

Modeling and Simulation of Vibrations of Non-Homogeneous Annular Plate of Quadratic Thickness Resting on Elastic Foundation

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Abstract: *Mathematical modeling is presented to analyze natural frequencies of vibrations of an isotropic annular plate of quadratic varying thickness resting on Winkler type elastic foundation where numerical simulation is carried out using quintic spline technique for three different combinations of edge conditions. Effect of elastic foundation, together with non-homogeneity variation, on the natural frequencies of vibration is illustrated for variety of thickness variation for the first three modes. To compare parametric effect on a specific plate, transverse displacements are presented in normalized form. Accuracy of the results and validity of numerical method is demonstrated by comparing the existing results in the literature.*

Keywords: *Quintic Spline, Elastic Foundation, Non-Homogeneity, Annular plate, Quadratic Thickness*

I. INTRODUCTION

Study of elastic foundation, non-homogeneity and/or solidity, and thickness variation on dynamic response of annular plates has immense importance due to its application in the conception and structure of machinery parts, thus making it necessary to research in this area which will help to improve the design of diaphragms of steam turbines, cylinder heads and piston heads in particular. In practical, modeling of the effect of foundation on structure is quite complex as phenomenon of actual response, and interaction between the system and foundation both are complicated to understand. However, Winkler, Pasternak and Vlasov, etc. are few proposed different types of foundation which approximate the actual behaviour of foundation. In this endeavor Bhattacharya [1] considered triangular plates with Vlasov foundation and Wang and Stephens [2] carried out their findings using Timoshenko beam resting on Pasternak foundation.

Many instances in real world such as highways with reinforced concrete pavements, building's foundation slabs and airport runways, etc., prove Winkler type foundation as more prominent foundation which is actually based on the virtual replacement of real foundation by a progression of vertical springs exclusive of shear interaction under the theory that it manifests the proportional deflection at each point. Many researchers (in 1970's and 1980's) started considering Winkler type foundation in their work, few are mentioned. Chonan [3] considered rectangular plates to analyze the natural frequencies while plates are assumed on 'Winkler elastic foundation'.

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Gupta et al., [4] studied the consequence of "Winkler" type foundation on vibrations of annular plate where the thickness is varying linearly, and Tomar et al. [5] carried their research on circular plates and studied the behavior of frequencies and frequency modes of free vibration with the effect of a Winkler type foundation.

Due to technical advancement where fibre-reinforced materials are highly used to maintain the flexibility under any possible operation, for instance use of diaphragms formed by different materials in pressure capsules, switches etc, it becomes necessary to examine the consequences of non-homogeneity on the vibrational behavior of annular plates. Non-homogeneity is also considered by the researchers in various ways in different models [6-10]. The presented work here deals with exponential variation of nonhomogeneity under the consideration of parameter μ in radial direction which is different from the density parameter η in the same direction. This provides a justified model as compared to the models where nonhomogeneity for Young's modulus variation $E = E_0 e^{\mu x}$, and density variation $\rho = \rho_0 e^{\eta x}$ are assumed in similar conduct.

Thus under all above consideration, a "mathematical model is formulated to study the vibrational behaviour of a non homogeneous annular plate of quadratic varying thickness which is resting on a Winkler type elastic foundation using quintic spline interpolation technique which provides the numerical simulation under three edge conditions (at inner and outer radii) namely (C-C), (C-F), and (C-S), here the symbols C, F and S stand for clamped, free and simply supported respectively".

II. METHODOLOGY

Considering an isotropic non-homogeneous annular plate (inner and outer radii b and a) of thickness $h(r)$, referred to a system of cylindrical coordinates (r, θ, z) . Then the mathematical model -fourth order partial differential equation- represents the phenomenon is as follows:

$$D \frac{\partial^4 w}{\partial r^4} + \frac{2}{r} \left[D + r \frac{\partial D}{\partial r} \right] \frac{\partial^3 w}{\partial r^3} + \frac{1}{r^2} \left[-D + r(2 + \nu) \frac{\partial D}{\partial r} + r^2 \frac{\partial^2 D}{\partial r^2} \right] \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3} \left[D - r \frac{\partial D}{\partial r} + r^2 \nu \frac{\partial^2 D}{\partial r^2} \right] \frac{\partial w}{\partial r} + \rho h \frac{\partial^2 w}{\partial t^2} + K_f w = 0, \quad (1)$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$, D , w , $E(r)$ and K_f are flexural rigidity, transverse deflection, Young's modulus and foundation constant, respectively.

To introduce non-homogeneity in plate material, let us assume that E and ρ are functions of space variable r . Now equation (1) becomes

$$\begin{aligned}
 & Eh^3 \frac{\partial^4 w}{\partial r^4} + \frac{2}{r} \left[Eh^3 + r \left(h^3 \frac{dE}{dr} + 3Eh^2 \frac{dh}{dr} \right) \right] \frac{\partial^3 w}{\partial r^3} \\
 & + \frac{1}{r^2} \left[-Eh^3 + r(2+\nu) \left(h^3 \frac{dE}{dr} + 3Eh^2 \frac{dh}{dr} \right) \right. \\
 & \left. + r^2 \left(h^3 \frac{d^2 E}{dr^2} + 6h^2 \frac{dE}{dr} \frac{dh}{dr} + 3E \left(2h \left(\frac{dh}{dr} \right)^2 + h^2 \frac{d^2 h}{dr^2} \right) \right) \right] \frac{\partial^2 w}{\partial r^2} \\
 & + \frac{1}{r^3} \left[Eh^3 - r \left(h^3 \frac{dE}{dr} + 3Eh^2 \frac{dh}{dr} \right) + r^2 \nu \left(h^3 \frac{d^2 E}{dr^2} + 6h^2 \frac{dE}{dr} \frac{dh}{dr} \right. \right. \\
 & \left. \left. + 3E \left(2h \left(\frac{dh}{dr} \right)^2 + h^2 \frac{d^2 h}{dr^2} \right) \right) \right] \frac{\partial w}{\partial r} \\
 & + 12\rho h(1-\nu) \frac{\partial^2 w}{\partial t^2} + K_f w = 0.
 \end{aligned} \tag{2}$$

Using $x = \frac{r}{a}, \bar{w} = \frac{w}{a}, \bar{h} = \frac{h}{a}$ equation (2) is non-dimensionalise and thickness quadratic variation is expressed as, $\bar{h} = h_0(1 + \alpha x + \beta x^2)$,

such that $|\alpha| \leq 1, |\beta| \leq 1$ and $\alpha + \beta > -1$, where h_0 is the thickness at inner edge, taper parameters are represented by α and β . Material non-homogeneity is assumed as

$$E = E_0 e^{\mu x}, \quad \rho = \rho_0 e^{\eta x},$$

where, μ and η are non-homogeneity parameters and ρ_0 is the density and E_0 is Young's Modulus at the inner edge.

$$\bar{w}(x, t) = W(x) e^{i\xi t}, \tag{3}$$

where ξ , radian frequency, provides harmonic motion and lead Eq. (2) to a liner ordinary differential equation

$$F_0 \frac{d^4 W}{dx^4} + F_1 \frac{d^3 W}{dx^3} + F_2 \frac{d^2 W}{dx^2} + F_3 \frac{dW}{dx} + F_4 W = 0, \tag{4}$$

where,

$$\begin{aligned}
 F_0 &= 1, \\
 F_1 &= 2(1 + Qx) / x, \\
 F_2 &= Q^2 + R + \{(2 + \nu)Q / x\} - \{1 / x^2\}, \\
 F_3 &= \{(1 - Qx) / x^3\} + \{\nu(Q^2 + R) / x\}, \\
 F_4 &= -\left\{ \Omega^2 e^{(\eta - \mu)x} / P^2 \right\} + \left\{ (12(1 - \nu^2) E_f / e^{\mu x} A^3 h_0^3) \right\},
 \end{aligned}$$

$$\Omega^2 = 12\rho_0 a^2 \xi^2 (1 - \nu^2) / E_0 h_0^2,$$

$$E_f = K_f a / E_0,$$

$$P = 1 + \alpha x + \beta x^2, \quad Q = \mu + \{(3\alpha + 6\beta x) / P\},$$

$$R = (6\beta - 3\alpha^2 - 6\beta^2 x^2 - 6\alpha\beta x) / P^2.$$

Eq.(4), which is quite complex to solve as it involves several plate parameters and thus it necessities to find a numerical

solution. Therefore further study is carried out using spline interpolation technique along with computer programming in order to get the results of desired accuracy. To make it possible boundary condition are imposed at the inner edge of annular plate, $X = \epsilon$, and at the outer edge $X = 1$.

In this method,

$$\bar{W}(X) = \sum_{i=0}^4 a_i (X - X_0)^i + \sum_{j=0}^{n-1} b_j (X - X_j)^5, \tag{5}$$

$$\text{where } (X - X_j)_* = \begin{cases} 0, & X \leq X_j \\ X - X_j, & X > X_j \end{cases},$$

A shape function and its first four derivatives resembles $W(x)$ and respective derivatives because $\bar{W}(X)$ so approximately represent the deflection function for free vibration, in the domain $[\epsilon, 1]$, where $\Delta X = (1 - \epsilon) / n$, and

$$X_k = k \Delta X, \quad (k = 0(1)n).$$

Thus eq.(5) helps us to reduce equation (4) in the form of

$$\begin{aligned}
 & F_4 a_0 + [F_4 (X_k - X_0) + F_3] a_1 + [F_4 (X_k - X_0)^2 \\
 & + 2F_3 (X_k - X_0) + 2F_2] a_2 + [F_4 (X_k - X_0)^3 + 3F_3 (X_k - X_0)^2 \\
 & + 6F_2 (X_k - X_0) + 6F_1] a_3 + [F_4 (X_k - X_0)^4 + 4F_3 (X_k - X_0)^3 \\
 & + 12F_2 (X_k - X_0)^2 + 24F_1 (X_k - X_0) + 24F_0] a_4 \\
 & + \sum_{j=0}^{n-1} \left\{ F_4 (X_k - X_0)_*^5 + F_3 (X_k - X_0)_*^4 + 20F_2 (X_k - X_0)_*^3 \right. \\
 & \left. + 60F_1 (X_k - X_0)_*^2 + 120F_0 (X_k - X_0)_* \right\} b_j = 0.
 \end{aligned} \tag{6}$$

Eq. (6) generates undetermined homogeneous system which can be express in a matrix representation

$$[A][B] = [0] \tag{7}$$

where, "A" is coefficient matrix of dimension $(n+1) \times (n+5)$ along with matrix "B" which is a column matrix contains $n+5$ unknowns (a_i 's and b_j 's).

For complete specification of mathematical model discussed above, appropriate boundary conditions must be imposed. Therefore, three cases of boundary conditions namely clamped-clamped, clamped-simply supported and clamped-free edge are considered.

Applying boundary conditionson both the edges using the expression $w = \frac{dW}{dX} = 0$; $w = \frac{d^2 W}{dX^2} + (\nu / X) \frac{dW}{dX} = 0$; and

$$\frac{d^2 W}{dX^2} + (\nu / X) \frac{dW}{dX} = \frac{d^3 W}{dX^3} + (1 / X) \frac{d^2 W}{dX^2} - (1 / X^2) \frac{dW}{dX} = 0$$

for clamped, simply supported and free edge, respectively, four additional equations will emerge with $(n+5)$ unknowns for different set of boundary and can be expressed via matrix representation as

$$[C][B] = [0] \tag{8}$$

Thus the Eq. (7) and Eq.(8) collectively produce a consistent system

$$\begin{bmatrix} A \\ C \end{bmatrix} [B] = [0]. \tag{9}$$

Eq. (9), offers a non-trivial solution when the characteristic determinant of above system vanishes, i.e.



$$\frac{A}{C} = 0 \quad (10)$$

III. RESULTS AND DISCUSSION

Equation (10) provides three different sets of frequency equations, for different choice of boundary conditions, to get the “values of frequency parameter Ω ” under a choice of plate parameters. Numerical simulation of vibrations of annular plates with three particular thickness variation $\alpha = -0.5; \beta = -0.1; \alpha = 0.0; \beta = 0.0;$ and $\alpha = +0.5; \beta = +0.1;$ is carried out for the discussion and these plates are name as Type-I, Type –II and Type-III. Natural frequencies of first three modes for these three plates are computed and presented for the permissible range of other plate parameters such as elastic foundation $E_f = 0.0(0.1)0.5;$ non-homogeneity parameter $\mu = -0.3(0.1)0.3;$ density parameter $\eta = 0.0(0.1)0.5;$ for all three edge conditions. During numerical computation, stability and accuracy of results up to four decimal places achieved by running computer programme for dividing the domain into different subintervals. In all the computation $n=400$ has been fixed which leads more stable and reliable results (Tables 1) as compared with existing results in literature.

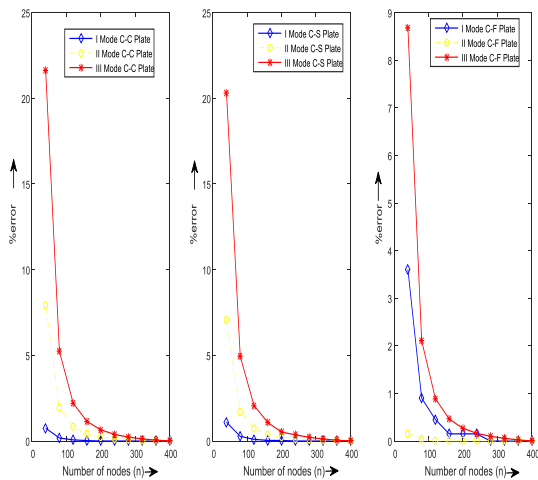


Figure 1. Stability analysis: Error versus number of subintervals (n) , (a) I-mode (b) II- mode (c) III- mode, for $\alpha=0.0, \beta=0.0;$ \diamond C-C plate , \circ C-S; $*$ C-F edge; Percentage error= $[(\Omega_n - \Omega_{400}) / \Omega_{400}] \times 100;$ $n=40(40)400.$

Table 1 Comparative results for uniform ($\alpha=0.0, \beta=0.0$) homogeneous ($\eta=0.0, \mu=0.0$), isotropic annular plate with $\epsilon=0.3$

Edge Conditions	Value take from References	Mode								
		C-C			C-S			C-F		
		I	II	III	I	II	III	I	II	III
$E_f=0.0$	Verma [1987]	45.3371	123.6191	246.6944	29.9689	100.6065	211.5629	6.6542	42.6156	123.5739
	Sharma [2006]	45.3462	123.3621	246.1573	29.9777	100.4228	211.1294	6.6604	42.6142	123.4661
	Present	45.3461	123.3631	246.1626	29.9776	100.4235	211.1336	6.6603	42.6142	123.4671
	Verma [1987]	46.5259	126.0531	246.9155	31.7385	101.1478	211.8208	12.3880	43.8782	124.0152
	Sharma [2006]	46.5347	125.7969	246.3790	31.7469	100.9651	211.3878	12.3920	43.8768	123.9076
	Present	46.5346	125.7979	246.3843	31.7468	100.9657	211.3921	12.3919	43.8767	123.9086
$E_f=0.01$	Verma [1987]	45.3371	123.6191	246.6944	29.9689	100.6065	211.5629	6.6542	42.6156	123.5739
	Sharma [2006]	45.3462	123.3621	246.1573	29.9777	100.4228	211.1294	6.6604	42.6142	123.4661
	Present	45.3461	123.3631	246.1626	29.9776	100.4235	211.1336	6.6603	42.6142	123.4671
	Verma [1987]	46.5259	126.0531	246.9155	31.7385	101.1478	211.8208	12.3880	43.8782	124.0152
	Sharma [2006]	46.5347	125.7969	246.3790	31.7469	100.9651	211.3878	12.3920	43.8768	123.9076
	Present	46.5346	125.7979	246.3843	31.7468	100.9657	211.3921	12.3919	43.8767	123.9086

Overall it has been observed that values of Ω for Type-II (Uniform) plate is smaller than Type-I ($\alpha = -0.5; \beta = -0.1;$) and greater than for Type-III ($\alpha = +0.5; \beta = +0.1;$) plate irrespective of other plate parameters. While value of Ω keeps on increasing with the rising values of elastic foundation E_f , non-homogeneity parameter μ and taper parameters α and β , it decreases with density parameter η .

Figures (2-4) show the parametric change for frequency parameter along with the elastic foundation, non-homogeneity parameter and density parameter respectively.

From figure 2(a, b, c), increasing behavior of Ω is observed with increasing value of E_f when plate vibrates in first, second and third mode, respectively. This rise in the value of Ω is subject to thickness variation and keeps on increasing with increasing value of α or β or both; irrespective of edge conditions of the plates. Also it is more obvious in case of plate vibrating in first mode as compared to second and third mode. The rate of increase of Ω is highest for C-F plate than that of other modes and edge conditions.

Figure 3(a) represents the change in frequency with respect to the change in density parameter η without compromising non-homogeneity of the material keeping $\mu=0.5$.



Decreasing trend has been observed in frequency with the escalating value of density or solidity parameter η . A similar tendency is noticed for second as well as third modes figure 3(b) and 3(c), however the rate of change in frequency parameter varies with thickness and mode of vibration as well. A surprising fact has been observed in case of C-F plate, frequencies with respect to thickness variation is just reverse in second and third mode that of first mode of vibration.

Figure 4(a,b,c) depict the effect of non-homogeneity variation, with the parametric value of μ , and its influence on first three frequencies and modes for the fixed value of density parameter keeping as $\eta=0.5$. An Increment blueprint has been presented for Ω and μ with the variation of thickness parameters α and β .

Nature of frequency parameter Ω astonishes under the edge condition C-F. Plate takes opposite (unexpected) shape in first mode with thickness variation.

Figure 5, 6 and 7 show the graphical represents of nodal circles for all three types of plates namely Type-I ($\alpha=-0.5, \beta=-0.1$), Type-II ($\alpha=0.0, \beta=0.0$) and Type-III ($\alpha=0.5, \beta=0.1$) of composite material ($\mu=0.5, \eta=0.5$) for Clamped-Clamped, Clamped-Simply supported and Clamped-Free plates, respectively. As the outer edge become thicker, nodal circles shift towards inner edge irrespective of boundary conditions. Clamped-Free edge plate behaves differently in first modes as illustrated by its normalized displacement,

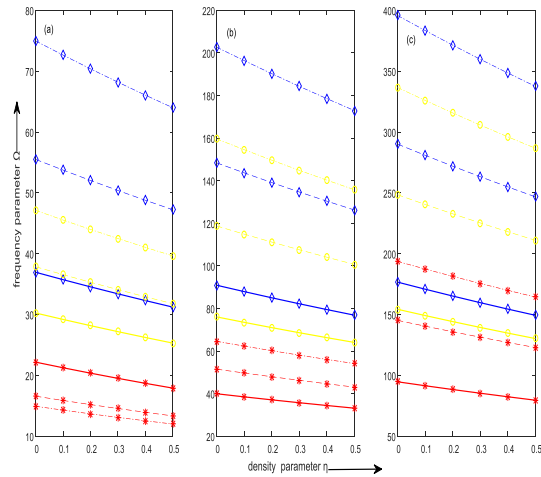


Figure 3 Variation in natural frequencies (a) I-mode (b) II- mode (c) III- mode with density parameter for $E_f=0.02$ and $\mu=0.5$, with edge conditions: “ \diamond C-C plate , \circ C-S plate; *C-F edge”; for different thickness: “—, $\alpha =-0.5, \beta=-0.1$; ----, $\alpha =0.0, \beta=0.0$; -.-. $\alpha =0.5, \beta=0.1$ ”.

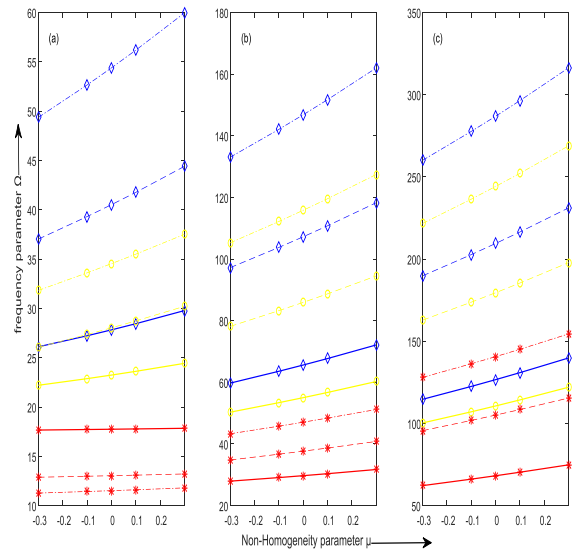


Figure 4. Variation in natural frequencies (a) I-mode (b) II- mode (c) III- mode, with non-homogeneity parameter when $E_f=0.02$, and $\eta=0.5$, with edge conditions: “ \diamond C-C plate , \circ C-S plate *C-F edge”; for different thickness: “—, $\alpha =-0.5, \beta=-0.1$; ----, $\alpha =0.0, \beta=0.0$; -.-. $\alpha =0.5, \beta=0.1$ ”;

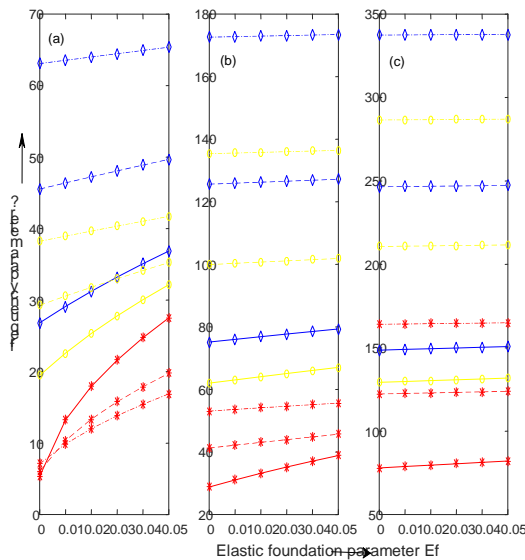


Figure 2. Variation in natural frequencies (a) I-mode (b) II- mode (c) III- mode, with elastic foundation when $\mu=0.5$ and $\eta=0.5$, with edge conditions: “ \diamond C-C plate ; \circ C-S plate; *C-F edge;” for different thickness: “—, $\alpha =-0.5, \beta=-0.1$; ----, $\alpha =0.0, \beta=0.0$; -.-. $\alpha =0.5, \beta=0.1$ ”

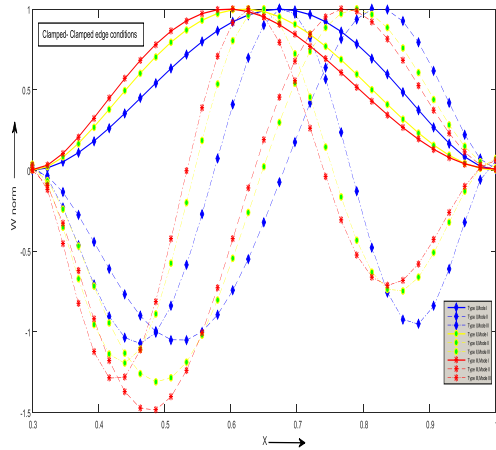


Figure 5. Nodal lines for C-C edge conditions, for $\mu=0.5$, $\eta=0.5$; for “—, I- mode;— —,II- mode, .-. .,III- mode;”with thickness variation: “ \diamond , (TYPE-I, $\alpha=-0.5$, $\beta=-0.1$); \circ , (TYPE-II, $\alpha=0.0$, $\beta=0.0$); *, (TYPE-III, $\alpha=0.5$, $\beta=0.1$)”.

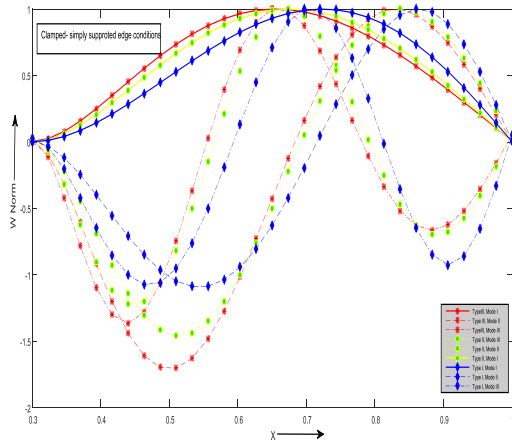


Figure 6. Nodal lines for C-S edge conditions, for $\mu=0.5$, $\eta=0.5$ for “—, I- mode;— —,II- mode, .-. .,III- mode;”with thickness variation: “ \diamond , (TYPE-I, $\alpha=-0.5$, $\beta=-0.1$); \circ , (TYPE-II, $\alpha=0.0$, $\beta=0.0$); *, (TYPE-III, $\alpha=0.5$, $\beta=0.1$)”.

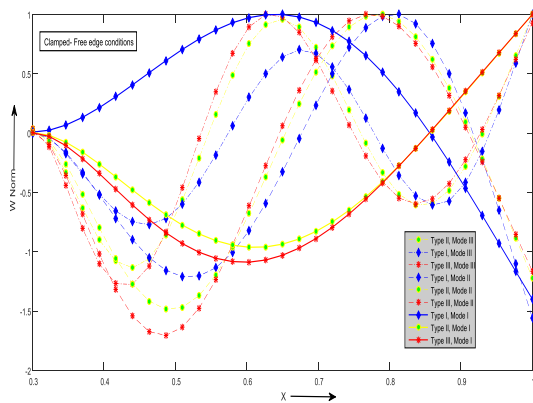


Figure 7. Nodal lines for C-F edge conditions, for $\mu=0.5$, $\eta=0.5$ for “—, I- mode;— —,II- mode, .-. .,III- mode;”with thickness variation: “ \diamond , (TYPE-I, $\alpha=-0.5$, $\beta=-0.1$); \circ , (TYPE-II, $\alpha=0.0$, $\beta=0.0$); *, (TYPE-III, $\alpha=0.5$, $\beta=0.1$)”.

IV. CONCLUSION AND FUTURE SCOPE

In the present work more complicating effects on natural frequencies and transverse modes of annular plates are accumulated and presented which will help design engineers to construct more reliable products and will motivate to rethink of their findings and constructions in this era of advancement. Hence, present modeling can be used as benchmark for annular plates and many other results can be evaluated for different choice of thickness variations (linear, parabolic, quadratic), for homogeneous/non-homogeneous with or without foundation.

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