Common Fixed Point Theorems for Generalized Cyclic Contraction Pair in Partial B-Metric Spaces

A. Jennie Sebasty Pritha, U.Karuppiah

Abstract: In this paper, we introduce the notion of generalized cyclic contraction pair with transitive mapping in partial b-metric spaces. Also, we establish some fixed point theorems for this contraction pair. Our results generalize and improve the result of Oratay, Sintunvarat and Je Cho (Fixed Point Theory App. 2015:164) in partial b-metric spaces.

I. INTRODUCTION

One of the recently popular topics in fixed point theory is to show the existence of fixed points of cyclic contraction mappings in several spaces. In 2013, Shukla [1] introduced the concept of partial b-metric spaces. There are number of generalizations of metric spaces and Banach contraction principle. In the sequel, Bakhtin [2] and Czerwik [3] introduced b-metric spaces as a generalization of metric spaces. On the other hand, Mathews [4] introduced the notion of partial metric spaces. Here, he replaced usual metric as partial metric space with the property that the self distance need not to be zero. In this paper, we generalize some fixed point results for cyclic contraction pair in partial b-metric space by using the condition of transitive mapping.

II. PRELIMINARIES

Definition 2.1. [5]
Let A and B be non-empty subsets of a metric space (X, d). A mapping \( f: A \cup B \rightarrow A \cup B \) is called a cyclic contraction if there exists \( k \in [0,1) \) such that
\[
d(f(x), f(y)) \leq kd(x, y)
\]
for all \( x \in A \) and \( y \in B \)

Definition 2.2. [1]
A partial b-metric on a non-empty set X is a function \( \rho: X \times X \rightarrow [0, \infty) \) such that for all \( x, y, z \in X \):

(i) \( \rho(x, y) = \rho(y, x) \)

(ii) \( \rho(x, x) \leq \rho(x, y) + \rho(y, x) \)

(iii) \( \rho(x, y) \leq \rho(x, z) + \rho(z, y) \)

(iv) there exists a real number \( s \geq 1 \) such that
\[
\rho(x, y) \leq s \left( \rho(x, z) + \rho(z, y) - \rho(z, z) \right)
\]

A partial b-metric space is a pair \((X, \rho)\) such that X is a non-empty set and \( \rho \) is a partial b-metric on X. The number \( s \) is called the coefficient of \((X, \rho)\).

Definition 2.3. [8]
The function \( \varphi: [0, \infty) \rightarrow [0, \infty) \) is called an altering distance function if the following properties hold:

(i) \( \varphi \) is continuous and non-decreasing;

(ii) \( \varphi(t) = 0 \) if and only if \( t = 0 \).

Definition 2.4. [6] and [7]
Let X be a non-empty set. The mapping \( \alpha: X \times X \rightarrow [0, \infty) \) is said to be transitive if for all \( x, y, z \in X \), \( \alpha(x, y) \geq 1, \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1 \).

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III. MAIN RESULTS

Let \((X, \rho)\) be a partial b-metric space with coefficient \( s \geq 1 \) and \( f, g: X \rightarrow X \) be two self mappings. For all \( x, y \in X \), we have

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Let \((X, \rho)\) be a partial b-metric space. Let A, B be \(X \times X \to [0, \infty), \psi, \phi : [0, \infty) \to [0, \infty)\) and \(f, g : X \to X\) be four mappings. The pair \((f, g)\) is called a cyclic generalized \(\alpha - (\psi, \phi, A, B)\)-contraction if

(i) \(\psi\) and \(\phi\) are altering distance functions.

(ii) \(A \cup B\) has a cyclic representation with respect to the pair \((f, g)\) that is, \(f(A) \subseteq B, g(B) \subseteq A\) and \(X = A \cup B\).

(iii) there exists \(0 < \delta < 1\) such that the following condition holds:

\[\forall x \in A, y \in B \text{ such that } \alpha(x, y) \geq 1 \text{ or } \alpha(y, x) \geq 1\]

This implies that

\[\psi(M(x, y)) \leq \delta \psi(M(x, y))\]  

where \(\varepsilon > 1\) is a real constant.

Theorem 3.1

Let \((X, \rho)\) be a \(Pb\)-complete partial b-metric space. Let \(A, B\) be the non-empty closed subsets of \(X\). Suppose that \(\alpha : X \times X \to [0, \infty), \psi, \phi : [0, \infty) \to [0, \infty)\) and \(f, g : X \to X\) such that the pair \((f, g)\) is \(\alpha - (A, B)\)-weakly increasing and the following condition holds:

the pair \((f, g)\) is cyclic generalized \(\alpha - (\psi, \phi, A, B)\)-contraction

\(f\) or \(g\) is continuous, \(\alpha\) is a transitive mapping.

If \(\{x_n\}\) is sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) and \(x_n \to z\) as \(n \to \infty\), then \(\alpha(z, z) \geq 1\).

Then \(f\) and \(g\) have a common fixed point in \(A \cap B\).

Proof. It is clear that from (1),

\[\psi(M(x, y)) \leq \delta \psi(M(x, y))\]  

for all \(x, y \in X\).

Now it follows immediately from (2) that \(z \in X\) is a fixed point of \(f\) if and only if \(z\) is a fixed point of \(g\). Choose \(x_0 \in A\), let \(x_1 = f(x_0)\) and since \(f(A) \subseteq B\), we have \(x_1 \in B\).

Let \(x_2 = g(x_1)\) and since \(g(B) \subseteq A\), we have \(x_2 \in A\).

Continuing this process, we can construct a sequence \(\{x_n\}\) in \(X\) such that \(x_{2n+1} = f(x_{2n}) \in B, x_{2n+2} = g(x_{2n+1}) \in A\) for all \(n \in \mathbb{N} \cup \{0\}\).

Since \(f\) and \(g\) are \(\alpha - (A, B)\) weakly increasing, we have

\[\alpha(fx_0, gx_0) \geq 1\]

\[\alpha(gx_0, fx_0) \geq 1\]

This implies that \(\alpha(x_1, x_2) \geq 1\) and \(\alpha(x_2, x_1) \geq 1\).

Continuing this process, we obtain

\[\alpha(x_n, x_{n+1}) \geq 1\]

for all \(n \in \mathbb{N} \cup \{0\}\). If \(x_{2n} = x_{2n+1}\) for some \(n\) or \(x_{2n+1} = x_{2n+2}\) for some \(n\), then obviously \(f\) and \(g\) have at least one common fixed point. Now we complete the proof by three steps:

Step 1: We prove that \(\lim_{n \to \infty} P_b(x_n, x_{n+1}) = 0\).

For each \(k \in \mathbb{N} \cup \{0\}\), we define \(P_b = P_b(x_k, x_{k+1})\).

Now assume that \(P_b = 0\) for some \(k_0 \in \mathbb{N} \cup \{0\}\). This implies that \(x_{k_0} = x_{k_0+1}\)

If \(k_0 = 2n\) for some \(n \in \mathbb{N}\), then \(x_{2n} = x_{2n+1}\)

Next to prove that \(x_{2n+1} = x_{2n+2}\). Since

\[\alpha(x_{2n+1}, x_{2n+2}) \geq 1\]

we have

\[\psi(SP_b(x_{2n+1}, x_{2n+2})) = \psi(SP_b(x_{2n+1}, x_{2n+2})) \leq \delta \psi(M(x_{2n+1}, x_{2n+2}))\]

\[\Rightarrow \delta \leq 1\]

This contradicts the fact that \(\delta < 1\).

Therefore

\[\psi(SP_b(x_{2n+1}, x_{2n+2})) = 0\]

Hence \(x_{2n+1} = x_{2n+2}\).

Similarly, if \(k_0 = 2n+1\) for some \(n \in \mathbb{N} \cup \{0\}\), then \(x_{2n+1} = x_{2n+2}\) gives \(x_{2n+2} = x_{2n+3}\).

Consequently, the sequence \(\{P_b\}\) becomes constant for \(k \geq k_0\), then \(\lim_{n \to \infty} P_b(x_k, x_{k+1}) = 0\).

Suppose \(P_b = P_b(x_k, x_{k+1}) > 0\), for all \(k \in \mathbb{N} \cup \{0\}\).

Now to prove \(P_b(x_{k+1}, x_{k+2}) \leq P_b(x_k, x_{k+1})\) for all \(k \in \mathbb{N} \cup \{0\}\).
Suppose \( P_b(x_{k+1}, x_{k+2}) > P_b(x_k, x_{k+1}) \).

If \( k \) is even, then \( k = 2n \), for some \( n \in \mathbb{N} \cup \{0\} \), then we have
\[ P_b(x_{2n}, x_{2n+1}) > P_b(x_{2n}, x_{2n+1}). \]
Since \( x_{2n} \in A \), \( x_{2n+1} \in B \) and \( \alpha(x_{2n}, x_{2n+1}) \geq 1 \), we have
\[ \psi(S'P_b(x_{2n}, x_{2n+1})) - \psi(S'P_b(x_{2n}, x_{2n+1})) 
\leq \partial \psi(M_b(x_{2n}, x_{2n+1})). \]
where
\[ M_b(x_{2n}, x_{2n+1}) = \max\left\{ P_b(x_{2n}, x_{2n+1}), P_b(x_{2n+2}, x_{2n+2}), P_b(x_{2n+1}, x_{2n+1}) \right\} \]
\[ = P_b(x_{2n}, x_{2n+1}) \]

Then we have
\[ \psi(S'P_b(x_{2n}, x_{2n+1})) \leq \psi(S'P_b(x_{2n}, x_{2n+1})) \leq \partial \psi(M_b(x_{2n}, x_{2n+1})). \]
Which is a contradiction. Therefore
\[ P_b(x_{2n+1}, x_{2n+2}) \leq P_b(x_{2n}, x_{2n+1}) = M_b(x_{2n+1}, x_{2n+2}) \]
for all \( n \in \mathbb{N} \cup \{0\} \).

If \( k \) is odd, then \( k = 2n+1 \) for some \( n \in \mathbb{N} \cup \{0\} \).
Therefore, we have
\[ P_b(x_{2n+2}, x_{2n+3}) > P_b(x_{2n+1}, x_{2n+2}) \]
Since \( x_{2n+2} \in A \), \( x_{2n+1} \in B \) and \( \alpha(x_{2n+1}, x_{2n+2}) \geq 1 \), we have
\[ \psi(S'P_b(x_{2n+1}, x_{2n+2})) \leq \partial \psi(M_b(x_{2n+1}, x_{2n+2})). \]
where
\[ M_b(x_{2n+1}, x_{2n+2}) = \max\left\{ P_b(x_{2n+1}, x_{2n+2}), P_b(x_{2n+2}, x_{2n+2}), P_b(x_{2n+1}, x_{2n+1}) \right\} \]
\[ = P_b(x_{2n+1}, x_{2n+2}) \]

Therefore
\[ \psi(S'P_b(x_{2n+1}, x_{2n+2})) \leq \partial \psi(M_b(x_{2n+1}, x_{2n+2})). \]
\[ \leq \psi(S'P_b(x_{2n+1}, x_{2n+2})) \leq \partial \psi(M_b(x_{2n+1}, x_{2n+2})). \]
Which is a contradiction. So we have
\[ P_b(x_{2n+2}, x_{2n+3}) \leq P_b(x_{2n+1}, x_{2n+2}) = M_b(x_{2n+1}, x_{2n+2}) \]
for all \( n \in \mathbb{N} \cup \{0\} \).

Hence
\[ P_b(x_{k+1}, x_{k+2}) \leq P_b(x_k, x_{k+1}) \]
holds. Then
\[ \{P_b(x_k, x_{k+1}) : k \in \mathbb{N} \cup \{0\}\} \]
is bounded below and non-increasing. Then there exists \( r \geq 0 \) such that
\[ \lim_{k \to \infty} P_b(x_k, x_{k+1}) = r \]
(5)
This implies
\[ \lim_{k \to \infty} M_b(x_k, x_{k+1}) = r \]
\[
P_b\left(z, x_{n+1}\right) = \max\left\{ P_b\left(x_m, x_n\right), P_b\left(x_m, x_{n+1}\right), P_b\left(x_n, x_{n+1}\right), P_b\left(x_n, x_m\right) \right\}
\]

Taking the limit supremum as \( k \to \infty \) in above inequality, from (7) and (13), we have

\[
\frac{\varepsilon}{2s} \leq \lim sup_{k \to \infty} P_b\left(x_{m+1}, x_{n+1}\right)
\]

Similarly, we have

\[
\lim sup_{k \to \infty} P_b\left(x_{m+1}, x_{n+1}\right) \leq s^3 \varepsilon
\]

It follows from (15) and (16) that

\[
\frac{\varepsilon}{2s} \leq \lim sup_{k \to \infty} P_b\left(x_{m+1}, x_{n+1}\right) \leq s^3 \varepsilon
\]

Since \( \alpha \) is transitive, we have

\[
\psi\left(s^3 P_b\left(x_{m+1}, x_{n+1}\right)\right) = \psi\left(s^3 P_b\left(x_{m+1}, x_{n+1}\right)\right) \leq \psi\left(M\left(x_{m+1}, x_{n+1}\right)\right)
\]

Where

\[
M\left(x_{m+1}, x_{n+1}\right) = \max\left\{ P_b\left(x_m, x_n\right), P_b\left(x_m, x_{n+1}\right), P_b\left(x_n, x_{n+1}\right), P_b\left(x_n, x_m\right) \right\}
\]

Letting the limit supremum as \( k \to \infty \) in the above equation and using (8), (7), (13), (14), (17), we have

\[
\varepsilon = \max\left\{ \frac{\varepsilon}{2s}, \frac{s^3 \varepsilon}{2s} \right\}
\]

\[
\lim sup_{k \to \infty} M\left(x_{m+1}, x_{n+1}\right) \leq \max\left\{ s^2 \varepsilon, s^3 \varepsilon \right\}
\]

\[
\varepsilon = s^2 \varepsilon
\]

Letting \( k \to \infty \) in (18), we obtain

\[
\psi\left(s^2 \varepsilon\right) = \psi\left(s^3 \varepsilon\right)
\]

\[
\leq \psi\left(s^3 \lim sup_{k \to \infty} P_b\left(x_{m+1}, x_{n+1}\right)\right)
\]

\[
\leq \delta \psi\left(\lim sup_{k \to \infty} M\left(x_{m+1}, x_{n+1}\right)\right)
\]

\[
\leq \delta \psi\left(s^2 \varepsilon\right)
\]

Since \( \delta < 1 \), we have \( \psi\left(s^2 \varepsilon\right) = 0 \) and hence \( \varepsilon = 0 \), which is a contradiction.

Therefore \( \{x_n\} \) is a \( \text{Pb} \) – Cauchy sequence.

Step 3: We prove that the existence of a common fixed point of \( f \) and \( g \).

Since \( (X, Pb) \) is a \( \text{Pb} \) complete and \( \{x_n\} \) is a then from definition and lemma of [9], \( \{x_n\} \) converges to \( z \in X \). Therefore, we have

\[
\lim_{n \to \infty} P_b\left(x_n, z\right) = \lim_{n \to \infty} P_b\left(x_n, x_{n+1}\right) = 0 = P_b\left(z, z\right)
\]

and

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} x_{n+1} = z.
\]

Since \( \{x_{2n}\} \) is a sequence in \( A \), \( A \) is closed and \( x_{2n} \to z \), we have \( z \in A \).

Also since \( \{x_{2n+1}\} \) is a sequence in \( B \), \( B \) is closed and \( x_{2n+1} \to z \), we have \( z \in B \).

To show that \( z \) is a fixed point of \( f \) and \( g \). Without loss of generality, we assume that \( f \) is continuous.

Since \( \{x_{2n}\} \to z \), we get \( x_{2n+1} = f x_{2n} \to f z \).

So, we have \( z = f z \). Now to prove that \( z = g z \).

Since \( z \in A, z \in B \) and \( \alpha (z, z) \geq 1 \), we have

\[
\psi\left(s^2 P_b\left(z, g z\right)\right) = \psi\left(s^2 P_b\left(f z, g z\right)\right)
\]

\[
\leq \delta \psi\left(M\left(z, z\right)\right)
\]

\[
= \delta \psi\left(M\left(z, z\right)\right)
\]

Since \( \delta < 1 \), it follows that \( P_b\left(z, g z\right) = 0 \). Thus \( z = g z \).

Therefore \( z \) is a common fixed point of \( f \) and \( g \). This completes the proof.

The previous theorem can be proved without assuming the continuity of \( f \) or \( g \). For instance, We assume that \( X \) satisfies following properties:

Definition 3.2

Let \((X, Pb)\) be a partial b-metric space and \(\alpha : X \times X \to [0, \infty)\) be a mapping. A space \( X \) satisfies the property \((Q_{Pb})\) if \(X\) is a sequence in \(X\) such that

\[
\alpha\left(x_n, x_{n+1}\right) \geq 1
\]

for all \(n \in \mathbb{N}\) and \(x_n \to x\) as \(n \to \infty\), then

\[
\alpha\left(x_n, x\right) \geq 1
\]

for all \(n \in \mathbb{N}\).

Theorem 3.2

Let \((X, Pb)\) be a complete partial b-metric with coefficient \(s \geq 1\) and \(A, B\) be a non-empty closed subsets of \(X\). If \(\alpha : X \times X \to [0, \infty), \psi : [0, \infty) \to [0, \infty)\) and \(f, g : X \to X\) are four mappings such that the pair \((f, g)\) is a cyclic \(-\) \((A, B)\)-weakly increasing and the following condition hold: \(i\) the pair \((f, g)\) is a cyclic \(-\) \((\psi, A, B)\)-contraction, \(ii\) \(X\) satisfies the property \((Q_{Pb})\), \(iii\) \(\alpha\) is a transitive mapping, \(iv\) If \(\{x_n\}\) is a sequence in \(X\) such that

\[
\alpha\left(x_n, x_{n+1}\right) \geq 1
\]

for all
\[ n \in \mathbb{N} \quad \text{and} \quad x_n \to z \quad \text{as} \quad n \to \infty, \quad \text{then} \quad \alpha(z, z) \geq 1. \]

Then f and g have a common fixed point in \( A \cap B \).

**Proof.** Now, we prove the reasoning of Theorem 3.1 step by step to construct a sequence \( \{x_n\} \) in \( X \) with
\[ \alpha(x_n, x_{n+1}) \geq 1, x_{2n} \in A, x_{2n+1} \in B \quad \text{for all} \quad n \in \mathbb{N} \]
and \( x_n \to z \) for some \( z \in X \). Since \( x_{2n} \to z, x_{2n+1} \to z \)
and A, B are closed in X. This implies \( z \in A \cap B \).

By using the property of \( \left( Q_{\rho} \right) \), we have \( \alpha(x_n, z) \geq 1 \)
for all \( n \in \mathbb{N} \). Since \( x_{2n} \in A, z \in B \) and \( x_n \),
\[ z \geq 1 \quad \text{for all} \quad n \in \mathbb{N}, \]
\[ \psi \left( S^P \left( x_{2n+1}, g z \right) \right) = \psi \left( S^P \left( f x_{2n}, g z \right) \right) \leq \delta \psi \left( M \left( x_{2n}, z \right) \right) \]
where
\[ M \left( x, z \right) = \max \left\{ P \left( x, z \right), P \left( x, f x_{2n} \right), P \left( z, g z \right), \frac{P \left( x, g z \right) + P \left( z, f x_{2n} \right)}{2} \right\} \]
\[ = \max \left\{ P \left( x, z \right), P \left( x, x_{2n+1} \right), P \left( z, g z \right), \frac{P \left( x, g z \right) + P \left( z, x_{2n+1} \right)}{2} \right\} \]
Taking the limit supremum as \( n \to \infty \) in the above inequality, we have
\[ \psi \left( S^P \left( z, g z \right) \right) = \psi \left( P \left( z, g z \right) \right) < \delta \psi \left( S^P \left( z, g z \right) \right) \]
Since \( \delta < 1 \) and \( s \geq 1 \), we have \( P_b \left( z, g z \right) = 0 \). Hence \( z = g z \).

Thus \( z \) is a common fixed point of \( f \) and \( g \). This completes the proof.

**IV. CONCLUSION**

In this paper we introduced and discussed about b-metric spaces and their properties.

**REFERENCES**


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