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Fekete – Szego Problem for Sakaguchi kind of Functions Related to Shell – like Curves Connected with Fibonacci Numbers

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Abstract**:** ***In this paper, it is attempted to introduce and investigate new subclasses of Sakaguchi kind of functions related to shell – likes curves connected with Fibonacci numbers. Furthermore, the estimates of first two coefficients of functions in these classes are obtained. Fekete – Szego inequalities for these function classes are also determined.***

Keywords **:** ***Fibonacci numbers, Sakaguchi kind of functions, Fekete – Szego inequalities.***

I. INTRODUCTION

Let $\Omega = \{z : |z| < 1\}$ denote the unit disc on the complex plane. The class of all analytic functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n \qquad (1)$$

in the open unit disc Ω with normalization $f(0) = f'(0) - 1 = 0$ is denoted by A and class $S \subset A$ is the class which consists of univalent functions in Ω.

The koebe one quarter theorem [3] ensures that the image of Ω under every univalent function $f \in A$ contains a disk of radius ¼. Thus every univalent function $f \in A$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \qquad (z \in \Omega) \qquad \text{and}$$

$$f(f^{-1}(w)) = w\left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right).$$

A function $f \in A$ is said to be bi – univalent on Ω if both f and f^{-1} are univalent in Ω. Let Σ denote the class of bi univalent functions as defined in the unit disk Ω. Since $f \in \Sigma$ has the Maclaurian Series given by (1) , a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) + \ldots.. \qquad (2)$$

One can see a short history and examples of function in the class Σ in [13]. Several authors have introduced and investigated subclasses of bi – univalent functions and obtained bounds for the initial coefficients (see [1, 2,8,13,14,15]).

An analytic function f is subordinate to an analytic function F in Ω, written as $f \,\pi\, F(z \in \Omega)$, provided there is an analytic function w defined on Ω with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = F(w(z))$. It follows from Schwarz Lemma that

$$f(z) \,\pi\, F(z) \Leftrightarrow f(0) = F(0) \qquad \text{and}$$

$$f(\Omega) \subset F(\Omega), \; z \in \Omega$$

(for details see [3,7]. The important subclasses of S in geometric function theory such that if $f \in A$ are recalled and

$$\frac{zf\ (z)}{f(z)} \,\pi\, p(z) \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \,\pi\, p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then it is said that f is star like and convex, respectively. These functions form known classes denoted by S^* and C, respectively. Recently, in [12], Sokol introduced the class SL of Shell – like functions on the set of functions $f \in A$ which is described in the following definitions:

Definition 1.1. The function $f \in A$ belongs to the class SL if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \,\pi\, \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}.$$

where

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$$\tau=\frac{(1-\sqrt{5})}{2}\approx -0.618.$$

It should be observed SL is a subclass of the star like functions S^*.

Later , Dziok et al. in [4] and [5] and Ozlem Guney et al. [9] defined and introduced various subclasses of bi-univalent function related to a shell – like curve connected with Fibonacci numbers, respectively.

Definition 1.2. The function $f\in A$ belongs to the class KSL of convex shell – like functions if it satisfies the condition that

$$1+\frac{zf''(z)}{f'(z)} \prec \tilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$$

with $\tau=\frac{(1-\sqrt{5})}{2}\approx -0.618$.

The function $\tilde{p}$ is not univalent in Ω, but it is univalent in the disc $|z|<\frac{(3-\sqrt{5})}{2}\approx 0.38$. For example, $\tilde{p}(0)=\tilde{p}(-1/2\tau)=1$ and $\tilde{p}\left(e^{\mp i\arccos(1/4)}\right)=\frac{\sqrt{5}}{5}$, and it may also be noticed that

$$\frac{1}{|\tau|}=\frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides [0,1] such that it fulfills the golden section. The image of the unit circle $|z|=1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x-\sqrt{5})y^2=(\sqrt{5}-2x)(\sqrt{5}x-1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0<r\le r_0=\frac{(3-\sqrt{5})}{2}\approx 0.38$. For $r_0<r<1$, it has a loop and for $r=1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2=1+\tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n=u_n\tau+u_{n-1}.$$

In [11], taking $\tau z=t$, Raina and Sokol showed that

$$\tilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}=\left(t+\frac{1}{t}\right)\frac{1}{1-t-t^2}$$

$$=\frac{1}{\sqrt{5}}\left(t+\frac{1}{t}\right)\left(\frac{1}{1-(1-\tau)t}-\frac{1}{1-\tau t}\right)$$

$$=\left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty}\frac{(1-\tau)^n-\tau^n}{\sqrt{5}}t^n \qquad (3)$$

$$=\left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty}u_nt^n=1+\sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^nz^n,$$

where

$$u_n=\frac{(1-\tau)^n-\tau^n}{\sqrt{5}},\ \tau=\frac{1-\sqrt{5}}{2}\ (n=1,2,....) \qquad (4)$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0=0, u_1=1, u_{n+2}=u_n+u_{n+1}$ for n= 0,1,2,…. And they got

$$\tilde{p}(z)=1+\sum_{n=1}^{\infty}\tilde{p}_nz^n=1+(u_0+u_2)\tau z+(u_1+u_3)\tau^2z^2+\sum_{n=3}^{\infty}(u_{n-3}+u_{n-2}+u_{n-1}+u_n)\tau^nz^n$$

$$=1+\tau z+3\tau^2z^2+4\tau^3z^3+7\tau^4z^4+11\tau^5z^5+.... \qquad (5)$$

Let $p(\beta)$, $0\le\beta<1$, denote the class of analytic functions p in Ω with $p(0)=1$, and $\mathrm{Re}\{p(z)\}>\beta$. Especially, p instead of $p(0)$ is used.

Theorem 1.3. [5] The function $\tilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$ belongs to the class $p(\beta)$ with $\beta=\frac{\sqrt{5}}{10}\approx 0.2236$.

Now the following lemma is given to prove the theorem.

Lemma 1.4. [10] Let $p\in P$ with $p(z)=1+c_1z+c_2z^2+....$ then

$$|c_n|\le 2, \text{ for } n\ge 1. \qquad (6)$$

In this present work, two subclasses of Sakaguchi kind of Σ associated with shell – like functions connected with Fibonacci numbers are introduced to obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for these function classes. Also, bounds for the Fekete – Szego functional $|a_3-\mu a_2^2|$ for each subclass are also given.

II. BI – UNIVALENT FUNCTION CLASS $SLM_{\lambda,t,\Sigma}(\tilde{p}(z))$

In this section, a new subclass of Sakaguchi kind of Σ is associated with shell – like functions connected with Fibonacci numbers in order to obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class of subordination.

Firstly, let $p(z)=1+p_1z+p_2z^2+.....$ and $p\,\pi\,\tilde{p}$. Then there exists an analytic functions *u* such that $|u(z)|<1$ in Ω and $p(z)=\tilde{p}(u(z))$. Therefore the function

$$h(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+.... \qquad (7)$$

is in the class $p(0)$. It follows that

$$u(z)=\frac{c_1z}{2}+\left(c_2-\frac{c_1^2}{2}\right)\frac{z^2}{2}+\left(c_3-c_1c_2-\frac{c_1^3}{4}\right)\frac{z^3}{2}+...... \qquad (8)$$

and

$$\tilde{p}(u(z))=1+\tilde{p}_1\left\{\frac{c_1z}{2}+\left(c_2-\frac{c_1^2}{2}\right)\frac{z^2}{2}+\left(c_3-c_1c_2-\frac{c_1^3}{4}\right)\frac{z^3}{2}+....\right\}$$
$$+\tilde{p}_2\left\{\frac{c_1z}{2}+\left(c_2-\frac{c_1^2}{2}\right)\frac{z^2}{2}+\left(c_3-c_1c_2-\frac{c_1^3}{4}\right)\frac{z^3}{2}+....\right\}^2$$
$$+\tilde{p}_3\left\{\frac{c_1z}{2}+\left(c_2-\frac{c_1^2}{2}\right)\frac{z^2}{2}+\left(c_3-c_1c_2-\frac{c_1^3}{4}\right)\frac{z^3}{2}+....\right\}^3+......$$

$$=1+\frac{\tilde{p}_1c_1z}{2}+\left\{\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right\}z^2+$$
$$\left\{\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^2}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right\}z^3+........ \qquad (9)$$

And similarly, there exists an analytic function v such that $|v(w)|<1$ in Ω and $p(w)=\tilde{p}(v(w))$. Therefore, the function

$$K(w)=\frac{1+v(w)}{1-v(w)}=1+d_1w+d_2w^2+... \qquad (10)$$

is in the class $P(0)$. It shows that

$$v(w)=\frac{d_1w}{2}+\left(d_2-\frac{d_1^2}{2}\right)\frac{w^2}{2}+\left(d_3-d_1d_2-\frac{d_1^2}{4}\right)\frac{w^3}{2}+....... \qquad (11)$$

And

$$\tilde{p}(v(w))=1+\frac{\tilde{p}_1d_1w}{2}+\left\{\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_1+\frac{d_1^2}{4}\tilde{p}_2\right\}w^2+$$
$$\left\{\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^2}{4}\right)\tilde{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_2+\frac{d_1^3}{8}\tilde{p}_3\right\}w^3+........ \qquad (12)$$

Definition 2.1. For $0\le\lambda\le1, |t|\le1$ but $t\neq1$ a function $f\in A$ of the form (1) is said to be in the class $SLM_{\lambda,t,\Sigma}(\tilde{p}(z))$ if the following subordination hold:

$$\frac{(1-t)(\lambda z^3f'''(z)+(1+2\lambda)z^2f''(z)+zf'(z))}{\lambda z^2(f''(z)-t^2f''(tz))+z(f'(z)-tf'(tz))}\,\pi\,\tilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \qquad (13)$$

and

$$\frac{(1-t)(\lambda w^3g'''(w)+(1+2\lambda)w^2g''(w)+wg(w))}{\lambda w^2(g''(w)-t^2g''(tw))+w(g'(w)-tg'(tw))}\,\pi\,\tilde{p}(w)=\frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2} \qquad (14)$$

Where $\tau=\frac{(1-\sqrt{5})}{2}\approx-0618$ where $z,w\in\Omega$ and g is given by (2).

In the following theorem, an attempt has been made to determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function $SLM_{\lambda,t,\Sigma}(\tilde{p}(z))$. Fekete – Szego functional $|a_3-\mu a_2^2|$ for this subclass is also obtained.

Theorem 2.2. Let f be given by (1) be in the class $SLM_{\lambda,t,\Sigma}(\tilde{p}(z))$. Then

$$|a_2|\le\frac{|\tau|}{\sqrt{4(1+\lambda)^2(1-t)^2+\{3(1+2\lambda)(2-t-t^2)-8(1+\lambda)^2(2-3t+t^2)\}\tau}} \qquad (15)$$

and

$$|a_3|\le\frac{|\tau|4(1+\lambda)^2((1-t)^2-2(2-3t+t^2)\tau)}{3(1+2\lambda)(2-t-t^2)\{4(1+\lambda)^2(1-t)^2+(3(1+2\lambda)(2-t-t^2)-8(1+\lambda)^2(2-3t+t^2))\tau\}}$$

(16)

Proof. Let $f\in SLM_{\lambda,t,\Sigma}(\tilde{p}(z))$ and $g=f^{-1}$. Considering (13) and (14), we have

$$\frac{(1-t)(\lambda z^3f'''(z)+(1+2\lambda)z^2f''(z)+zf'(z))}{\lambda z^2(f''(z)-t^2f''(tz))+z(f'(z)-tf'(tz))}=\tilde{p}(u(z)) \qquad (17)$$

and

$$\frac{(1-t)(\lambda w^3g'''(w)+(1+2\lambda)w^2g''(w)+wg(w))}{\lambda w^2(g''(w)-t^2g''(tw))+w(g'(w)-tg'(tw))}=\tilde{p}(v(w)) \qquad (18)$$

Where $\tau=\frac{(1-\sqrt{5})}{2}\approx-0618$ where $z,w\in\Omega$ and *g* is given by (2).

Since

$$\frac{(1-t)(\lambda z^3f'''(z)+(1+2\lambda)z^2f''(z)+zf'(z))}{\lambda z^2(f''(z)-t^2f''(tz))+z(f'(z)-tf'(tz))}$$
$$=1+2(1+\lambda)(1-t)a_2z+(3(1+2\lambda)(2-t-t^2)a_3-4(1-t^2)(1+\lambda)^2a_2^2)z^2+....$$

and

$$\frac{(1-t)(\lambda w^3g'''(w)+(1+2\lambda)w^2g''(w)+wg(w))}{\lambda w^2(g''(w)-t^2g''(tw))+w(g'(w)-tg'(tw))}$$
$$=1-2(1+\lambda(1-t)a_2w+((6(1+2\lambda)(2-t-t^2)-4(1-t^2)(1+\lambda)^2)a_2^2)-3(1+2\lambda)(2-t-t^2)a_3)w^2+.....$$

Thus we have

$$1+2(1+\lambda)(1-t)a_2z+(3(1+2\lambda)(2-t-t^2)a_3-4(1-t^2)(1+\lambda)^2a_2^2)z^2+....$$
$$=1+\frac{\tilde{p}_1c_1z}{2}+\left\{\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right\}z^2+$$
$$\left\{\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^2}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right\}z^3+........ \qquad (19)$$

and

$$1-2(1+\lambda(1-t)a_2w+((6(1+2\lambda)(2-t-t^2)-4(1-t^2)(1+\lambda)^2)a_2^2)-3(1+2\lambda)(2-t-t^2)a_3)w^2+.....$$

$$=1+\frac{\tilde{p}_1 d_1 w}{2}+\left\{\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_1+\frac{d_1^2}{4}\tilde{p}_2\right\}w^2+$$
$$\left\{\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^2}{4}\right)\tilde{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_2+\frac{d_1^3}{8}\tilde{p}_3\right\}w^3+\ldots\ldots \quad (20)$$

It follows from (19) and (20) that

$$2(1+\lambda)(1-t)a_2=\frac{c_1\tau}{2} \quad (21)$$

$$3(1+2\lambda)(2-t-t^2)a_3-4(1-t^2)(1+\lambda)^2a_2^2=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tau+\frac{c_1^2}{4}3\tau^2, \quad (22)$$

and

$$-2(1+\lambda)(1-t)a_2=\frac{d_1\tau}{2} \quad (23)$$

$$\left(2\left(3(1+2\lambda)(2-t-t^2)-2(1-t^2)(1+\lambda)^2\right)a_2^2\right)-3(1+2\lambda)(2-t-t^2)a_3$$
$$=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{d_1^2}{4}3\tau^2 \quad (24)$$

From (21) and (23), we have

$$c_1=-d_1 \quad (25)$$

and

$$8a_2^2=\frac{(c_1^2+d_1^2)\tau^2}{4(1+\lambda)^2(1-t)^2} \quad (26)$$

Now, by summing (22) and (24), we obtain

$$\left(6(1+2\lambda)(2-t-t^2)-8(1+\lambda)^2(1-t^2)\right)a_2^2=\frac{1}{2}(c_2+d_2)\tau-\frac{1}{4}(c_1^2+d_1^2)\tau+\frac{3}{4}(c_1^2+d_1^2)\tau^2 \quad (27)$$

By putting (26) in (27) , we have

$$2\left(4(1+\lambda)^2(1-t)^2+\left(3(1+2\lambda)(2-t-t^2)-8(1+\lambda)^2(2-3t+t^2)\right)\tau\right)a_2^2=\frac{1}{2}(c_2+d_2)\tau^2. \quad (28)$$

Therefore, using Lemma1.4, we obtain

$$|a_2|\le\frac{|\tau|}{\sqrt{4(1+\lambda)^2(1-t)^2+\left(3(1+2\lambda)(2-t-t^2)-8(1+\lambda)^2(2-3t+t^2)\right)\tau}} \quad (29)$$

Now, so as to find the bound on $|a_3|$, let's subtract from (22) and (24). So, we find

$$6(1+2\lambda)(2-t-t^2)a_3-6(1+2\lambda)(2-t-t^2)a_2^2=\frac{1}{2}(c_2-d_2). \quad (30)$$

Hence , we get

$$6(1+2\lambda)(2-t-t^2)|a_3|\le 2|\tau|+6(1+2\lambda)(2-t-t^2)|a_2^2| \quad (31)$$

Then, in view of (29), we obtain

$$|a_3|\le\frac{|\tau|4(1+\lambda)^2\left((1-t)^2-2(2-3t+t^2)\tau\right)}{3(1+2\lambda)(2-t-t^2)\left\{4(1+\lambda)^2(1-t)^2+\left(3(1+2\lambda)(2-t-t^2)-8(1+\lambda)^2(2-3t+t^2)\right)\tau\right\}} \quad (32)$$

□

If we can take the parameter $\lambda=0$ in the above theorem, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $KL_{\Sigma,t}(\tilde{p}(z))$.

Corollary 2.3. Let f given by (1) be in the class $KSL_{\Sigma}(\tilde{p}(z))$. Then

$$|a_2|\le\frac{|\tau|}{\sqrt{4(1-t)^2+\left(3(2-t-t^2)-8(2-3t+t^2)\right)\tau}} \quad (33)$$

And

$$|a_3|\le\frac{|\tau|4\left((1-t)^2-2(2-3t+t^2)\tau\right)}{3(2-t-t^2)\left\{4\,(1-t)^2+\left(3(2-t-t^2)-8(2-3t+t^2)\right)\tau\right\}} \quad (34)$$

Taking $t = 0$, we get the following corollary which is obtained by [9].

Corollary 2.4. Let f given by (1) be in the class $KSL_{\Sigma}(\tilde{p}(z))$. Then

$$|a_2|\le\frac{|\tau|}{\sqrt{4-10\tau}} \quad (35)$$

and

$$|a_3|\le\frac{|\tau|(1-4\tau)}{3(2-5\tau)} \quad (36)$$

III. FEKETE- SZEGO INEQUALITIES FOR THE FUNCTION CLASS $SLM_{\lambda,t,\Sigma}(\tilde{p}(z))$

Fekete and Szego [6] introduced the generalized functional $|a_3-\mu a_2^2|$,where μ is some real number. Due to Zaprawa [15], in the following theorem we determine the Fekete – Szego functional for $f\in SLM_{\lambda,t,\Sigma}(\tilde{p}(z))$.

Theorem 3.1. let f given by (1) be in the class $SLM_{\lambda,t,\Sigma}(\tilde{p}(z))$ and $\mu\in\Re$. Then we have

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{|\tau|}{3(1+2\lambda)(2-t-t^2)}, & |\mu-1|\le\dfrac{A_1}{A_2}\\[2ex] \dfrac{4(1-\mu)\tau^2}{2A_1}, & |\mu-1|\ge\dfrac{A_1}{A_2}\end{cases}$$

Where

$$A_1=8(1+\lambda)^2(1-t)^2+\left(6(1+2\lambda)(2-t-t^2)-16(1+\lambda)^2(2-3t+t^2)\right)\tau$$

and

$$A_2=6(1+2\lambda)(2-t-t^2)|\tau|$$

Proof . From (28) and (30) the following equation is obtained

$$a_3-\mu a_2^2=(1-\mu)\frac{(c_{2+}+d_2)\tau^2}{2\left(8(1+\lambda)^2(1-t)^2+\left\{6(1+2\lambda)(2-t-t^2)-16(1+\lambda)^2(2-3t+t^2)\right\}\tau\right)}+\frac{(c_2-d_2)\tau}{12(1+2\lambda)(2-t-t^2)}$$
$$=\left(\frac{(1-\mu)\tau^2}{2\left(8(1+\lambda)^2(1-t)^2+\left\{6(1+2\lambda)(2-t-t^2)-16(1+\lambda)^2(2-3t+t^2)\right\}\tau\right)}+\frac{\tau}{12(1+2\lambda)(2-t-t^2)}\right)c_2$$
$$+\left(\frac{(1-\mu)\tau^2}{2\left(8(1+\lambda)^2(1-t)^2+\left\{6(1+2\lambda)(2-t-t^2)-16(1+\lambda)^2(2-3t+t^2)\right\}\tau\right)}-\frac{\tau}{12(1+2\lambda)(2-t-t^2)}\right)d_2 \quad (37)$$

Therefore

$$a_3-\mu a_2^2=\left(h(\mu)+\frac{\tau}{12(1+2\lambda)(2-t-t^2)}\right)c_2+\left(h(\mu)-\frac{\tau}{12(1+2\lambda)(2-t-t^2)}\right)d_2 \quad (38)$$

where

$$h(\mu)=\frac{(1-\mu)\tau^2}{2\left(8(1+\lambda)^2(1-t)^2+\left\{6(1+2\lambda)(2-t-t^2)-16(1+\lambda)^2(2-3t+t^2)\right\}\tau\right)} \quad (39)$$

Then, by taking modulus of (38), we conclude that

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{3(1+2\lambda)(2-t-t^2)}, 0 \leq \dfrac{|\tau|}{12(1+2\lambda)(2-t-t^2)} \\ 4|h(\mu)|, |h(\mu)| \geq \dfrac{|\tau|}{12(1+2\lambda)(2-t-t^2)} \end{cases}$$

□

Taking $\mu=1$, the following corollary is obtained.

Corollary 3.2. If $f \in SLM_{\lambda,t,\Sigma}(\tilde{p}(z))$, then

$$\left|a_3-a_2^2\right| \leq \frac{|\tau|}{3(1+2\lambda)(2-t-t^2)}. \quad (40)$$

If we take the parameter $\lambda=0$ in the above theorem, we have the following the Fekete – Szego inequality for the function class $KSL_{\Sigma}(\tilde{p}(z))$.

Corollary 3.3. Let f given by (1) be in the class $KSL_{\Sigma}(\tilde{p}(z))$ and $\mu \in \Re$

Then we have

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{3(2-t-t^2)}, |\mu-1| \leq \dfrac{8(1-t)^2+\left(6(2-t-t^2)-16(2-3t+t^2)\right)\tau}{6(2-t-t^2)|\tau|} \\ \dfrac{4(1-\mu)\tau^2}{2\left(8(1-t)^2+\left(6(2-t-t^2)-16(2-3t+t^2)\right)\tau\right)}, |\mu-1| \geq \dfrac{8(1-t)^2+\left(6(2-t-t^2)-16(2-3t+t^2)\right)\tau}{6(2-t-t^2)|\tau|} \end{cases}$$

Taking *t* = *0* we get the following corollary which is obtained by [9].

Corollary 3.4. Let f given by (1) be in the class $KSL_{\Sigma}(\tilde{p}(z))$ and $\mu \in \Re$. Then we have

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{6}, |\mu-1| \leq \dfrac{2-5\tau}{3|\tau|} \\ \dfrac{(1-\mu)\tau^2}{2(2-5\tau)}, |\mu-1| \geq \dfrac{2-5\tau}{3|\tau|} \end{cases}$$

IV. CONCLUSION

It is attempted to introduce and investigate new subclasses of Sakaguchi kind of functions related to shell – likes curves connected with Fibonacci numbers. Furthermore, the estimates of first two coefficients of functions in these classes are obtained. Fekete – Szego inequalities for these function classes are also determined.

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