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Numerical Computation of Delay Differential Equation using Laplace Transform and Lambert W Function

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***Abstract:** In this paper a novel approach using Laplace transform for the solution of delay differential equation with a single delay based on Lambert W function has been investigated. An obtained result is extended to the n^{th} order DDEs. Numerical examples have been provided to illustrate the obtained result.*

***Keywords:** Delay Differential Equation, Lambert W function, Laplace transform*

I. INTRODUCTION

An appearance of time delays in a system is unavoidable. It will appear frequently in the field of engineering and science and deteriorate the main system performances. Time delay systems belong to the class of infinite dimensional systems. It could be expressed by delay differential equations. DDEs are generally solved by numerical methods such as the Least Squares Method, Pade Approximation Method, Adomian Decomposition Method, Homotopy Perturbation Method, Laplace Transform Method. In 1997, W.H. Enright *et.al.*, found a novel approach for solving neutral delay differential equations by continuous Runge-Kutta formula [1]. In 2003, an analytical method on the basis of Lambert W function was developed to find the solution of DDEs by Ulsoy *et.al.*,[2]. F. Karako *et.al.*, implemented Differential Transform Method (DTM) to obtain an exact, analytical, and numerical solutions of both linear and nonlinear equations in 2009[3]. The numerical solution of delay differential equation can be found using Coupled Block Method by Hue Chi San in 2011 [4]. Adomian decomposition method (ADM) was presented to solve both linear and nonlinear delay differential equation by Ogunfiditimi, F.O. in 2015 [5]. An optimal perturbation iteration method, was developed to find an approximate solutions of delay differential equations by Necdet Bildik *et.al.*,in 2017 [6]. An analytical approach using Laplace transform for solving linear systems of DDEs was investigated based on the matrix Lambert W function method by sun Yi *et.al.*, in 2007[7].

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In this paper an analytical solution of linear system of delay differential equation with single delay is discussed using Laplace transform based on Lambert function.

II. PROPOSED METHODOLOGY

A. *First order delay differential equation*

Consider the following first order delay differential equation

$$\dot{x} + ax(t) - Kx(t-\tau) = 0, \quad t > 0$$
$$x(t) = g(t), \quad t\epsilon[-\tau, 0) \qquad (1)$$
$$x(t) = x_0, t = 0$$

where a and K are real constants, g(t) is an initial function and x_0 is an initial value . Now,

$$L[x(t-\tau)] = \int_0^\infty e^{-st} x(t-\tau)dt$$
$$= \int_0^\tau e^{-st}x(t-\tau)dt + \int_\tau^\infty e^{-st}x(t-\tau)dt$$
$$= \int_0^\tau e^{-st}g(t)dt + \int_\tau^\infty e^{-s\tau}e^{-st_1}x(t_1)dt$$
$$= G(s) + e^{-s\tau}X(s)$$

Taking Laplace Transform on (1),

$$L[\dot{x}] + aL[x(t)] - KL[x(t-\tau)] = 0$$

$$[s - Ke^{-s\tau} + a]X(s) - x_0 - KG(s) = 0$$
$$X(s) = \frac{x_0 + KG(s)}{s - Ke^{-s\tau} + a} \qquad (2)$$

The solution of eqn(1) with respect to Lambert W function is

$$x(t) = \sum_{-\infty}^{\infty} e^{S_k t} C_k, \text{ where } S_k = \frac{1}{\tau} W_k(b\tau e^{a\tau}) - a$$

To find the co-efficient C_k,

$$L[x(t)] = L[\sum_{-\infty}^{\infty} e^{S_k t} C_k]$$
$$X(s) = \frac{\sum_{-\infty}^{\infty} C_k n_k(s)}{d(s)} \qquad (3)$$

Where

$$d(s) = \ldots(s - S_{-1})(s - S_0)(s - S_1)\ldots = \prod_{-\infty}^{\infty}(s - S_k)$$
$$n_k(s) = \frac{d(s)}{s - S_k} = \cdots(s - S_{k-2})(s - S_{k-1})(s - S_{k+1})(s - S_{k+2})\ldots$$

From equations (2) and (3),

$$d(s) = \prod_{-\infty}^{\infty}(s - S_k) = J(s)(s - Ke^{-s\tau} + a) \qquad (4)$$
$$\sum_{-\infty}^{\infty} C_k n_k(s) = J(s)(x_0 + KG(s)) \qquad (5)$$

Here J(s) represents a polynomial in s,

$n_k(s = S_l) = 0$, when $k \neq l$

$= \cdots (S_l - S_{l-2})(S_l - S_{l-1})(S_l - S_{l+1})(S_l - S_{l+2}) \ldots$, when k= l.

We compute C_k from (5),

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$C_0 = \frac{J(S_0)(x_0+KG(S_0))}{n_0(S_0)}$

$= \frac{J(S_0)(x_0+KG(S_0))}{\ldots(S_0-S_{-2})(S_0-S_{-1})(S_0-S_1)(S_0-S_2)\ldots}$

$C_1 = \frac{J(S_1)(x_0+KG(S_1))}{n_1(S_1)}$

$= \frac{J(S_1)(x_0+KG(S_1))}{\ldots(S_1-S_{-2})(S_1-S_{-1})(S_1-S_1)(S_1-S_2)\ldots}$ ………

Using L, Hospital's rule,

$J(s) = \lim_{s\to S0} \frac{\prod_{-\infty}^{\infty}(s-s_k)}{s-Ke^{-s\tau}+a}$

$= \frac{\ldots(S_0-S_{-2})(S_0-S_{-1})(S_0-S_1)(S_0-S_2)\ldots}{1+K\tau e^{-S_0\tau}}$

Consequently, $C_0 = \frac{x_0+KG(S_0)}{1+K\tau e^{-S_0\tau}}$, $C_1 = \frac{x_1+KG(S_1)}{1+K\tau e^{-S_1\tau}}$ hence $C_k = \frac{x_k+KG(S_k)}{1+K\tau e^{-S_k\tau}}$

B. Second order delay differential equation

Consider the following second order delay differential equation

$x''(t) + ax'(t) + bx(t) - Kx(t-\tau) = 0, t > 0$

$x(t) = g(t)$ and $x'(t) = f(t)$, $t\epsilon[-\tau, 0]$ (6)

$x(t) = x_0$ and $x'(t) = 0$, t = 0

Taking Laplace transform on (6),

$X(s) = \frac{(s+a)x_0+x'(0)+KG(s))}{s^2+as+b+Ke^{-s\tau}}$ (7)

The solution of eqn(6) in respect of Lambert W function is

$x(t) = \sum_{-\infty}^{\infty} e^{S_k t} C_k$ where $S_k = \frac{2}{\tau} W_k(\frac{\tau}{2} e^{((\frac{\tau}{2})\propto)(\sqrt{K})}) - \alpha$

Then the Laplace transform of x(t) is given by

$X(s) = \frac{\sum_{-\infty}^{\infty} C_k n_k(s)}{d(s)}$ (8)

From equations (7) and (8),

$d(s) = \prod_{-\infty}^{\infty}(s - S_k) = J(s)(s^2 + as + b + Ke^{-s\tau})$ (9)

$\sum_{-\infty}^{\infty} C_k n_k(s) = J(s)((s+a)\ x_0 + x'(0)+KG(s))$ (10)

From (10),

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$C_0 = \frac{J(S_0)((S_0+a)x_0+x'(0)+KG(S_0))}{n_0(S_0)}$

$= \frac{J(S_0)((S_0+a)x_0+x'(0)+KG(S_0))}{\ldots(S_0-S_{-2})(S_0-S_{-1})(S_0-S_1)(S_0-S_2)\ldots}$

$C_1 = \frac{J(S_1)((S_1+a)x_0+x'(0)+KG(S_1))}{n_1(S_1)}$

$= \frac{J(S_1)((S_1+a)x_0+x'(0)+KG(S_1))}{\ldots(S_1-S_{-2})(S_1-S_{-1})(S_1-S_1)(S_1-S_2)\ldots}$

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From (9)

$J(s) = \lim_{s\to S_0} \frac{\prod_{-\infty}^{\infty}(s-s_k)}{s^2+as+b-Ke^{-s\tau}}$

$= \frac{\ldots(S_0-S_{-2})(S_0-S_{-1})(S_0-S_1)(S_0-S_2)\ldots}{2S_0+a+K\tau e^{-S_0\tau}}$

So that $C_0 = \frac{(S_0+a)x_0+x'(0)+KG(S_0)}{2S_0+a+K\tau e^{-S_0\tau}}$,

$C_1 = \frac{(S_1+a)x_0+x'(0)+KG(S_1)}{2S_1+a+K\tau e^{-S_1\tau}}$ hence $C_k = \frac{(S_k+a)x_0+x'(0)+KG(S_k)}{2S_k+a+K\tau e^{-S_k\tau}}$

C. Third order delay differential equation

Consider the following third order delay differential equation

$x'''(t) + ax''(t) + bx'(t) + cx(t) - Kx(t-\tau) = 0, t > 0$

$x(t) = g(t),\ \ x'(t) = f(t)\ and\ x''(t) = h(t),\ \ t\epsilon(-\tau, 0)$(11)

$x(t) = x_{0,}\ \ x'(t) = 0\ and\ x''(t) = 0,\ \ t = 0$

Taking Laplace transform on (11),

$X(s) = \frac{(s^2+as+b)x_0+(s+a)x_0'+x''(0)+KG(s)}{s^3+as^2+bs+c-Ke^{-s\tau}}$ (12)

The solution of eqn.(11) in relation to Lambert W function is

$x(t) = \sum_{-\infty}^{\infty} e^{S_k t} C_{k,}$ where $S_k = \frac{3}{\tau} W_k(\frac{\tau}{3} e^{((\tau/_3)\alpha)(\sqrt[3]{K})}) - \alpha$

Using Laplace Transform

$X(s) = \frac{\sum_{-\infty}^{\infty} C_k n_k(s)}{d(s)}$ (13)

From (12) and (13),

$d(s) = \prod_{-\infty}^{\infty}(s - S_k) = J(s)(s^3 + as^2 + bs + c - Ke^{-s\tau})$ (14)

$\sum_{-\infty}^{\infty} C_k n_k(s) = J(s)((s^2 + as + b)x_0 + (s + a)x_0' + x''(0) + KG(s))$

(15)

From (15),

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$C_0 = \frac{J(S_0)((S_0^2+aS_0+b)x_0+(S_0+a)x_0'+x''(0)+KG(S_0)}{n_0(S_0)}$

$= \frac{J(S_0)((S_0^2+aS_0+b)x_0+(S_0+a)x_0'+x''(0)+KG(S_0))}{\ldots(S_0-S_{-2})(S_0-S_{-1})(S_0-S_1)(S_0-S_2)\ldots}$

$C_1 = \frac{J(S_1)((S_1^2+aS_1+b)x_0+(S_1+a)x_0'+x''(0)+KG(S_1)}{n_1(S_1)}$

$= \frac{J(S_1)((S_1^2+aS_1+b)x_0+(S_1+a)x_0'+x''(0)+KG(S_1))}{\ldots(S_1-S_{-2})(S_1-S_{-1})(S_1-S_1)(S_1-S_2)\ldots}$

From (14),

$$J(s) = lim_{s\to S_0} \frac{\prod_{-\infty}^{\infty}(s - S_k)}{s^3 + as^2 + bs + c - Ke^{-s\tau}}$$

$$= \frac{\ldots(S_0-S_{-2})(S_0-S_{-1})(S_0-S_1)(S_0-S_2)\ldots}{3S_0^2+2aS_0+b+K\tau e^{-S_0\tau}}$$

Therefore $C_0 = \frac{(S_0^2+aS_0+b)x_0+(S_0+a)x_0'+x''(0)+KG(S_0)}{3S_0^2+2aS_0+b+K\tau e^{-S_0\tau}}$,

$C_1 = \frac{(S_1^2+aS_1+b)x_0+(S_1+a)x_0'+x''(0)+KG(S_1)}{3S_1^2+2aS_1+b+K\tau e^{-S_1\tau}}$ hence

$C_k = \frac{(S_k^2+aS_k+b)x_0+(S_k+a)x_0'+x''(0)+KG(S_k)}{3S_k^2+2aS_k+b+K\tau e^{-S_k\tau}}$

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Hence in general we conclude that for n[th] order,

$$C_k = \frac{a_0 x(0) + a_1 x'(0) + \cdots + x^{n-1}(0) + coe.of x(t-\tau)G(S_k)}{nS_k^{n-1} + (n-1)cof.of x^{n-1} S_k^{n-2} + \cdots + \tau cof.of x(t-\tau)e^{-S_k \tau}}$$

Where $a_0 = \left(S_k^{n-1} + cof.of x^{n-1}(t) S_k^{n-2} + \cdots + cof.of x'(t)\right)$, $a_1 = \left(S_k^{n-2} + cof.of x^{n-1}(t) S_k^{n-3} + \cdots + cof.of x''(t)\right)$

III. NUMERICAL EXAMPLES

A. Example

Consider the following DDE

$$x'(t) + 3x(t) = 5x(t-\tau)$$

with initial point $x'(0) = 1$ and initial function $\varphi(t) = e^{3t}$.

Now, $G(S_k) = \int_0^\tau e^{-S_k t} \varphi(t-\tau) dt$

$$= e^{-3\tau} \int_0^\tau e^{(3-S_k)t}$$

$$G(S_k) = \frac{e^{-3\tau}}{3-S_k}\left[e^{(3-S_k)\tau} - 1\right]$$

The eigenvalues of the above DDE is calculated by $S_k = (1/\tau)W(be^{a\tau}\tau) - a$ here $S_0 = 0.3889$; $S_1 = -0.1704 + 5.2099i$; $S_{-1} = -0.1704 - 5.2099i$

From the above values $G(S_0) = 0.2405$; $G(S_1) = -0.1020 + 0.1611i$; $G(S_{-1}) = -0.1020 - 0.1611i$.

And $C_k = \frac{x_0 + KG(S_k)}{1 + K\tau e^{-S_k T}}$ here $C_0 = 0.5018$; $C_1 = 0.1453 + 0.0127i$; $C_{-1} = 0.1453 - 0.0127i$.

The solution is

$x(t) = (0.1453 - 0.0127i)e^{(-0.1704 - 5.2099i)t} + (0.5018)e^{(0.3889)t} + (0.1453 + 0.0127i)e^{(-0.1704 + 5.2099it}$

The complete solution of the above DDE in terms of Lambert function is $x(t) = \cdots + (0.2572 - 0.2737i)e^{(-0.1704 - 5.2099i)t} + (0.4856)e^{(0.3889)t} + (0.2572 + 0.2737i)e^{(-0.1704 + 5.2099i)} + \cdots$

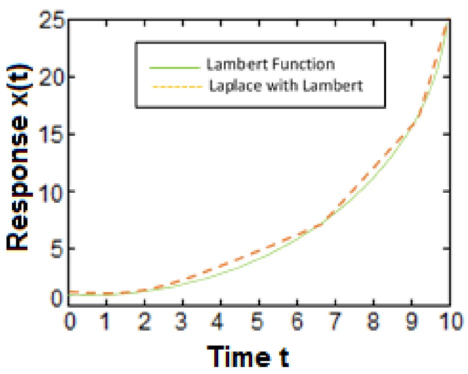

Fig.1.Solutions obtained using Lambert W Function and Laplace Transform combined with Lambert W Function

From the Fig.1, it shows that the solution obtained using Lambert W Function and Laplace Transform combined with the Lambert function is almost same.

B. Example

Consider the second order DDE

$$x''(t) + 2x'(t) + x(t) = 0.2x(t-1)$$

with an initial points $x(0) = 1$, $x'(0) = 1$ and an initial function $\varphi(t) = te^{-t}$.

Here $G(S_k) = \frac{e^\tau}{S_k + 1}\left[-\tau - \frac{e^{-(S_k+1)\tau}}{S_k+1} + \frac{1}{S_k+1}\right]$

Where $S_k = \frac{2}{\tau} W_k\left(\frac{\tau}{2} e^{((\tau/2)\alpha)(\sqrt{K})}\right) - \alpha$

Using the above, $G(S_0) = -1.1380$; $G(S_1) = -4.8657 + 0.2612i$; $G(S_{-1}) = -4.8657 - 0.2612i$.

And $C_k = \frac{(S_k + a)x_0 + x'(0) + KG(S_k))}{2S_k + a + K\tau e^{-S_k \tau}}$, here $C_0 = 1.633$; $C_1 = -0.0477 - 0.0960i$; $C_{-1} = -0.0477 + 0.0960i$

The solution is

$x(t) = (-0.0477 + 0.0960i)e^{(-6.1720 - 8.3115i)t} + (1.633)e^{(-0.4421)t} + (-0.0477 - 0.0960i)e^{(-6.1720 + 8.3115i)t}$

The complete solution of the above DDE in terms of Lambert function is

$x(t) = (0.0214 - 0.0074i)e^{(-6.1720 - 8.3115i)t} + (-0.0428)e^{(-0.4421)t} + (0.0214 + 0.0074i)e^{(-6.1720 + 8.3115i)t}$

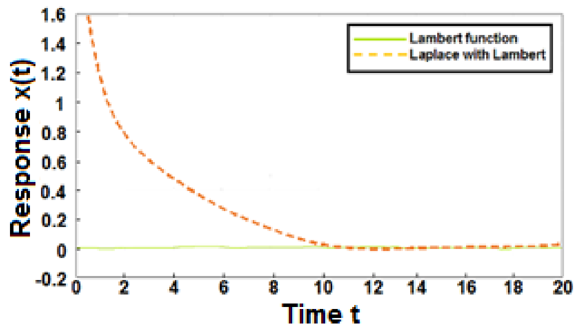

Fig.2.Solutions of second order DDE using Lambert W Function and Laplace Transform combined with Lambert Function Fig.2 shows the solution of second order DDE obtained by Lambert W function and Laplace transform combined with the Lambert W function is same after the certain stage.

C. Example

Consider the third order DDE

$x'''(t) - 9x''(t) + 15x'(t) + 25x(t) = 2x(t - \tau)$ with an initial values $x(0) = 1$, $x'(0) = 1$, $x''(0) = 1$ and the initial function $\varphi(t)) = Ae^{-t} + (Bt + C)e^{5t}$. which is the preshape function for the given DDE.

Here

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$$G(S_k) = \frac{1.47e^{\tau}}{-(S_k+1)}\left(e^{-(S_k+1)\tau} - 1\right) + \frac{4.82e^{-5\tau}}{S_k-5}\left(-\frac{e^{(S_k-5)\tau}}{S_k-5} - \tau + \frac{1}{S_k-5}\right) + \frac{0.47e^{-5\tau}}{-(S_k-5)}\left[e^{-(S_k-5)\tau} - 1\right]$$ and

$$S_k = \frac{3}{\tau} W_k\left(\frac{\tau}{3} e^{((\tau/3)\propto)(\sqrt[3]{K})}\right) - \propto.$$

Here $S_0 = 3.4050, S_1 = -7.6238 + 11.9580i, S_{-1} = -7.6238 - 11.9580i$ and $G(S_0) = 0.8585$, $G(S_1) = -1.6704e + 01 + 1.3674e + 02i, G(S_{-1}) = -1.6704e + 01 - 1.3674e + 02i$

C_k can be computed using the below formula

$$C_k = \frac{(S_k^2 + aS_k + b)x(0) + (S_k + a)x'(0) + x''(0) + KG(S_k)}{3S_k^2 + 2aS_k + b + K\tau e^{-S_k\tau}}.$$ We get

$$C_0 = 0.6056, C_1 = -0.0131 + 0.0049i, C_{-1} = -0.0131 - 0.0049i$$

The solution is

$$x(t) = (-0.0131 - 0.0049i)e^{(-7.6238-11.9580i)t} + (0.6056)e^{(3.4050)t} + (-0.0131 + 0.0049i)e^{(-7.6238+11.9580i)t}$$

IV. RESULT ANALYSIS

In the current investigation first order, second order and third order differential equations with delayed argument are solved and extended to the nth order DDEs as well, that is done by using Laplace Transform Method combined with Lambert W function. The characteristic equation of linear delay differential equation is transcendental and has infinite number of roots. This may lead to a major challenge in solving DDEs. Here we apply the Laplace transform connected with Lambert function. First order, second order and third order linear DDEs have been solved using the above procedure.

V. CONCLUSION

In this paper the numerical solution of delay differential equation with a single delay is discussed with an approach using Laplace Transform based on Lambert function. This approach has been extended to the nth order system of delay differential equation. Numerical examples are given to support our result.

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