Hop Excellence of Paths

P. Getchial Pon Packiavathi, Balamurugan, R.B. Gnana Jothi

Abstract: *Excellent graphs introduced by[4] Dr.N. Sridharan and M.Yamuna, had been the motivation for many researchers and so many papers have been written as a result of deep and vast study of excellent graphs. [3] deals with a new type of excellence, namely hop excellence, based on hop domination closely related with distance-2 domination. A graph G is said to be hop excellent if every vertex* $u \in V$ *belongs to a minimal hop domination set* $(\gamma_h - set)$ *of G. This paper aims to study hop excellence of paths.*

Index Terms: *Dominating Set, Excellent Graphs and Hop Dominating Set..*

I. INTRODUCTION

In 2015, C.Natarajan and S.K. Ayyaswamy [1] introduced a new distance related domination parameter called the hop domination in graphs. A subset $S \subset V$ of a graph G is a hop dominating set of G if for every $v \in V - S$, there exists $u \in S$ such that d(u, v) = 2. The minimum cardinality of a hop dominating set of G is called the hop domination number and is denoted by $\gamma_h(G)$. In [4] N. Sridharan and M. Yamuna introduced Excellent - Just Excellent - Very Excellent Graphs in 1980. A graph is said to be excellent if given any vertex x then there is a γ-set of G containing x. A graph G is said to be just excellent if to each $u \in V$, there is a unique γ-set of G containing u. An excellent graph G is said to be very excellent, if there is a γ-set S of G such that to each vertex $u \in V - S$, there exist a vertex $v \in S$ such that $(S - v) \cup \{u\}$ is a γ-set of G.A γ-set S of G satisfying this property is called a very excellent γ-set of G.

In [3] we introduced a new type of excellence. A graph G is said to be hop excellent if every vertex $u \in V$ belongs to a minimal hop dominating set $(\gamma_h - set)$ of G. Some families of hop excellent graphs have been identified and some elementary properties of hop excellence are dealt with.

II. MAIN RESULTS

Theorem :2.1

P_n, the path on n vertices is not hop excellent when $n \equiv 0,1,3,5 \ (mod\ 6)$.

Proof:

$$\gamma_h(P_n) = \begin{cases} 2r & if\ n = 6r \\ 2r+1 & if\ n = 6r+1 \\ 2r+2 & if\ n = 6r+s, 2 \le s \le 5 \end{cases}$$

[1]

Let $V(P_n) = \{v_1, v_2, \ldots.. v_n\}$ and

E(P_n) $= \{v_i v_{i+1} / \ 1 \le i \le n-1\}$

Case 1:

n $\equiv 0(mod\ 6)$

Let n = 6r

Then $\gamma_h(P_n)$ = $2r = \frac{6r}{3}$

Each vertex in a γ_h-set should hop dominate two vertices apart from it and no two vertices in

γ_h-set can hop dominate same vertices.

Hence pendant vertices as well as support vertices cannot appear in any γ_h-set.

Therefore Path with n $\equiv 0(mod 6)$ are not hop excellent.

Case 2:

$n \equiv 5$ (mod 6)

Let n = 6r+5; Then γ_h = 2r+2.

6r+ 5 = (2r+1)3 + 2

Let D be a $\gamma_h -$ set.

At most one vertex from $\{v_1, v_2, v_{n-1}, v_n\}$ can be present in D.

Suppose $v_1 \in$ D, $v_2 \notin$ D

v_1 hop dominates v_3,

$v_2 \notin$ D $\Rightarrow$ $v_4 \in$ D.

v_3 is already hop dominated. If at all $v_5 \in$ D, it hop dominates v_5 *and* v_7 only.

Hence $v_5 \notin$ D.

v_4 hop dominates v_6,

$v_3, v_5 \notin$ D $\Rightarrow v_7 \in$ D.

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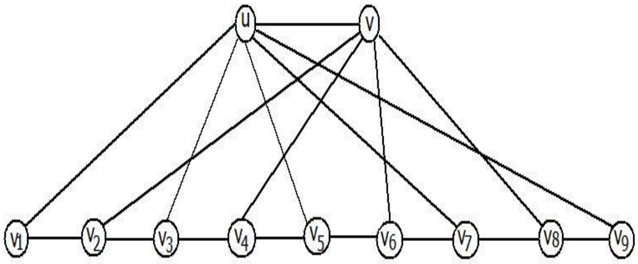
Proceeding like this $v_{3k+1} \in$ D for k = 0, 1, 2….
$v_{n-1} = v_{6r+4} = v_{3(2r+1)+1} \in$ D
$\Rightarrow\Leftarrow$ ($\because$ $v_{n-1} \notin$ D)
Hence path with $n \equiv 5$ (mod 6) is not hop excellent.
Case 3:
n $\equiv 1$ (mod 6)
Let n = 6r+1 ; Then $\gamma_h(P_n)$= 2r+1
$P_n = v_1 v_2 \dots\dots v_{6r+1}$
Note that any hop dominating set of P_n should contain at least two vertices from $\{v_1, v_2, v_3, v_4\}$.
Let D be any γ_h-set of P_n
$v_1, v_3 \in D \Rightarrow D \setminus v_1$ is also a hop dominating set.
$\Rightarrow\Leftarrow$
Similarly v_2, v_4 can not appear together in D.
Claim:
v_2 is a bad vertex
Suppose $v_1, v_2 \in D$. Then $v_3, v_4 \notin D$
Let $D' = D \setminus \{v_1, v_2\}$
$|D'| = |D| - 2 = 2r - 1$
Consider of $P_n' = v_5 v_6 \dots\dots v_{6r+1}$
Clearly D' is a hop dominating set of P_n'.

$$\Rightarrow\Leftarrow \left(\begin{array}{l} \because l(P_n') = 6r+1-4 = 6r-3 \equiv 3 (mod\ 6) \\ \Rightarrow \gamma_h(P_n') = 2(r-1)+2 = 2r \end{array} \right)$$

Suppose $v_2, v_3 \in D. Then\ v_1, v_4 \notin D$
Let $D' = D \setminus \{v_2, v_3\}$
$|D'| = |D| - 2 = 2r - 1$
If $v_5 \in D'$, then D' is a hop dominating set of P_n'= $v_5 v_6 \dots\dots v_{6r+1}$
$\Rightarrow\Leftarrow$
If $v_5 \notin D'$, D' is a hop dominating set of P_n''= $v_6 v_7 \dots\dots v_{6r+1}$
$l(P_n'') = 6r + 1 - 5 = 6r - 4 = 6(r-1) + 2$
$\Rightarrow \gamma_h(P_n'') = 2(r-1) + 2 = 2r$
$\Rightarrow\Leftarrow$
Therefore v_2 can not appear in any $\gamma_h -$ set and so v_2 is a bad vertex.
Hence path with $n \equiv 1$ (mod 6) is not hop excellent.
Case 4:
$n \equiv 3$ (mod 6)
Let n = 6r+3; Then $\gamma_h(P_n)$ = 2r+2.
6r+ 3 = (2r-1)3+ (3 x 2)
Claim:
v_5 is a bad vertex
If possible let D be a $\gamma_h -$ set of P_n containing v_5.
Note that D should contain at least two vertices from $\{v_1, v_2, v_3, v_4\}$
Case 1:
D contains two vertices from $\{v_1, v_2, v_3\}$
Consider D' = D $\setminus \{v_1, v_2, v_3\}$
Then $|D'|$ = $|D|$ – 2
Since $v_5 \in D$, D' should be a $\gamma_h -$ set of P_n' = $v_5 v_6 \dots\dots v_{6r+3}$
$l(P_n') =$ 6r + 3 – 4
= 6r – 1
$\equiv$ 5 (mod 6)
By case 2, for a path P_n with $l(P_n) \equiv$ 5 (mod 6), pendant vertex is a bad vertex.
So $v_5 \notin D$
$\Rightarrow\Leftarrow$
Case 2:
Suppose $v_1, v_4 \in D$, Then $v_3, v_2 \notin D$.
Subcase 1:
$v_6 \in D$ or $v_8 \in D$
Let D' = D $\setminus \{v_1, v_4\}$
$|D'|$ = 2r
D' is a hop dominating set of P_n' = $v_5 v_6 \dots\dots v_{6r+3}$
$l(P_n') =$ 6r + 3 – 4
= 6r – 1
$\equiv$ 5 (mod 6)
$\gamma_h(P_n')$ = 2(r-1)+2 =2r
Hence D' is a $\gamma_h -$set of P_n'
By case 2, pendant vertex of P_n' cannot be in D'
. $\Rightarrow\Leftarrow (\because v_5 \in D')$
Subcase 2:
$v_6 \notin D$ or $v_8 \notin D$, then $v_{10} \in D$
If $v_7 \in D, D''$ = D $\setminus \{v_1, v_4, v_5\}$
$|D''|$ = 2r-1
D'' is a hop dominating set of P_n'' = $v_7 v_8 \dots\dots v_{6r+3}$
$l(P_n'') =$ 6r + 3 – 6
= 6r – 3
= 6(r-1)+3
$\equiv$ 3 (mod 6)
$\gamma_h(P_n'')$ = 2(r-1)+2 =2r
$\Rightarrow\Leftarrow (\because |D''| = 2r - 1)$
If $v_7 \notin D$ and $v_9 \notin D$,
D'' is a hop dominating set of
P_n''' = $v_9 v_{10} \dots\dots v_{6r+3}$
$l(P_n''') =$ 6r + 3 – 8
= 6r – 5
= 6(r-1)+1
$\equiv$ 1 (mod 6)
$\gamma_h(P_n''')$= 2(r-1)+1 =2r-1
Hence D'' is a $\gamma_h -$set of P_n'''.

By Case 3: support vertex cannot be in any $\gamma_h -$set .
$\therefore\ v_{10} \notin D''$
$\Rightarrow\Leftarrow (\because v_{10} \in D'')$
Hence path with $n \equiv 3$ (mod 6) is not hop excellent.
Theorem : 2.2
P_n, the path on n vertices is hop excellent when $n \equiv 2\ (mod\ 6)$.
Proof:
n $\equiv$ 2 (mod 6)
Let n = 6r +2, then $\gamma_h(P_n)$ = 2r +2.
Let $P_n = v_1 v_2 \dots \dots v_{6r+2}$ and $P_n' = v_5 v_6 \dots \dots v_{6r+2}$
$l(P_n') =$ 6r-2 = 6(r-1) +4 $\equiv$ 4 (mod 6).
It has been proved that in [3], P_n is hop excellent when $n \equiv$ 4(mod 6).
P_n' is hop excellent and $\gamma_h(P_n')$ = 2(r-1)+2 = 2r.
Let H be any $\gamma_h -$ set of P_n'.
H$\cup \{v_1, v_4\}$ and H $\cup \{v_2, v_3\}$ are $\gamma_h -$ sets of P_n.
($\because\ \gamma_h(P_n)$ = 2r +2)
Hence path with $n \equiv 2$ (mod 6) is hop excellent.
Theorem: 2.3
Any path can be embedded in a hop excellent graph.
Proof :
Let P_n be a path of n vertices with $V(P_n) = \{v_1, v_2, \dots\dots v_n\}$ and
E(P_n) = $\{v_i v_{i+1} / 1 \leq i \leq n-1\}$.
Now construct a graph H as follows,
Let {u ,v} be a set disjoint with $V(P_n)$.
Let V(H) = $V(P_n) \cup \{u, v\}$ and E(H) =
E(P_n) $\cup \{uv_i / for\ odd\ i\} \cup \{vv_i / for\ even\ i\}$.
Clearly P_n is an induced subgraph of H.
For $1 \leq i \leq n, d(u, v_i) = \begin{cases} 1 & if\ i\ is\ odd \\ 2 & if\ i\ is\ even \end{cases}$
and $d(v, v_i) = \begin{cases} 1 & if\ i\ is\ even \\ 2 & if\ i\ is\ odd \end{cases}$
Note that {u, v} is a hop dominating set of H .
So the vertices u and v are hop excellent.
For any odd i, $\{v_i, u\}$ is a γ_h set of H.
For any even i, $\{v_i, v\}$ is a γ_h set of H.
Hence any P_n is hop excellent graph.
Therefore H is hop excellent.

Ilustration: H_9



In H_9, The γ_h - sets are
{u ,v},
$\{v_1, u\}, \{v_3, u\}, \{v_5, u\}, \{v_7, u\}, \{v_9, u\}, \{v_2, v\}, \{v_4, v\}, \{v_6, v\}, \{v_8, v\}$.

III. RESULTS & CONCLUSION

Paths have been completely categorized according to their behavior of being hop excellent. Embedding property has also been studied for paths. This work can be extended to some other families of graphs resulting to generalization of these results.

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