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Bounds for the Second Hankel Determinant of Certain Univalent Functions

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Abstract*: *We study the estimates for the Second Hankel determinant of analytic functions. Our class includes (j,k)-convex, (j,k)-starlike functions and Ma-Minda starlike and convex functions..

Index Terms*: *Hankel determinant,, starlike functions, Ma-Minda starlike and convex functions.

I. INTRODUCTION

Let A denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \ (1.1)$$

which are analytic in the open unit disk U = $\{z : z \in$ C *and* $|z| < 1\}$, and S denote the
subclass of A consisting of all function which are univalent in U.

We denote by $\mathcal{S}^*,\ \mathcal{K},\ \mathcal{S}_s^*,\ \mathcal{K}_s$ the familiar subclasses consisting of functions which, respec-
tively, starlike, convex, starlike with respect to symmetric points and convex with respect to symmetric points in U.

In 1976, Noonan and Thomas [13] stated the q^{th} Hankel determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ as

$$\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}, \ a_1 = 1 \quad ,$$

This determinant has also been considered by several authors. For example Noor in [12] determined the rate of growth $H_q(k)$ as $k \to \infty$ for functions f given by (1.1) with bounded boundary. Ehrenborg in [2] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Hayman in [5].

Easily, one can observe that the Fekete and Szeg¨*o* functional is $H_2(1)$. Fekete and Szeg¨*o* [3]

then further generalised the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in$ S.

For our discussion in this paper, we consider the Hankel determinant tn the case of $q = 2$ and $n = 2$, known as second Hankel determinant:

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2| \quad ,$$

and obtain an upper bound to $H_2(2)$ for $f(z) \in S^{j,k}$ and $f(z) \in K^{j,k}$. Janteng et al. [7] have considered the functional $|H_2(2)|$ and found a sharp bound, the subclass of S as $<\{f^0(z)\} > 0$. In their work, they have shown that if $f \in$ S, then $|H_2(2)| \leq 4/9$. These authors [6, 7] also studied the second Hankel determinant and sharp bound for the classes of starlike and convex functions, close-to-starlike and close-to-convex functions with respect to symmetric points have shown that $|H_2(2)| \leq 1$, $|H_2(2)| \leq 1/8$, $|H_2(2)| \leq 1$, $|H_2(2)| \leq 1/9$, respectively.

Definition 1.1. *Let k be a positive integer. A domain* D *is said to be k-fold symmetric if a rotation of* D *about the origin through an angle* $\frac{2\pi}{k}$ *carries* D *onto itself. A function f is said to be k-fold symmetric in* U *if for every z in* U

1. $\quad$ *2πi 2πi*

f(e k z) = e k f(z).

The family of all k-fold symmetric functions is denoted by S^k *and for k* = 2 *we get class of the odd univalent functions.*

The notion of (j, k)-symmetrical functions ($k = 2,3,\ldots$;$j = 0,1,2,\ldots,k-1$) is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.

The theory of (j, k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings [11].

Definition 1.2. *Let* $\varepsilon = (e^{\frac{2\pi i}{k}})$ *and* $j = 0,1,2,..,k-1$ *where k* ≥ 2 *is a natural number. A*

function f : U 7→ C *is called* (j,k)*-symmetrical if*

$f(\varepsilon z) = \varepsilon^j f(z)$, $z \in$ U.

We note that the family of starlike functions with respect to (j,k)-symmetric points is denoted be $S^{(j,k)}$. Also, $S^{(0,2)}$, $S^{(1,2)}$ and $S^{(1,k)}$ are called even, odd and k-symmetric functions respectively. We have the following decomposition theorem.

Definition 1.3. *A function* $f(z) \in$ A *is in the class* $S_{j,k}{}^*$ *if*

$$\Re\left\{\frac{zf'(z)}{f_{j,k}(z)}\right\} > 0 \quad ,$$

where $f_{j,k}$ *defined by* (1.2).

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Definition 1.4. *A function* $f(z) \in A$ *is in the class* $K_{j,k}$ *if*

$$\Re\left\{\frac{(zf'(z))'}{f'_{j,k}(z)}\right\} > 0,$$

where $f_{j,k}$ *defined by* (1.2).

Theorem 1.5. *[11] For every mapping* $f : D \mapsto C$*, and* D *is a k-fold symmetric set, there exists exactly the sequence of* (j,k)*- symmetrical functions* $f_{j,k}$*,*

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z),$$

$$(f \in \mathcal{A};\ k = 1,2,...;\ j = 0,1,2,...,k-1) \quad (1.2)$$

From (1.2) we can get

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj}\left(\sum_{n=1}^{\infty} a_n(\varepsilon^v z)^n\right),$$

then

$$f_{j,k}(z) = \sum_{n=1}^{\infty} \delta_{n,j} a_n z^n,\ a_1 = 1$$

$$\psi \quad \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{(n-j)v} = \begin{cases} 1, & n = lk + j \\ 0, & n \neq lk + j; \end{cases} \quad (1.3)$$

Definition 1.6. [1] *Let* P *denote the class of analytic functions* $p : U \to C$*,* $p(0) = 1$*, and* $<\{p(z)\} > 0$*, then* $p(z) \prec \frac{1+z}{1-z}$.

The class P can be completely characterized in terms of subordination.

We need the following lemmas to derive our results.

Lemma 1.7. [1] *If the function* $p \in$ P *is given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ..., \quad (1.4)$$

then the following sharp estimate holds:

$$|c_n| \le 2 \ (n = 1,2,...).$$

Lemma 1.8. [4] *If the function* $p \in$ P *is given by the series* (1.4), *then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \quad (1.5)$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \quad (1.6)$$

for some x,z *with* $|x| \le 1$ *and* $|z| \le 1$.

II. MAIN RESULTS

Definition 2.1. *Let* $\phi : U \to$ C *be analytic, and let the Maclaurin series of* ϕ *is given by*

$$\phi(z) = 1 + D_1 z + D_2 z^2 + D_3 z^3 + ... \quad (D_1, D_2 \in R, D_1 > 0). \quad (2.1)$$

The class $S^{*(j,k)}(\phi)$ *of starlike functions with respect to* ϕ *consists of functions* $f \in A$ *satisfying the subordination*

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \varphi(z).$$

Theorem 2.2. *Let the function* $f \in S^*(\phi)$ *be given by* (1.1)

2. *If* D_1, D_2 *and* D_3 *satisfy the conditions*

$$\beta D_1^2 + 2\alpha|D_2| + \gamma D_1 \le 0,$$

$$\left|\psi_1(3-\psi_3)^2 D_1 D_3 + \delta D_1^4 + \beta D_1^2 D_2 - \psi_1^2(4-\psi_4)(2-\psi_2)D_2^2\right| - (\gamma - \lambda)D_1^2 \le 0,$$

then the second Hankel determinant satisfies

$$|a_2 a_4 - a_3^2| \le \frac{D_1^2 \psi_1^2}{(3-\psi_3)^2}.$$

3. *If* D_1, D_2 *and* D_3 *satisfy the conditions*

$$\beta D_1^2 + 2\alpha|D_2| + \gamma D_1 \ge 0,$$

$$\left|\psi_1(3-\psi_3)^2 D_1 D_3 - \psi_1^2(4-\psi_4)(2-\psi_2)D_2^2 + \delta D_1^4 + \beta D_1^2 D_2\right| - \alpha D_1|D_2| - \frac{1}{2}\beta D_1^3 - \frac{1}{2}\lambda D_1^2 \ge 0,$$

or the conditions

$$\beta D_1^2 + 2\alpha|D_2| + \gamma D_1 \le 0,$$

$$\left|\psi_1(3-\psi_3)^2 D_1 D_3 + \delta D_1^4 + \beta D_1^2 D_2 - \psi_1^2(4-\psi_4)(2-\psi_2)D_2^2\right| - (\gamma - \lambda)D_1^2 \ge 0,$$

then the second Hankel determinant satisfies

$$|a_2 a_4 - a_3^2| \le \frac{1}{(4-\psi_4)(3-\psi_3)^2(2-\psi_2)}\left|\psi_1(3-\psi_3)^2 D_1 D_3 - \psi_1^2(4-\psi_4)(2-\psi_2)D_2^2 + \delta D_1^4 + \beta D_1^2 D_2\right|.$$

$$|a_2 a_4 - a_3^2| \le \frac{D_1^2}{(4-\psi_4)(3-\psi_3)^2(2-\psi_2)} \cdot \frac{\left(\begin{array}{r} (4-\psi_4)\psi_1^2(2-\psi_2)\left|\psi_1(3-\psi_3)^2 D_1 D_3 + \delta D_1^4 + \beta D_1^2 D_2 - \psi_1^2(4-\psi_4)(2-\psi_2)D_2^2\right| \\ -\alpha^2 D_2^2 - \alpha(3-\psi_3)^2\psi_1^2 D_1|D_2| - \frac{1}{2}\beta(1+2\lambda)D_1^3 \\ + \left(\gamma(4-\psi_4)\psi_1^2(2-\psi_2) + \frac{\lambda^2}{4}\right)D_1^2 - \frac{\beta^2}{4}D_1^4 \end{array}\right)}{\left|\psi_1(3-\psi_3)^2 D_1 D_3 + \delta D_1^4 + \beta D_1^2 D_2 - \psi_1^2(4-\psi_4)(2-\psi_2)D_2^2\right| - 2\alpha D_1|D_2| - \beta D_1^3 - \gamma D_1^2}$$

4. *If* D_1, D_2 *and* D_3 *satisfy the*

$$\beta D_1^2 + 2\alpha|D_2| + \gamma D_1 > 0,$$

$$\left|\psi_1(3-\psi_3)^2 D_1 D_3 - \psi_1^2(4-\psi_4)(2-\psi_2)D_2^2 + \delta D_1^4 + \beta D_1^2 D_2\right| - \alpha D_1|D_2| - \frac{1}{2}\beta D_1^3 - \frac{1}{2}\lambda D_1^2 \ge 0,$$

then the second Hankel determinant satisfies

where

$$\alpha = \psi_1(3-\psi_3)^2 - \psi_1^2(4-\psi_4)(2-\psi_2), \quad \beta = \psi_1^2\left(\frac{(2\psi_3 + 3\psi_2 - 2\psi_1\psi_2)(3-\psi_3) - 2\psi_2(4-\psi_4)(2-\psi_2)}{(2-\psi_2)}\right),$$

$$\gamma = \psi_1^2(3-\psi_3)^2 - \psi_1^2(4-\psi_4)(2-\psi_2),$$

$$\delta = \frac{\psi_1^2\psi_2\psi_3(3-\psi_3)(2-\psi_2) - \psi_1^2\psi_2(4-\psi_4)}{(2-\psi_2)} \text{ and}$$

$$\lambda = \psi_1^2(3-\psi_3)^2 - 2\psi_1^2(4-\psi_4)(2-\psi_2). \quad (2.2)$$

Proof. Since $f \in S^{*(j,k)}(\phi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$\frac{zf'(z)}{f_{j,k}(z)} = \varphi(w(z)). \quad (2.3)$$

Define the function p_1 by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + ...,$$

or, equivantely

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + ...\right) \quad (2.4)$$

Then p_1 is analytic in U with $p_1(0) = 0$ and has a positive real part in U. By using(2.4) together with (2.1), it is evident that

$$\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}D_1 c_1 z + \left(\frac{1}{2}D_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}D_2 c_1^2\right)z^2 + ... \quad (2.5)$$

By (2.3) we have

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$$\frac{zf'(z)}{f_{j,k}(z)} = 1 + \frac{1}{2}D_1c_1z + \left(\frac{1}{2}D_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}D_2c_1^2\right)z^2 + ..., \quad (2.6)$$

so that

$$\left(\frac{z + 2a_2z^2 + 3a_3z^3 + ...}{\psi_1z + \psi_2a_2z^2 + \psi_3a_3z^3 + ...}\right) = 1 + \frac{1}{2}D_1c_1z + \left(\frac{1}{2}D_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}D_2c_1^2\right)z^2 + ...$$

which implies the equality

$$z + 2a_2z^2 + 3a_3z^3 + ... = \psi_1z + \left(\psi_2a_2 + \frac{1}{2}D_1c_1\psi_1\right)z^2 + \left(\psi_3a_3 + \frac{1}{2}\psi_2D_1c_1a_2 + \frac{1}{2}D_1\psi_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}D_2\psi_1c_1^2\right)z^3 + ...,$$

Equating the coefficients on both sides we have

$$a_2 = \frac{D_1c_1\psi_1}{2(2-\psi_2)},$$

$$a_3 = \frac{1}{4(3-\psi_3)}\left(\left(\frac{(D_1^2\psi_1\psi_2}{(2-\psi_2)} - D_1\psi_1 + D_2\psi_1\right)c_1^2 + 2D_1\psi_1c_2\right),$$

$$a_4 = \frac{1}{8(4-\psi_4)(3-\psi_3)(2-\psi_2)}\Big[\big(-2(3-\psi_3)(2-\psi_2)D_2 + \psi_1(3-\psi_3)(2-\psi_2)D_1 + \psi_1\psi_2\psi_3D_1^3 - (\psi_1\psi_3(2-\psi_2)) +$$

$$\psi_1\psi_2(3-\psi_3))D_1^2 + (\psi_1\psi_3(2-\psi_2) + \psi_1\psi_2(3-\psi_3))D_1D_2 + (2-\psi_2)(3-\psi_3)c_1^3 + 4\psi_1(2-\psi_2)(3-\psi_3)B_1c_3$$

$$+2((\psi_1\psi_3(2-\psi_2) + \psi_1\psi_2(3-\psi_3)D_1^2 - (2\psi_1(2-\psi_2)(3-\psi_3)D_1 + 2(2-\psi_2)(3-\psi_3)D_2))c_1c_2\Big].$$

Therefore where *α,β,γ,δ and δ* are given by (2.2).
Let

$$d_1 = 4(3-\psi_3)\psi_1^2D_1, \qquad d_2 = \frac{2}{(3-\psi_3)}\left(\beta D_1^2 + 2\alpha D_2 - 2\gamma D_1\right), \qquad d_3 = -\frac{4\psi_1^2(4-\psi_4)(2-\psi_2)}{(3-\psi_3)}D_1,$$

$$d_4 = \frac{1}{(3-\psi_3)}\left(\delta D_1^3 + \gamma D_1 - 2\alpha D_2 + \psi_2(3-\psi_3)^2D_3 - \psi_1^2(4-\psi_4)(2-\psi_2)\frac{D_2^2}{D_1} - \beta D_1^2 + \beta D_1D_2\right)$$

(2.7)

and $T = \frac{D_1}{16(4-\psi_4)(3-\psi_3)(2-\psi_2)}$.

Then

$$\left|a_2a_4 - a_3^2\right| = T\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|. \quad (2.8)$$

Since the function $p(e^{i\theta}z)(\theta \in R)$ is in the class P for any $p \in P$, there is no loss of generality in assuming $c_1 > 0$. Write $c_1 = c, c \in [0,2]$. Substituting the values of c_2 and c_3 respectively from (1.5) and (1.6) in (2.8), we obtain

$$|a_2a_4 - a_3^2| = \frac{T}{4}\Big|c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4-c^2)(d_1 + d_2 + d_3)$$

$$+(4-c^2)x^2\left(-d_1c^2 + d_3(4-c^2)\right) + 2d_1c(4-c^2)(1-|x|^2z)\Big|.$$

Replacing $|x|$ by μ and substituting the values of d_1,d_2,d_3 and d_4 from (2.7) yield

$$|a_2a_4 - a_3^2| \le \frac{T}{4}\left[\frac{c^4}{(3-\psi_3)}\left|4\delta D_1^3 + 4(3-\psi_3)^2D_3 - 4\psi_1^2(4-\psi_4)(2-\psi_2)\frac{D_2^2}{D_1} + 4\beta D_1D_2\right| + \frac{2\mu c^2(4-c^2)}{(3-\psi_3)}\left(\alpha|D_2| + 2\beta D_1^2\right)\right.$$

$$\left.+\frac{4\mu^2(4-c^2)}{(3-\psi_3)}\left(\lambda D_1c^2 + 16\psi_1^2(4-\psi_4)(2-\psi_2)D_1\right) + 8(3-\psi_3)\psi_1^2D_1c(4-c^2)(1-\mu^2)\right].$$

$$= T\left[\frac{c^4}{4(3-\psi_3)}\left|4\delta D_1^3 + 4(3-\psi_3)^2D_3 - 4\psi_1^2(4-\psi_4)(2-\psi_2)\frac{D_2^2}{D_1} + 4\beta D_1D_2\right| + 2(3-\psi_3)\psi_1^2D_1c(4-c^2) + \frac{\mu c^2(4-c^2)}{(3-\psi_3)}\right.$$

$$\left.\left(\beta D_1^2 + 2\alpha|D_2|\right) + \frac{\mu^2(4-c^2)D_1}{(3-\psi_3)}\left(\gamma c^2 - 2(3-\psi_3)^2\psi_1^2c + 4(4-\psi_4)(2-\psi_2)\psi_1^2\right)\right].$$

$\equiv F(c,\mu)$, (2.9)

where *α,β,γ,δ and δ* are given by (2.2).

Again, differentiating $F(c,\mu)$ in(2.9) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\left[\frac{c^2}{(3-\psi_3)}(4-c^2)\left(\beta D_1^2 + 2\alpha|D_2|\right) + \frac{2\mu(4-c^2)D_1}{(3-\psi_3)}\left(\gamma c^2 - 2(3-\psi_3)^2\psi_1^2c + 4(4-\psi_4)(2-\psi_2)\psi_1^2\right)\right]$$

(2.10)

Then, for $0 < \mu < 1$, $0 < q < 1$ and for any fixed c with $0 < c < 2$, it is clear from (2.18) that $\frac{\partial F}{\partial \mu} > 0$, that is, $F(c,\mu)$ is an increasing function of μ. Hence, for fixed $c \in [0,2]$, the maximum of $F(c,\mu)$ occurs at $\mu = 1$, and
$maxF(c,\mu) = F(c,1) \equiv G(c)$.

Also note that

$$G(c) = T\left[\frac{c^4}{(3-\psi_3)}\left(\left|4\delta D_1^3 + 4\beta D_1D_2 + 4(3-\psi_3)^2D_3 - 4\psi_1^2(4-\psi_4)(2-\psi_2)(D_2^2/D_1)\right| - \beta D_1^2 - 2\alpha|D_2| - \gamma D_1\right)\right.$$

$$\left.+\frac{4c^2}{(3-\psi_3)}\left(\beta D_1^2 + 2\alpha|D_2| + \lambda D_1\right) + 16\frac{(4-\psi_4)(2-\psi_2)}{(3-\psi_3)}\psi_1^2D_1\right]$$

,

where $\lambda = (3-\psi_3)^2\psi_1^2 - 2(4-\psi_4)(2-\psi_2)\psi_1^2$, $\alpha, \beta, \gamma, \delta$ *and* δ are given by (2.2). Let

$$P = \frac{1}{(3-\psi_3)}\left(\left|4\delta D_1^3 + 4\beta D_1D_2 + 4(3-\psi_3)D_3 - 4\psi_1^2(4-\psi_4)(2-\psi_2)(D_2^2/D_1)\right| - \beta D_1^2 - 2\alpha|D_2| - \gamma D_1\right),$$

$$Q = \frac{4}{(3-\psi_3)}\left(\beta D_1^2 + 2\alpha|D_2| + \lambda D_1\right), \qquad R = 16\frac{\psi_1^2(4-\psi_4)(2-\psi_2)}{(3-\psi_3)}D_1.$$

(2.11)

Since

$$\max(Pt^2 + Qt + R) = \begin{cases} R, & Q \le 0, P \le \frac{-Q}{4}; \\ 16P + 4Q + R, & Q \ge 0, P \ge \frac{-Q}{8} or Q \le 0, P \ge \frac{-Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \le \frac{-Q}{8}; \end{cases} \quad (2.12)$$

where $0 \le t \le 4$.

Then we have

$$|a_2a_4 - a_3^2| \le \frac{D_1}{16(4-\psi_4)(3-\psi_3)(2-\psi_2)}\begin{cases} R, & Q \le 0, P \le \frac{-Y}{4}; \\ 16P + 4Q + R, & Q \ge 0, X \ge \frac{-Q}{8} \text{ or } Q \le 0, P \ge \frac{-Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \le \frac{-Q}{8}; \end{cases}$$

where P,$Q$and R are given by (2.19).

Remark 1

- As $q \to 1$ Theorem 2.2 reduces to Theorem 1 in [10].
- As $q \to 1$ and $B_1 = B_2 = B_3 = 2$, Theorem 2.2 reduces to Theorem 3.1 in [7].

Definition 2.3. *Let $\phi : U \to C$ be analytic, and let $\phi(z)$ be given as in* (2.1). *The class $K^{j,k}(\phi)$ of (j,k)-convex symmetrical functions with respect to ϕ consists of functions f satisfying the subordination*

$$1 + \frac{zf''(z)}{f'_{j,k}(z)} \prec \varphi(z).$$

Theorem 2.4. *Let the function $f \in K^{j,k}(\phi)$ be given by* (1.1).

1. If D_1,D_2 and D_3 satisfy the conditions $\left(\frac{6(\psi_3+2\psi_2)-16}{2}\right)D_1^2\psi_1^2 + 4|D_2|\psi_1^2 - 2D_1\psi_1^2 \le 0$ *and*

$$\left|6D_1D_3\psi_1^2 + \left(3\psi_1^2(2\psi_1 + \psi_3) - 8\psi_1^2\psi_2\right)D_1^2D_2 + \left(3\psi_1^2\psi_2\psi_3 - 4\psi_1^2\psi_2\right)D_1^4 + 2(7\psi_2 - 3\psi_3 - 4)\psi_1^2D_1^3 - 6D_2^2\psi_1^2\right| - 4D_1^2\psi_1^2 \le 0.$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \le \frac{D_1^2}{36}\psi_1^2.$$

2. If D_1,D_2 and D_3 satisfy the conditions

$$\left(\frac{6(\psi_3+2\psi_2)-16}{2}\right) D_1^2\psi_1^2 + 4|D_2|\psi_1^2 - 2D_1\psi_1^2 \geq 0$$ *and*

$$2\left|6D_1D_3\psi_1^2 + \left(3\psi_1^2(2\psi_1+\psi_3) - 8\psi_1^2\psi_2\right)D_1^2D_2 + \left(3\psi_1^2\psi_2\psi_3 - 4\psi_1^2\psi_2\right)D_1^4 + 2(7\psi_2 - 3\psi_3 - 4)\psi_1^2D_1^3 - bD_2^2\psi_1^2\right|$$
$$-\left(\frac{6(\psi_3+2\psi_2)-16}{2}\right)D_1^3\psi_1^2 - 4D_1|D_2|\psi_1^2 - 6D_1^2\psi_1^2 \geq 0,$$

or the conditions

$$\left(\frac{6(\psi_3+2\psi_2)-16}{2}\right) D_1^2\psi_1^2 + 4|D_2|\psi_1^2 - 2D_1\psi_1^2 \leq 0$$ *and*

$$\left|6D_1D_3\psi_1^2 + \left(3\psi_1^2(2\psi_1+\psi_3) - 8\psi_1^2\psi_2\right)D_1^2D_2 + \left(3\psi_1^2\psi_2\psi_3 - 4\psi_1^2\psi_2\right)D_1^4 + 2(7\psi_2 - 3\psi_3 - 4)\psi_1^2D_1^3 - D_2^2\psi_1^2\right| - 4D_1^2\psi_1^2 \leq 0.$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{D_1}{144}\left|6D_1D_3\psi_1^2 + \left(3\psi_1^2(2\psi_1+\psi_3) - 8\psi_1^2\psi_2\right)D_1^2D_2 + \left(3\psi_1^2\psi_2\psi_3 - 4\psi_1^2\psi_2\right)D_1^4 + 2(7\psi_2 - 3\psi_3 - 4)\psi_1^2D_1^3 - 4D_2^2\psi_1^2\right|.$$

3. If D_1,D_2 and D_3 satisfy the conditions

$$\left(\frac{6(\psi_3+2\psi_2)-16}{2}\right) D_1^2\psi_1^2 + 4|D_2|\psi_1^2 - 2D_1\psi_1^2 > 0$$ *and*

$$2\left|6D_1D_3\psi_1^2 + \left(3\psi_1^2(2\psi_1+\psi_3) - 8\psi_1^2\psi_2\right)D_1^2D_2 + \left(3\psi_1^2\psi_2\psi_3 - 4\psi_1^2\psi_2\right)D_1^4 + 2(7\psi_2 - 3\psi_3 - 4)\psi_1^2D_1^3 - bD_2^2\psi_1^2\right|$$
$$-\left(\frac{6(\psi_3+2\psi_2)-16}{2}\right)D_1^3\psi_1^2 - 4D_1|D_2|\psi_1^2 - 6D_1^2\psi_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{D_1^2\psi_1^2}{576}\,\frac{\begin{pmatrix} 16\left|6D_1D_3 + \gamma D_1^2D_2 + \beta D_1^4 + \delta D_1^3 - 4D_2^2\right| \\ -12\alpha D_1^3 - 48D_1|D_2| \\ -36D_1^2 - \alpha D_1^4 - 8\alpha D_1^2|D_2| - 16D_2^2 \end{pmatrix}}{\left|6D_1D_3 + \gamma D_1^2D_2 + \beta D_1^4 + \delta D_1^3 - 4D_2^2\right| - \alpha D_1^3 - 4D_1|D_2| - 2D_1}$$

where

$$\alpha = \left(\frac{6(\psi_3+2\psi_2)-16}{2}\right), \qquad \beta = (3\psi_2\psi_3 - 4\psi_2),$$

$\gamma = (3(2\psi_1 + \psi_3) - 8\psi_2)$,

$\delta = 2(7\psi_2 - 3\psi_3 - 4)$. (2.13)

Proof. Since $f \in K^{j,k}(\phi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$1 + \frac{zf''(z)}{f'_{j,k}(z)} = \varphi(w(z)) = 1 + \frac{1}{2}D_1c_1z + \left(\frac{1}{2}D_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}D_2c_1^2\right)z^2 + \ldots \quad (2.14)$$

so that

$$\left(\frac{2a_2z + 6a_3z^2 + 12a_4z^3 + 20a_5z^4 + \ldots}{\psi_1 + 2\psi_2a_2z + 3\psi_3a_3z^2 + 4\psi_4a_4z^3 + \ldots}\right) = 1 + \frac{1}{2}D_1c_1z + \left(\frac{1}{2}D_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}D_2c_1^2\right)z^2 + \ldots$$

which implies the equality

$$2a_2z + 6a_3z^2 + 12a_4z^3 + 20a_5z^4 + \ldots) = \left(1 + \frac{1}{2}D_1c_1\psi_1\right)z + \left(a_2D_1c_1\psi_2 + \psi_1\left(\frac{1}{2}D_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}D_2c_1^2\right)\right)z^2$$
$$+\left(\frac{3}{2}a_3D_1c_1\psi_3 + \psi_2a_2\left(\frac{1}{2}D_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}D_2c_1^2\right) + \psi_1\left(D_1\left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right) + D_2c_1\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) + \frac{D_3c_1^3}{8}\right)\right)z^3 + \ldots$$

Equating the coefficients on both sides we have

$$a_2 = \frac{D_1c_1\psi_1}{4},$$

$$a_3 = \frac{1}{24}\left(\left(D_1^2\psi_2 - D_1 + D_2\right)c_1^2\psi_1 + 2D_1c_2\psi_1\right),$$

$$a_4 = \frac{\psi_1}{192}\left[\left(-4D_2 + 2D_1 + D_1^3\psi_2\psi_3 - 3D_1^2\psi_3 + (2\psi_1 + \psi_3)D_1D_2 + 2D_3\right)c_1^3 + 2\left((\psi_3 + 2\psi_2)D_1^2 - 4D_1 + 4D_2\right)c_1c_2 + 8D_1c_3\right].$$

Therefore

$$a_2a_4 - a_3^2 = \frac{D_1\psi_1^2}{768}\left[c_1^4\left(\frac{-4}{3}D_2 + \frac{2}{3}D_1 + \frac{D_1^3}{3}\beta - \frac{D_1^2}{3}(9\psi_3 - 8\psi_2) + \frac{D_1D_2}{3}\gamma + 2D_3 - \frac{4D_2^2}{3D_1}\right) + \frac{2c_1^2c_2}{3}\left(\alpha D_1^2 - 4D_1 + 4D_2\right) + 8D_1c_1c_3 - \frac{16}{3}D_1c_2^2\right].$$

where

α,β,γ are given by (2.13) By writing

$$d_1 = 8D_1, \quad d_2 = \frac{2}{3}\left(2D_1^2 - 4D_1 + D_2\right), \quad d_3 = \frac{-16D_1}{3},$$

$$d_4 = \left(\frac{-4}{3}D_2 + \frac{2}{3}D_1 + XD_1^3 - (9\psi_3 - 8\psi_2)\frac{D_1^2}{3} + \frac{Y}{3}D_1D_2 + 2D_3 - 4\frac{D_2^2}{3D_1}\right) \quad (2.15)$$

$T = \frac{D_1}{768}$.

we have

$$\left|a_2a_4 - a_3^2\right| = T\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right| \quad (2.16)$$

Since the function $p(e^{i\theta}z)(\theta \in R)$ is in the class P for any $p \in$ P, there is no loss of generality in assuming $c_1 > 0$. Write $c_1 = c, c \in [0,2]$. Substituting the values of c_2 and c_3 respectively from (1.5) and (1.6) in (2.16), we obtain

$$|a_2a_4 - a_3^2| = \frac{T}{4}\left|c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4-c^2)(d_1 + d_2 + d_3)\right.$$
$$\left.+(4-c^2)x^2\left(-d_1c^2 + d_3(4-c^2)\right) + 2d_1c(4-c^2)(1-|x|^2z)\right|.$$

Replacing $|x|$ by μ and substituting the values of d_1,d_2,d_3 and d_4 from (2.15) yield

$$|a_2a_4 - a_3^2| \leq \frac{T(\psi_1^2)}{4}\left[c^4\left|\frac{4}{3}\beta D_1^3 + \frac{4}{3}\gamma D_1D_2 + 8D_3 - \frac{16D_2^2}{3D_1} + \frac{4}{3}\delta B_1^2\right| + \frac{4\mu c^2(4-c^2)}{3}\alpha D_1^2 + \frac{8}{3}|D_2|\right.$$
$$\left.+\mu^2(4-c^2)\left(\frac{8}{3}D_1c^2 + \frac{64}{3}D_1\right) + 16D_1c(4-c^2)(1-\mu^2)\right].$$
$$= T\left[\frac{c^4}{3}\left|\beta D_1^3 + \gamma D_1D_2 + 6D_3 - 4(D_2^2/D_1) + \delta D_1^2\right| + 4D_1c(4-c^2) + \frac{\mu c^2(4-c^2)}{3}\left(\alpha D_1^2 + 4|D_2|\right) + \frac{2D_1}{3}\right.$$
$$\left.\frac{\mu^2(4-c^2)D_1}{3}((c-2)(c-4))\right]$$

.

$\equiv F(c,\mu)$, (2.17)

where α,β,γ *and* δ are given by (2.13).

Again, differentiating $F(c,\mu)$ in(2.17) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\left[\frac{c^2}{3}(4-c^2)\left(\alpha D_1^2 + 4|D_2|\right) + \frac{2\mu(4-c^2)D_1}{3}((c-2)(c-4))\right]. \quad (2.18)$$

Then, for $0 < \mu < 1$, $0 < q < 1$ and for any fixed c with $0 < c < 2$, it is clear from (2.18) that $\frac{\partial F}{\partial \mu} > 0$, that is, $F(c,\mu)$ is an increasing function of μ. Hence, for fixed $c \in [0,2]$, the maximum of $F(c,\mu)$ occurs at $\mu = 1$, and

$maxF(c,\mu) = F(c,1) \equiv G(c)$.

Also note that

$$G(c) = T\left[\frac{c^4}{3}\left(\left|\beta D_1^3 + \gamma D_1D_2 + 6D_3 - 4(D_2^2/D_1) + \delta D_1^2\right| - \alpha D_1^2 - 4|D_2| - 2D_1\right)\right.$$
$$\left.+\frac{4c^2}{3}\left(\alpha D_1^2 + 4|D_2| - 2D_1\right) + \frac{64}{3}D_1\right],$$

where α,β,γ *and* δ are given by (2.13). Let

$$P = \frac{1}{3}\left(\left|\beta D_1^3 + \gamma D_1D_2 + 6D_3 - 4(D_2^2/D_1) + \delta D_1^2\right| - \alpha D_1^2 - 4|D_2| - 2D_1\right)$$

$$Q = \frac{4}{3}\left(\alpha D_1^2 + 4|D_2| - 2D_1\right), \qquad R = \frac{64}{3}D_1.$$,

(2.19)

By using (2.12) we get

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{768}\begin{cases} Z, & Q \leq 0, P \leq \frac{-Q}{4}; \\ 16X + 4Y + Z, & Q \geq 0, X \geq \frac{-Q}{8} \text{ or } Q \leq 0, P \geq \frac{-Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq \frac{-Q}{8}; \end{cases}$$

Where, X,Y and Z are given by (2.19).

Remark 2:

- As $q \to 1$ Theorem 2.4 reduces to Theorem 2 in [10].
- As $q \to 1$ for the choice $\varphi(z) = ((1+z)/(1-z))$, Theorem 2.4 reduces to Theorem 3.2 in [7].

III. CONCLUSION

We study the estimates for the Second Hankel determinant of analytic functions. Our class includes (j,k)-convex, (j,k)-starlike functions and Ma-Minda starlike and convex functions..

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