Computing Cluster Centers of Triangular Fuzzy Numbers using Innovative Metric Distance

S. Sreenivasan, B. J. Balamurugan

Abstract: In this paper we compute cluster centers of triangular fuzzy numbers through fuzzy c means clustering algorithm and kernel based fuzzy c means clustering algorithm. An innovative distance between the triangular fuzzy numbers is used and the distance is complete metric on triangular fuzzy numbers. The set of triangular fuzzy numbers and an another set with the same triangular fuzzy numbers by including an outlier or noisy point as an additional triangular fuzzy number are taken to find the cluster centers using MATLAB programming. An example is given to show the effectiveness between the algorithms.

Keywords: Fuzzy c means clustering algorithms, Fuzzy Clustering, Kernel function, Triangular fuzzy numbers.

I. INTRODUCTION

In the history of fuzzy clustering, fuzzy c means (FCM) clustering procedure introduced by Dunn [3] and improved by Bezdek [1] are most used and discussed. However, FCM is not good if the cluster of points containing outlier or noisy points. A clustering algorithm is good if it is strong to deal with the real time problems.

In this paper we use KFCM on triangular fuzzy numbers and discuss the robustness of the fuzzy c means type algorithms with the new vertex distance. And we used the complete metric distance to find the distance between triangular fuzzy numbers.

The distance which is complete and metric on triangular fuzzy numbers is described in section II. Based on the complete metric distance two clustering algorithms on triangular fuzzy numbers are given in section III. Numerical example is given to compare the effectiveness of the two algorithms, FCM and KFCM in section IV. Conclusions are given in section V.

II. A COMPLETE METRIC DISTANCE FOR TRIANGULAR FUZZY NUMBERS

We consider the triangular fuzzy numbers on \( R = (-\infty, \infty) \).

Definition 1: A fuzzy set \( \tilde{A} = (A^L, A^C, A^R) \) is said to be triangular fuzzy number, if the membership function of the triangular fuzzy number is

\[
\tilde{A}(x) = \begin{cases} 
0, & x < A^L \\
\frac{x - A^L}{A^C - A^L}, & A^L \leq x \leq A^C \\
\frac{x - A^R}{A^C - A^R}, & A^C \leq x \leq A^R \\
0, & x > A^R 
\end{cases}
\]

Let the set of all fuzzy numbers be \( F(R) \). To run the clustering algorithms on the set \( F(R) \) we apply the distance \( d \) [2] defined as follows:

Definition 2: The distance between two triangular fuzzy numbers \( \tilde{A} = (A^L, A^C, A^R) \) and \( \tilde{B} = (B^L, B^C, B^R) \) which is complete and metric is defined as

\[
d^2(\tilde{A}, \tilde{B}) = \frac{1}{3}[(A^L - B^L)^2 + (A^C - B^C)^2 + (A^R - B^R)^2]
\]

This is an efficient method to calculate the distance between two triangular fuzzy numbers is used in this paper. Yang and Ko [9] proved that \( (F(R), d) \) is a complete and metric.

III. FUZZY CLUSTERING ALGORITHMS

In this section we recall FCM clustering algorithm[1] and KFCM clustering algorithm[10], [11].

Let \( X = \{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, ..., \tilde{X}_n\} \) be the set of fuzzy numbers in \( F(R) \) with \( \tilde{X}_k = (X^L_k, X^C_k, X^R_k) \), \( 1 \leq k \leq n \). Let the number of clusters be \( n \). Let \( V = \{V_i | 1 \leq i \leq c\} \) is the set of centers, where \( \tilde{V}_i = (V^L_i, V^C_i, V^R_i) \) and \( d_{ik} \) be the distance between \( \tilde{X}_k \) and \( \tilde{V}_i \).

A. Fuzzy c Means Clustering

The fuzzy c means clustering algorithm divides \( X \) into \( c \) fuzzy subsets by minimizing the function

\[
J_{FCM}(U, V) = \sum_{i=1}^{c} \sum_{k=1}^{n} u_{ik}^m d_{ik}^2
\]

where \( d_{ik} = d(\tilde{V}_i, \tilde{X}_k) \) and \( u_{ik} \) is the membership value of the fuzzy number \( \tilde{X}_k \) in cluster \( i \) with \( \sum_{i=1}^{c} u_{ik} = 1 \), the fuzziness index \( m \in [1, \infty) \) and \( U = (u_{ik})_{n \times c} \) is a fuzzy c partition matrix.

The Parameters of FCM are calculated by improving the
function min $J$ step by step according to the formulas below:

$$V_i^L = \frac{\sum_{k=1}^{n} u_{ik}^m X_k^L}{\sum_{k=1}^{n} u_{ik}^m}$$

$$V_i^C = \frac{\sum_{k=1}^{n} u_{ik}^m X_k^C}{\sum_{k=1}^{n} u_{ik}^m}$$

$$V_i^R = \frac{\sum_{k=1}^{n} u_{ik}^m X_k^R}{\sum_{k=1}^{n} u_{ik}^m}$$

$$u_{ik} = \frac{d_{ik}^{-1}}{\sum_{j=1}^{m} d_{ik}^{-1}}$$

Based on these formulas, on triangular fuzzy numbers we use the following fuzzy c means clustering algorithm.

Step 1: Let the fuzziness index be $m > 1$, let the number of partitions $c = \{2, 3, 4, \ldots, (n-1)\}$ and let any $\varepsilon > 0$.

Choose $U^{(0)}$ be the fuzzy c partition matrix initially and let $t = 0$.

Step 2: Calculate cluster centers $V^{(l)} = \{\tilde{V}_i^{l} | 1 < i \leq c\}$ using $U^{(l)}$ and equations (1), (2) and (3).

Step 3: Improve $U^{(l)}$ by $U^{(l+1)}$ using $V^{(l)}$ and equation (4).

Step 4: Compute $E^k = \max_{i,k} \{||u_{ik}^{(l+1)} - u_{ik}^{(l)}||\}$, if $E^k \leq \varepsilon$, stop.

Otherwise set $t^{(i+1)} = t^{(i)}$ and move to step 2.

B. Kernel Fuzzy c Means Clustering

Let the unlabeled set $X = \{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \ldots, \tilde{X}_n\}$ in the $M$-dimensional space $R^M$, let

$\Phi : R^M \rightarrow H, \tilde{X} \rightarrow \Phi (\tilde{X})$

We can use the kernel function $K(\tilde{X}_i, \tilde{X}_j) = \Phi (\tilde{X}_i) \Phi (\tilde{X}_j)$ to find the dot product in the high dimensional feature space.

Examples of kernel function

- Linear: $K(\tilde{X}_i, \tilde{X}_j) = \tilde{X}_i^T \tilde{X}_j$
- Polynomial: $K(\tilde{X}_i, \tilde{X}_j) = (\gamma \tilde{X}_i^T \tilde{X}_j + c)^d, \gamma > 0, d \in N$
- Sigmoid: $K(\tilde{X}_i, \tilde{X}_j) = \tanh(\gamma \tilde{X}_i^T \tilde{X}_j + c), \gamma > 0$
- RBF: $K(\tilde{X}_i, \tilde{X}_j) = \exp(-\gamma \|\tilde{X}_i - \tilde{X}_j\|^2), \gamma > 0$

where $\gamma, c, d$ are kernel parameters.

Since,

$$\|\Phi (\tilde{X}_i) - \Phi (\tilde{V}_j)\|^2 = (\Phi (\tilde{X}_i) - \Phi (\tilde{V}_j))^T (\Phi (\tilde{X}_i) - \Phi (\tilde{V}_j))$$

$$= \Phi (\tilde{X}_i)^T \Phi (\tilde{X}_i) - \Phi (\tilde{X}_i)^T \Phi (\tilde{V}_j) - \Phi (\tilde{V}_j)^T \Phi (\tilde{X}_i) + \Phi (\tilde{V}_j)^T \Phi (\tilde{V}_j)$$

$$= K(\tilde{X}_i, \tilde{X}_i) + K(\tilde{V}_j, \tilde{V}_j) - 2K(\tilde{X}_i, \tilde{V}_j)$$

when the kernel function is chosen as RBF, $K(\tilde{X}_i, \tilde{X}_i) = 1, K(\tilde{V}_j, \tilde{V}_j) = 1$, then

$$\|\Phi (\tilde{X}_i) - \Phi (\tilde{V}_j)\|^2 = 2(1 - K(\tilde{X}_i, \tilde{V}_j))$$

The kernel fuzzy c means clustering algorithm divides $X$ into $c$ fuzzy subsets by minimizing the function

$$J_{KFCM}(U, V) = \sum_{i=1}^{n} \sum_{k=1}^{m} u_{ik}^m (1 - K(\tilde{X}_k, \tilde{V}_i))$$

The Parameters of kernel fuzzy c means are calculated by improving the function min $J$ step by step according to the formulas below:

$$V_i^L = \frac{\sum_{k=1}^{m} u_{ik}^m K(\tilde{X}_k, \tilde{V}_i) X_k^L}{\sum_{k=1}^{m} u_{ik}^m K(\tilde{X}_k, \tilde{V}_i)}$$

$$V_i^C = \frac{\sum_{k=1}^{m} u_{ik}^m K(\tilde{X}_k, \tilde{V}_i) X_k^C}{\sum_{k=1}^{m} u_{ik}^m K(\tilde{X}_k, \tilde{V}_i)}$$

$$V_i^R = \frac{\sum_{k=1}^{m} u_{ik}^m K(\tilde{X}_k, \tilde{V}_i) X_k^R}{\sum_{k=1}^{m} u_{ik}^m K(\tilde{X}_k, \tilde{V}_i)}$$

$$u_{ik} = \frac{1}{\sum_{j=1}^{m} (1 - K(\tilde{X}_k, \tilde{V}_j))}$$

Based on these formulas, on triangular fuzzy numbers we use the following kernel fuzzy means clustering algorithm.

Step 1: Let the fuzziness index be $m > 1$, let the number of partitions $c = \{2, 3, 4, \ldots, (n-1)\}$ and let any $\varepsilon > 0$.

Choose $U^{(0)}$ be the fuzzy c partition matrix initially and let $t = 0$.

Step 2: Calculate cluster centers $V^{(l)} = \{\tilde{V}_i^l | 1 < i \leq c\}$ using $U^{(l)}$ and equations (5), (6) and (7).

Step 3: Improve $U^{(l)}$ by $U^{(l+1)}$ using $V^{(l)}$ and equation (8).
Compute $E^{i} = \text{Max}_{j,k} \left| u_{i}^{(i+1)} - u_{i}^{(i)} \right|$, if $E^{i} \leq \varepsilon$, stop.
Otherwise set $U^{(i+1)} = U^{(i)}$ and move to step 2.

In the experiment, we used RBF kernel is used with the parameter $\gamma$ defined by

$$
\gamma = \left( \sum_{k=1}^{n} d \left( \overline{X}_{k}, \overline{W} \right) \right)^{-1} 
$$

with $\overline{W} = (\overline{W}^{L}, \overline{W}^{C}, \overline{W}^{R})$ is the arithmetic mean, where

$$
\overline{W}^{L} = \frac{\sum_{k=1}^{n} X_{k}^{L}}{n},
\overline{W}^{C} = \frac{\sum_{k=1}^{n} X_{k}^{C}}{n},
\overline{W}^{R} = \frac{\sum_{k=1}^{n} X_{k}^{R}}{n}.
$$

### IV. NUMERICAL EXAMPLE

We run both the algorithms FCM and KFCM using the metric distance $d$ to compare the effectiveness with a numerical example. We implement the algorithms with $m = 2$ and $\varepsilon = 0.00001$. Consider the data set $D$ given by Hung and Yang [5] consisting of 20 triangular fuzzy numbers given in “Fig. 1”.

For the set $D_1$ given in “Fig. 1”, the suitable number of clusters $c = 2$. Therefore, on the set $D_1$ with the number of clusters $c = 2$ we run both the FCM and KFCM algorithms. The corresponding results are given in “Fig. 3” and “Table. I”.

Consider the set $D_2$ consisting $D_1$ and one more point (99.29, 100, 101.79) called outlier point shown in “Fig. 2”. Now on the set $D_2$ with the number of clusters $c = 2$ we run both the FCM and KFCM algorithms. The corresponding results are shown in “Fig. 4” and “Table. II”.

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<th>Table- I: Cluster centers in set $D_1$</th>
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<th>Table- II: Cluster centers in set $D_2$</th>
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<td>$V_1$</td>
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When we run FCM and KFCM algorithms on data set $D_1$ with the distance $d$, the cluster centers are almost same. While running FCM algorithm on the data set $D_2$ (With outlier point) gives poor result, but the KFCM algorithm on $D_2$ provides almost the same result of $D_1$. That is the centers obtained by FCM algorithm on $D_2$ are away from the clusters whereas the centers obtained by KFCM algorithm on $D_2$ are within the cluster and coincides with the centers of $D_1$ data.

### V. CONCLUSION

We use the innovative metric distance $d^2(\hat{A},\hat{B})$ to find the cluster centers. We run FCM algorithm and KFCM algorithm using MATLAB. We have theoretically verified that KFCM algorithm gives better result than FCM with noisy point and outliers. KFCM performs well for the sets $D_1$ and $D_2$ with the distance $d^2(\hat{A},\hat{B})$ examined in this paper.
REFERENCES


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