\Delta\left(\frac{1}{r_3(n)}\left(\Delta\left(\frac{1}{r_2(n)}\left(\Delta\left(\frac{1}{r_1(n)}(\Delta x(n))^{\alpha_1}\right)\right)^{\alpha_2}\right)^{\alpha_3}\right)\right)+q(n)f(x[g(n)])=0,$$

where α >1. In particular we establish the discrete oscillation using Riccati Techniques and Conjugacy criteria. The Proofs of all the results in this paper are based on the Riccati technique.

Keywords* : *Conjugacy criteria, Discrete oscillation, Fourth order difference equation, Riccati techniques.

I. INTRODUCTION

Consider the fourth order non linear difference equation

$\mathbb{M}_4\mathcal{X}(n)+q(n)f(\mathcal{X}[g(n)])=0, n\in\mathbb{N}=\{0,1,2,\ldots\ldots\}$
(1)

where

$\mathbb{M}_0\mathcal{X}(n)=\mathcal{X}(n), \mathbb{M}_{\mathbb{K}}\mathcal{X}(n)=\frac{1}{r_{\mathbb{K}(n)}}\left(\Delta\mathbb{M}_{\mathbb{K}-1}\mathcal{X}(n)\right)^{\alpha\mathbb{K}}$,
$\mathbb{K}=1,2,3,\ldots\ldots$ (2)

$\mathbb{M}_4\mathcal{X}(n)=\Delta\mathbb{M}_3\mathcal{X}($and Δ is the forward difference operator. It is known, see [Tipler, 1978], that the most of the basic oscillatory properties of the linear differential equation

$(P(t)y')'+q(t)y=0$ (3)

can be extended (generalized and discretized) to the difference equation

$\Delta(p_{\mathbb{K}}\psi(\Delta y_{\mathbb{K}}))+q_{\mathbb{K}}\psi(y_{\mathbb{K}+1})=0$(4)

where *pk* and *qk*are real-valued sequences defined on with r_K= 0 and, from those we mention the possibility to use the Riccati Technique in the Oscillatory theory that is based on the equivalence between a non oscillatory and a solvability of the generalized Riccati difference equation. This technique is used for establishing the so called Hart man-wintner lemma for equation (3) with $r(t)\equiv 1$, see [Kwong et al.,]. Here we shall assume that

(i) $p_i(n), q(n):\mathcal{N}\to\mathcal{R}^+=(0,\infty), i=1,2,3$

and

$$\sum_{\mathbb{k}=n_0>0}^{\infty} r_i^{\frac{1}{\alpha_i}}(\mathbb{K})=\infty, i=1,2,3,$$

$\mathcal{G}(n):\mathcal{N}\to\mathcal{R}, \{\mathcal{G}(n)\}\lim_{n\to\infty}\mathcal{G}(n)=\infty$ (3)

(ii) $\mathcal{F}\in\mathcal{C}(\mathcal{R},\mathcal{R}), \mathcal{X}\mathcal{F}(\mathcal{X})>0,\ and\ \mathcal{F}'(\chi)\geq 0, for\ \chi\neq 0$

(iii) $\alpha_{i,}\ i=1,2,3,$are the ratios of positive odd integers.

The domain $D(L4)$ of $L4$ is defined to be the set of all function $x(n):[nx,\infty)\to$ R, $nx\geq n0\geq 0$ such that $Lj\,x(n)$, $0\leq j\leq 4$ exist on $[nx,\infty)$. A solution $\{x(n)\}$ of equation (1) is called oscillatory if for any $m\in$N there exist m, $m2\geq m$ such that $x(m1)x(m2)<0$, otherwise it is non oscillatory. equation (1) is called B-oscillatory if all its bounded solutions are oscillatory. Our approach to the discrete oscillation and conjucation of equation (1) will be based largely on a discrete version of the Riccati equation. The necessary and the sufficient condition for the oscillation of equation (1) is expressed in terms of the existence of a certain solution of the generalized Riccati difference equation which is in the summation form. Determining oscillation and non oscillation criteria for difference equation has received a great deal of attention in the last few years. See, for example the monographs [Chen and Erbe, 1989, Cheng, 1994, Dosly and Rehak, 1998b, Dosly and Rehak, 1998a, Erbe and Yan, 1992, Hooker et al., 1987, Kwong et al., , Selvaraj and Lovenia,] and the references cited there in. This interest is motivated by the importance of difference equations is the numerical solution of differential equation. Compared to first and second order difference equations, the study of higher order equations and in particular fourth order equations of type equation (1) with αi = 1, i = 1, 2, 3 and some $\alpha i(n)\equiv 1$, i = 1, 2, 3 has received considerably less attention (see [Hooker and Patula, 1981, Patula, 1979, Selvaraj and Daphy, 2011, Taylor, 1993, Tipler, 1978, Wintner, 1951, Mary, 1987]) and the referenced cited there in.

Our purpose of this paper is to obtain some sufficient conditions for the discrete oscillation of all bounded solutions of equation (1). And also we discuss some conjugacy criteria for the equation of type (1).

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II. MAIN RESULTS

Consider the following inequality

$\mathrm{M}_4 x(n) + q(n) f(x[g(n)]) = 0, n \in \mathbb{N} = \{0,1,2, \dots \dots\}$

(6)

If (6) is non oscillatory, then the solution *xn* is said to be principal or recessive in case

$$\sum_{n=N}^{\infty} \alpha_n x_n x_{n+1} = +\infty$$

for some $\mathbb{N} \geq$ and is said to be non-principal or dominant in case

$$\sum_{n=N}^{\infty} \alpha_n x_n x_{n+1} < +\infty \quad (7)$$

Definition 2.1 *(i) An interval* $(m, m+1]$, *is said to contain the generalized zero of a solution of* (6), *if* $x_m \neq 0$ *and* $a_m x_m x_{m+1} \leq 0$.

(i) Equation (6) is said to be disconjugate on the discrete interval [m, n] provided any solution of this equation has at most one generalized zero on $(\mathrm{m}, \mathrm{m}+1]$. And the solution x satisfying xm = 0 has no generalized zero on $(\mathrm{m}, \mathrm{m}+1]$. Otherwise (6) is said to be conjugate on [m, n].

(ii) Equation (6) in said to be non oscillatory if there exist m ∈N . such that this equation is disconjugate on [m, n] for every n > m.

(iii) The application of the following result is usually referred to as the Riccati technique.

Proposition 2.2 *Equation* (1) *is non oscillatory if and only if there exist a sequence W and m*∈N *such that*

$$\mathcal{R}[W_k] = \Delta W_k + q_k + s(W_k, r_k)$$

with $r_k + W_k > 0$ *for* $\mathbb{K} \geq m$, *where*

$$s(W_k, r_k) = W_k \left(1 - \frac{r_k}{\lambda(\lambda^{-1}(r_k) + \lambda^{-1}(W_k))}\right)$$

λ^{-1} *being the inverse of* λ,

i.e., $\lambda^{-1}(x) = |x|^{\alpha-1}$ *sgn x and β is the conjugate number of α, ie.,* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

At the end of this section we describe some properties of the function, assuming $W_k = x, r_k =$

$$s(x, y) = x\left(1 - \frac{y}{\lambda(\lambda^{-1}(x) + \lambda^{-1}(y))}\right)$$

Lemma 2.3 *Suppose that α >1. The function s(x, y) has the following properties:*

(i) $s(x, y)$ *is continuously differentiable on*

$\mathcal{D} = \{(x, y) \in \mathcal{R} * \mathcal{R}, x \neq -y\}$

(ii) Let y > 0. Then $S_x(x, y) \geq 0$ *for* $x + y > 0$, *where* $S_x(x, y) = 0$ *if and only if* $x = 0$.

(iii) Let $x + y > 0$. *Then* $S_y(x, y) \geq 0$, *where the equality holds if and only if x = 0.*

(iv) $S(x, y) \geq 0$ *for* $x + y > 0$, *where the equality holds if and only if x = 0.*

(v) Suppose that the Sequence (xk , yk), K = 1, 2, . . . , is such that xK + yK> 0and there exists a constant M > 0 such that yK ≤ M for K = 1, 2, 3, ... ThenS(xk, yk, ∞) → 0 implies xk → 0. Moreover, liminfyk≥ 0.

k→0

(vi) Let $\bar{s}(x, y) = x - s(x, y)$. *Then* $\bar{s}(x, y) = \bar{s}(y, x)$ *on D and* $\bar{s}_x(x, y) \geq 0$ *for* $x + y > 0$, *where the equality holds if and only if* $y = 0$. *If* $y \leq 1$, *then* $\bar{s}(x, y) < 1$ *for all* $x + y > 0$.

III. DISCRETE OSCILLATION AND NON OSCILLATION CRITERIA

We shall be interested in using the Riccati equation along with certain averaging techniques to obtain some new discrete oscillation and non oscillation criteria.

We denote by *S* the set of all real sequences $\mathbb{C} = \{\mathbb{C}_n\}_n^{\infty}$ such $\lim_{n\to\infty} \sum_{j=0}^{n} \mathbb{C}_j \equiv \sum_{j=0}^{\infty}$ exists (finite).

We shall also need the following conditions which will be imposed in the theorems to follow:

(U1) $\lim_{n\to\infty} \sup n^{-\frac{3}{2}} \sum_{j=0}^{n} r_j < +\infty$ *(U2)* $\limsup_{n\to\infty} \left(\frac{1}{n}\right) \sum_{j=0}^{n} r_j < +\infty$ *(U3)* There exists *M* >0 with $0 < r_n \leq M$ for all $n \geq 0$. Clearly, (*A*3) ⇒(*A*2) ⇒(*A*1).

Theorem 3.1 *Assume* that (*U*1) *holds and that* (1) *is non oscillatory. Then the following statements are equivalent:*

(i) $\lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{i=0}^{n} \sum_{i=0}^{l} q_i$ *exists (finite).*

(ii) The sequence $q = \{q_i\}_{i=0}^{\infty}$ *satisfies*

$$\liminf_{n\to\infty} \left(\frac{1}{n}\right) \sum_{i=0}^{n} \sum_{i=0}^{l} q_i > -\infty \quad (9)$$

(iii) For any non oscillatory solution x with $x_n x_{n+1} > 0$, $n \geq \mathbb{N}$, *for some* $\mathbb{N} \geq 0$, *the sequence* $u_n = \frac{r_n \Delta x_n}{x_n}, n \geq \mathbb{N}$, *satisfies*

$$\sum_{i=0}^{l} \frac{u^2{}_i}{u_i + r_i} < +\infty \quad (10)$$

Proof: Clearly (*i*) → (*ii*). To show that (*ii*) → (*iii*) suppose to the contrary that there is a non oscillatory solution *x*(*n*) of (1) such that $u_n = \frac{r_n \Delta x_n}{x_n} > -r$ for $n \geq$ and

$$\sum_{i=N}^{\infty} \frac{\mathcal{U}^2{}_i}{\mathcal{U}_i + r_i} = +\infty \quad (11)$$

$$\mathcal{U}_{n+1} + \sum_{i=N}^{n} \frac{\mathcal{U}_j{}^2}{\mathcal{U}_j + r_j} + \sum_{i=N}^{\infty} q_i = \mathcal{U}_n$$

and therefore

$$\frac{1}{n}\sum_{i=N}^{n} -\mathcal{U}_{i+1} = \frac{1}{n}\sum_{i=N}^{n}\sum_{i=N}^{K}\frac{\mathcal{U}_i^2}{\mathcal{U}_i+r_j} + \frac{1}{n}\sum_{i=N}^{n}\sum_{i=N}^{K} q_i - \left(\frac{n-N+1}{n}\right)\mathcal{U}_n, \text{ for n } \geq \mathbb{N} \quad (13)$$

From (9),(11),(13) we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=N}^{n}(-\mathcal{U}_{i+1}) = +\infty \quad (14)$$

and here

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=N}^{n}|\mathcal{U}_j| = +\infty \quad (15)$$

Let $\mathcal{V}_n = \frac{\mathcal{U}_n{}^2}{\mathcal{U}_n + r_n}$ for $n \geq \mathbb{N}$. Then $\mathcal{V}_n \geq 0$ and $\mathcal{V}_n = 0$ if and only if $\mathcal{U}_n = 0$. Let $b_n = \frac{\mathcal{U}_n{}^2}{\mathcal{V}_n}$ if $\mathcal{U}_n \neq 0$ and $b_n = 0$ if $\mathcal{U}_n = 0$. Then we have $r_n \geq b_n - \mathcal{V}_n$ and hence

$$n^{-3/2}\sum_{i=N}^{n} r_i \geq n^{-\frac{3}{2}}\sum_{i=N}^{n} a_i + n^{-\frac{3}{2}}\sum_{j=N}^{n} -\mathcal{U}_j \quad (16)$$

Now from (16), (U 1) and $an \geq 0$ it follows that

$$\limsup_{n\to\infty} n^{-\frac{3}{2}}\sum_{i=N}^{n} -\mathcal{U}_i < \infty \quad (17)$$

(ii) we obtain

$$\limsup_{n\to\infty} n^{-\frac{3}{2}}\sum_{i=N}^{n}\sum_{i=N}^{n} \mathcal{V}_i < +\infty \quad (18)$$

we have

$$\limsup_{n\to\infty} n^{-\frac{3}{2}}\sum_{i=N}^{n} \mathcal{V}_i < +\infty \quad (19)$$

on the other hand, from (19) there is an M >0 such that

$$n^{-3/2}\sum_{i=N}^{n} r_i \geq \frac{1}{M}\left(\frac{1}{n}\sum_{i=N}^{n}|\mathcal{U}_j|\right)^2 \quad (20)$$

So from (15)-(17) and (20) we have

$$\lim_{n\to\infty} n^{-\frac{3}{2}}\sum_{i=N}^{n} r_i = +\infty$$

which contradicts (U 1). There fore (ii) ⇒(iii), (iii) ⇒(i) Let u be the sequence in (iii) and let $B_n = \sum_{i=N}^{n}|\mathcal{U}_j|$ then we have $(\sum_{i=N}^{n}\mathcal{U}_i)^2 \leq B_n{}^2 \leq 2\mathbb{K}max\{B_n, \sum_{i=N}^{n} r_i\}$ where $\mathbb{K} = \sum_{j=N}^{\infty}\mathcal{R}_j$, $\mathcal{R}_n = \sum_{i=N}^{n} r_i$. Hence, we have

$$B_n \leq max\left\{2\mathbb{K}, \left((2\mathbb{K}\sum_{i=N}^{n} r_i)^{\frac{1}{2}}\right)\right\} \quad (21)$$

It follows from (U 1) and (21) that $\lim_{n\to\infty}\frac{1}{n}B_n = 0$ so that $\lim_{n\to\infty}\frac{1}{n}\sum_{i=N}^{n} -\mathcal{U}_{i+1} = 0$.

Corollary 3.2 *Let (U 1) hold. Then* (1) *is oscillatory in case either of the following holds*

$$-\infty < inf\,\frac{1}{n}\sum_{l=0}^{n}\sum_{i=N}^{l} q_i < \limsup_{n\to\infty}\frac{1}{n}\frac{1}{n}\sum_{l=0}^{n}\sum_{i=0}^{l} q_i \quad (22)$$

(or)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n}\sum_{i=0}^{l} q_i = +\infty \quad (23)$$

Remark 3.3 *The Criteria given in corollary* (3.2) *are discrete analogues of the oscillation criteria of Hartman [Dosly and Rehak, 1998a] and wintner [Taylor, 1993] for equation* (3) *with* $r(\epsilon) \equiv 1$

Theorem 3.4 *(i) Let(U 1) and* (9) *holds. Then if* (1) *is non oscillatory we can define the constant*

$$\psi \equiv \lim_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n}\sum_{i=0}^{l} q_i \quad (24)$$

such that

$$\mathcal{U}_n = \psi - \sum_{i=0}^{n-1}\sum q_i + \sum_{i=n}^{\infty}\frac{\mathcal{U}_i{}^2}{\mathcal{U}_i + r_i} \geq 0; \quad n \geq \mathbb{N} \quad (25)$$

(ii) If there exists a sequence u, un >−rn, n ≥ N ≥ 0 and a constant ψ such that

$$\mathcal{U}_n \geq \psi - \sum_{i=0}^{n-1}\sum q_i + \sum_{i=n}^{\infty}\frac{\mathcal{U}_i{}^2}{\mathcal{U}_i + r_i} \geq 0$$

(or)

$$\mathcal{U}_n \leq \psi - \sum_{i=0}^{n-1}\sum q_i + \sum_{i=n}^{\infty}\frac{\mathcal{U}_i{}^2}{\mathcal{U}_i + r_i} \leq 0 \quad (26)$$

then(1) *is non oscillatory.*

Theorem 3.5 *Let* (U 2) *hold. If the sequence q satisfies*

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n}\sum_{i=0}^{l} q_i = -\infty \quad (27)$$

and

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{l=0}^{n}\sum_{i=0}^{l} q_i > -\infty \quad (28)$$

then(1) *is oscillatory.*

Proof: suppose to the contrary that (1) is non oscillatory and let $Un = rn\Delta xn/xn$ for $n \geq 0$. Now since condition (U 1) follows from (U 2), theorem (3.1) and (3.4) imply that $\sum_{i=N}^{\infty}\frac{\mathcal{U}_i{}^2}{\mathcal{U}_i + r_i} = +\infty$ but

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=N}^{n}(-\mathcal{U}_{i+1}) \geq \liminf_{n\to\infty}\frac{1}{n}\sum_{i=n}^{\infty}\frac{\mathcal{U}_i{}^2}{\mathcal{U}_i + r_i} + \limsup_{n\to\infty}\frac{1}{n}\sum_{i=N}^{n}\sum_{i=N}^{l} q_i -$$

which is impossible from

$-uj+1 < rj+1$. This completes the proof.

Theorem 3.6 *Let* $(U\ 3)$ *hold and assume* (1) *is non oscillatory. Then the following statements are equivalent*

(i) $q \in \mathcal{C}$

(ii) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n}\sum_{j=0}^{k} q_j$ *exist (finite)*

(iii) The sequence q satisfies (9)

(iv) For any non oscillatory solution x, with $x_n x_{n+1} > 0,\ n \ge \mathbb{N}$ *for some* $\mathbb{N} \ge 0$, *the sequence* $x_n = r_n \Delta x_n / x_n$, *satisfies* (10).

Proof: Obviously (*i*) ⇒(*ii*) ⇒(*iii*). Theorem (3.1) shows that (*iii*) and (*iv*) are equivalent. The proof of (*iv*) → (*i*) in immediate by letting $n \to \infty$ and observing that (*iv*) implies that $u_n \to 0$ as $n \to \infty$.
This completes the proof.

Remark 3.7 *We observe therefore that theorem* (3.6) *show that if* $(U\ 3)$ *holds and* (1) *is non oscillatory then the sequence q either* converges *or diverges to* $-\infty$. *This is not true in the differential equations case (cf [Tipler, 1978]). The following Oscillation tests are immediate.*

Corollary 3.8 *Let* (*A*3) *and* (9) *hold. If* $q \notin \mathcal{C}$ *then* (1) *is oscillatory.*

Corollary 3.9 *Let* (*A*3) *hold. If*

$$-\infty = \liminf_{n\to\infty} \sum_{j=0}^{n} q_j < \limsup_{n\to\infty} \sum_{j=0}^{n} q_j \quad (30)$$

Then (1) *is oscillatory.*

Theorem 3.10 *If* there *exist two sequence* $\{N_k\}$ *and* $\{n_k\}$ *of integers* $n_k \ge N_k + 1$ *such that* $N_k \to \infty$ *as* $K \to \infty$ *and*

$$\sum_{j=N_k}^{n_k-1} q_j \ge r_{N_k} + r_{n_k} \quad (31)$$

Then (1) *in oscillatory.*

IV. CONJUGACY CRITERIA

Theorem 4.1 *Suppose that* $q_k \not\equiv 0$ *and* $\sum_{j=-\infty}^{\infty} q_j = \lim_{k\to\infty}\sum_{j=-k}^{k} q_j$ *exists (as a finite number). If*

$$\sum_{j=-\infty}^{\infty} q_j \ge 0 \quad (32)$$

equation(1) *is conjugate on* $\mathbb{Z}$.

Proof: Suppose for the contrary, that equation (1) is not conjugate on Z. Then it is disconjugate and hence there exists a solution *x* of (1) such that $r_k x_k x_{k+1} >$ for all $k \in$. We can assume that there exists a solution *w* of the generalized Riccati difference equation with $r_k + w_k >$, $k \in$, which is related to (1) by $w_k = r_k \varphi(\Delta y_k / y_k)$.
Let *m*, *n* be integers such that $0 \le m < n$.

$$w_n - w_m = \sum_{j=m}^{n-1} q_j - \sum_{j=m}^{n-1} s(w_j, r_j)$$

$$w_{-m} - w_{-n} = \sum_{j=-n}^{m-1} p_j \sum_{j=m}^{n-1} s(w_j, r_j)$$

Letting $n \to \infty$ and putting $m = 0$ in the above equation we have

$$-w_0 = -\sum_{j=0}^{\infty} q_j - \sum_{j=0}^{\infty} s(w_j, r_j)$$

and

$$w_0 = -\sum_{j=-\infty}^{-1} p_j - \sum_{j=-\infty}^{-1} s(w_j, r_j)$$

Since w_k and also w_k and also w_{-k} tend to zero as $k \to \infty$ from the addition of last two inequalities we have

$$\sum_{j=-\infty}^{\infty} q_j = -\sum_{j=-\infty}^{\infty} s(w_j, r_j)$$

which is a contradiction to (32).

Corollary 4.2 *Let* $m_1, m_2 \in \mathbb{N}$ *be such that* $m_2 \ge m_1 + 2$. *A sufficient condition for* (1) *to be conjugate on the interval* $[m_1, m_2]$ *is that either*

$$\sum_{j=m_1}^{m_2} q_j \ge r_{m_1} + r_{m_2+1}$$

(or)

$$q_{m_i} \ge r_{m_i} + r_{m_i+1}, i = 1,2,3, \ldots$$

Theorem 4.3 *Suppose that* $q_k \ge 0$ *for* $k \ge$. *A sufficient* condition *for conjugacy of* (1) *in an interval* $[n, +\infty)$, $n \in$, *is that there exist integers l, m with n < l < m such that*

$$\frac{1}{(1-n)^{\alpha-1}} < \sum_{k=l}^{m} q_k$$

Proof: We will show that the solution *x* of (1) given by the initial condition $x_n = 0$, $x_{n+1} = 1$ has a generalized zero in (n, ∞). Then without loss of generality we can assume $x_k > 0$ in (n, ∞) and $\Delta x_k \ge 0$ in $[n, \infty)$, since if $\Delta x_k < 0$ at some point in (n, ∞), we would have a generalized zero in (n, ∞) by the condition $q_k \ge 0$.

$$(\Delta x_m)^{\alpha-1} = (\Delta x_l)^{\alpha-l} - \sum_{k=l}^{m} (x_k)^{\alpha-1} q_k .$$

Since $q_k \ge 0$, using the discrete version of the lagrange mean value theorem we have,

$$(\Delta x_l)^{\alpha-1} - \sum_{k=l}^{m} (x_k)^{\alpha-1} q_k \le (\Delta x_l)^{\alpha-1} - \sum_{k=l}^{m} (x_l)^{\alpha-1} q_k = (\Delta x_l)^{\alpha-1} [1$$

By hypothesis, the factor in brackets is negative. If $\Delta x_l > 0$, then $\Delta x_{m+1} < 0$, implying a generalized zero in $(m + 1, \infty)$. *x* has a generalized zero in $(m + 1, \infty)$ and so (1) in conjugate in $[n, \infty)$.

Remark 4.4 *The results*

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presented in preposition (2.2) *and lemma* (2.3) *are applied to check the conjugacy criteria for* (1).

V. CONCLUSION

Here the complete study made on the discrete oscillation and conjugacy criteria using Riccati techniques. The research could be further extended in future for oscillation of higher order non linear as well as linear difference equation.

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