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Equivalent Stiffness Methods for Free Vibration and Bending of Bimodular Composite Laminated Thick Curved Beam

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Abstract*: In this research work an attempt has been made to analyzed the bending and free vibration behavior of bimodular composite material laminated thick curved beam based on first order shear deformation theory (FSDT). The effect of coupling parameter is more severe in bimodular laminated beam as compare to unimodular laminated beam. Therefore, equivalent stiffness methods (VS and RC) are used to incorporate the effect of all the coupling parameters. In the present analysis uniformly distributed load in transverse direction is considered. Hamilton's principle is applied to elaborate governing equations from energy functional, and the equations are solved by analytical method. The position of neutral axis, transverse deflection, through the thickness strain distribution and frequencies for positive and negative side bending are presented for different bimodular composite material laminated curved beam. The results are presented for different stacking sequence and geometric parameter of the beam. The non-dimensional neutral surface location, positive and negative side bending deflections and positive and negative half cycle frequencies are having same magnitude for $[0^0]_4$ or $[0^0/90^0]_s$ laminated curved beam of Material 1 (Aramid Rubber), and for Material 2 (Polyester Rubber) they are not same.

***Keywords:* bimodular, curved beam, angle-ply, equivalent stiffness, first order shear deformation theory**

I. INTRODUCTION

In the modern era composite materials are extensively used as structural element in different field (aerospace, automobile, submarine etc.). In most of the cases, the behavior of composite materials in tension and compression are same those composite are called unimodular composites. There are a few composites which shows different behavior in tension and compressions are called bimodular composites as shown in figure 1. The bimodularity ratio (E_{1t}/E_{1c}) are not same for all the bimodular composite materials. The bimodularity ratio of Aramid Rubber and Polyester Rubber [1] are approximately 300 and 14 respectively. Jones [2] proposed a simplified bilinear model for bimodular composite material. Jones bilinear model has been used by all the research work related to analyses of bimodular composite structures. Available literature [1-3] developed various material models to assign tensile and compression properties to a bimodular composite laminate; among all the material models, Bert's fiber direction strain governed model has been used by most of the researchers due to it's simplicity.

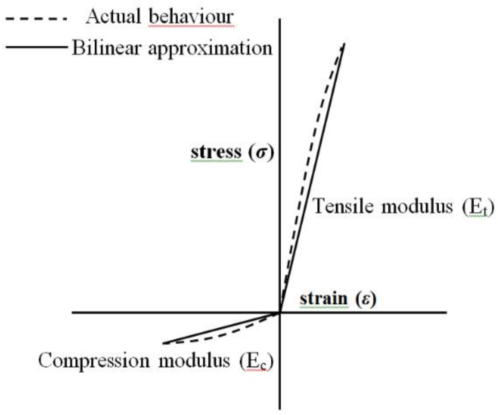

Fig. 1. Bilinear approximation and actual behaviour of the bimodular material

Murthy and Rao [4] presented a finite element analysis of bimodular composite material laminated straight beam of fixed-free boundary condition for different loading and lay-up. It is observed that the loading, lay-up scheme have significant effect on neutral surface location along the beam length. Bert and Gordaninejad [5] presented the bending behavior of bimodular material straight beam using the Transfer matrix method (TMM), The TMM results were found to be very close to the closed form solution results and also this method was proved to be computationally effective.

The available literature in the field of dynamic analysis of bimodular composite material laminated beam is meager. Bert and Tran [6] analyzed the transient response of single-layered bimodular material clamped-clamped thick beam using TMM and Newmark beta method. Rebello et. al. [7] presented an analytical and experimental investigation of free vibration frequencies for first three modes of three-layered bimodular sandwich straight beam (core is of isotropic bimodular, and facings are of orthotropic bimodular material) using Timoshenko beam theory. Hajianmaleki and Qatu [8] studied the behavior of unimodular composite material laminated curved and straight beam using analytical method. The equivalent stiffness parameters were used to calculate the deflections and free vibration frequencies of simply supported beam.

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Hajianmaleki and Qatu [9] used equivalent stiffness parameters $\overline{A}_{11}, \overline{B}_{11}, and\ \overline{D}_{11}$ or $\widetilde{A}_{11}, \widetilde{B}_{11}, and\ \widetilde{D}_{11}$ to include the effect of various coupling instead of normal A_{11}, B_{11} and D_{11}, and presented a comparative study of bending deflection and free vibration frequencies obtained from equivalent stiffness parameters and normal stiffness parameters (A_{11}, B_{11} and D_{11}).

The free vibration and bending analysis of curved beam of orthotropic bimodular composite material is not available in literature. In this paper the free vibration and bending analysis of curved beam of orthotropic bimodular composite material have been presented. The FSDT is assumed for present analysis and equivalent stiffness parameters have been used to calculate free vibration frequencies, and transverse deflections under uniformly distributed transverse load. The governing equations are derived from energy functional and are solved analytically.

II. FORMULATION

A simply supported laminated thick curved beam of composite material have been considered with R, a, h, b as radius of curvature, curved length, thickness and width respectively as shown in figure 2. The displacement field is assumed as:

$$u(x,y,z,t)=u_0(x,t)+z\psi(x,t)$$
$$v(x,y,z,t)=0 \tag{1}$$
$$w(x,y,z,t)=w_0(x,t)$$

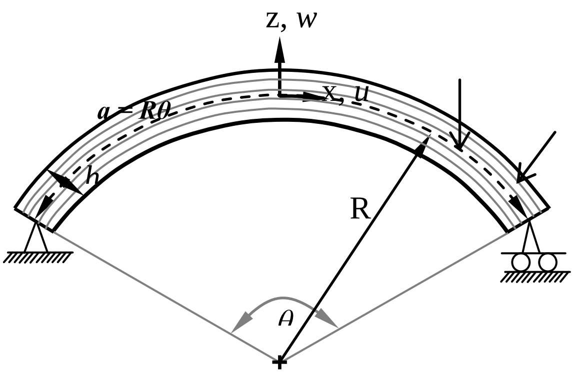

Fig. 2. Geometry configuration of simply supported laminated curved beam

The strain at the arbitrary point of the beam is assumed in the terms of the mid surface strain (ε_0) and curvature change (κ):

$$\varepsilon=\frac{1}{1+z/R}\left(\varepsilon_0+z\kappa\right) \tag{2}$$

where, $\varepsilon_0=\frac{\partial u_0}{\partial x}+\frac{w_0}{R}$ and $\kappa=\frac{\partial \psi}{\partial x}$, u_0 and w_0 are mid surface displacements of beam in respective direction.

The shear strain at neutral axis is:

$$\gamma=\frac{\partial w_0}{\partial \alpha}+\psi-\frac{u_0}{R} \tag{3}$$

For n number of laminated layers, the above equation is written in terms of A_{11}, B_{11}, D_{11} and A_{55} as:

$$\left[N,M,Q\right]=b\int_{h/2}^{h/2}\left[\sigma,\sigma z,\tau\right]dz \tag{4}$$

The above equation is written in terms of A_{11}, B_{11}, D_{11} and A_{55} as:

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix}=\begin{bmatrix} A_{11} & B_{11} & 0 \\ B_{11} & D_{11} & 0 \\ 0 & 0 & A_{55} \end{bmatrix}\begin{bmatrix} \varepsilon_0 \\ \kappa \\ \gamma \end{bmatrix} \tag{5}$$

where,

$$A_{11}=R\sum_{k=1}^{N} bE_x^k \ln\left(\frac{R+h_k}{R+h_{k-1}}\right)$$

$$B_{11}=R\sum_{k=1}^{N} bE_x^k \begin{pmatrix} (h_k-h_{k-1})- \\ R\ln\left(\frac{R+h_k}{R+h_{k-1}}\right) \end{pmatrix}$$

$$D_{11}=R\sum_{k=1}^{N} b\,E_x^k \begin{pmatrix} \frac{1}{2}\left(h_k^2-h_{k-1}^2\right)-R\left(h_k-h_{k-1}\right) \\ +R^2\ln\left(\frac{R+h_k}{R+h_{k-1}}\right) \end{pmatrix}$$

$$A_{55}=\frac{5}{4}\sum_{k=1}^{N} b\overline{Q}_{55}^k \begin{pmatrix} (h_k-h_{k-1})+ \\ \frac{4}{3h^2}\left(h_k^3-h_{k-1}^3\right) \end{pmatrix}$$

The governing equations of motion by applying Hamilton's principle from energy functional:

$$\frac{\partial N}{\partial \alpha}+\frac{Q}{R}=\overline{I}_1\frac{\partial^2 u}{\partial t^2}+\overline{I}_2\frac{\partial^2 \psi}{\partial t^2}-p_\alpha \tag{6}$$

$$-\frac{N}{R}+\frac{\partial Q}{\partial \alpha}=\overline{I}_1\frac{\partial^2 w}{\partial t^2}-p_n \tag{7}$$

$$\frac{\partial M}{\partial \alpha}-Q=\overline{I}_2\frac{\partial^2 u}{\partial t^2}+\overline{I}_3\frac{\partial^2 \psi}{\partial t^2} \tag{8}$$

where,

$$\overline{I}_1=I_1+\frac{I_2}{R}+\frac{I_3}{R^2}$$

$$\overline{I}_2=I_2+\frac{I_3}{R}$$

$\overline{I}_3=I_3$ and

$$\left[I_1,I_2,I_3\right]=\sum_{K=1}^{N} b\rho^k \begin{bmatrix} \left(h_k-h_{k-1}\right), \\ \frac{1}{2}\left(h_k^2-h_{k-1}^2\right), \\ \frac{1}{3}\left(h_k^3-h_{k-1}^3\right) \end{bmatrix}$$

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Using the Equation (5), Equation of motion (6), (7) and (8) rewritten in matrix form as:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}\begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix} - \begin{bmatrix} \overline{I}_1 & 0 & \overline{I}_2 \\ 0 & -\overline{I}_1 & 0 \\ \overline{I}_2 & 0 & \overline{I}_3 \end{bmatrix}\frac{\partial^2}{\partial t^2}\begin{bmatrix} u_0 \\ w_0 \\ \psi \end{bmatrix} = \begin{bmatrix} -p_x \\ p_x \\ 0 \end{bmatrix} \tag{9}$$

where,

$$L_{11} = A_{11}\frac{\partial^2}{\partial x^2} - \frac{A_{55}}{R^2}$$

$$L_{12} = L_{21} = \left(\frac{A_{11}}{R} + \frac{A_{55}}{R}\right)\frac{\partial}{\partial x}$$

$$L_{13} = L_{31} = B_{11}\frac{\partial^2}{\partial x^2} + \frac{A_{55}}{R}$$

$$L_{22} = \frac{A_{11}}{R^2} - A_{55}\frac{\partial^2}{\partial x^2}$$

$$L_{23} = L_{32} = \left(\frac{B_{11}}{R} - A_{55}\right)\frac{\partial}{\partial x}$$

$$L_{33} = D_{11}\frac{\partial^2}{\partial x^2} - A_{55}$$

A. Vinson-Sierakowski equivalent stiffness method (VS)

Vinson and siearakowski [9] developed an equivalent stiffness method. In which the equivalent stiffness parameters are calculated as:

$$\left[\overline{A}_{11}, \overline{B}_{11}, \overline{D}_{11}\right] = \sum_{K=1}^{N} bE_x^k \begin{bmatrix} \left(h_k - h_{k-1}\right), \\ \frac{1}{2}\left(h_k^2 - h_{k-1}^2\right), \\ \frac{1}{3}\left(h_k^3 - h_{k-1}^3\right) \end{bmatrix} \tag{10}$$

where, where, N_e (N_e=N+1) is the number of effective layers, N is the number of layers. For bimodular material, some part of the laminate (ε_x<0) will be assigned compressive material property and remaining part of the laminate (ε_x>0) will be assigned tensile property and

$$\frac{1}{E_x^k} = \frac{\cos^4\theta^k}{E_{11n}^k} + \left[\frac{1}{G_{12n}^k} - \frac{2\upsilon_{12n}^k}{E_{11n}^k}\right]\cos^2\theta^k \sin^2\theta^k + \frac{\sin^4\theta^k}{E_{22n}^k}$$

Here, $E_{11}^k, E_{22}^k, G_{12}^k$, υ_{12}^k and θ^k are elastic modulus in X-direction, elastic modulus in Y-direction, in-plane shear modulus, in-plane poison ratio and ply-angle of the k^{th} layer respectively. The subscript n=c stands for the compressive property, and n=t stands for the tensile property.

B. Rios- Chan equivalent stiffness method (RC)

Rios and Chan [10] also developed another equivalent stiffness method. They computed the unified [*ABD*] matrix for finding the equivalent stiffness parameter as below:

$$\begin{bmatrix} N_X \\ N_Y \\ N_{XY} \\ M_X \\ M_Y \\ M_{XY} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{26} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}\begin{bmatrix} \varepsilon_X \\ \varepsilon_Y \\ \gamma_{XY} \\ \kappa_X \\ \kappa_Y \\ 2\kappa_{XY} \end{bmatrix} \tag{11}$$

The [*ABD*] matrix components can be obtained for unimodular as well as bimodular material with required modification. The equivalent stiffness parameters are determined as:

$$\tilde{A}_{11} = \frac{1}{a_{11} - \frac{b_{11}^2}{d_{11}}} \tag{12}$$

$$\tilde{B}_{11} = \frac{1}{b_{11} - \frac{a_{11}d_{11}}{b_{11}}} \tag{13}$$

$$\tilde{D}_{11} = \frac{1}{d_{11} - \frac{b_{11}^2}{a_{11}}} \tag{14}$$

where, $a_{11}=J_{11}$, $b_{11}=J_{14}$, $d_{11}=J_{44}$ *and* $[J]=[ABD]^{-1}$

C. Solution Methodology

For simply supported beam, boundary conditions are:

$$w = N_x = M_x = 0 \text{ at } x = -\frac{a}{2}, \frac{a}{2}$$

For axial and transverse displacement, the general solution assumed as following equation so that the boundary conditions are satisfied.

$$\left[u_0, w_0\right] = \sum_{m=1}^{M}\begin{bmatrix} A_m \sin(\alpha_m x), \\ C_m \cos(\alpha_m x) \end{bmatrix}\sin\omega t \tag{15}$$

where, $\alpha_m = \frac{m\pi}{a}$, A_m, C_m are constant, m is integer and a is curve beam length.

The expansion of external force in terms of Fourier series in x can be written as:

$$\left[p_z\right] = \sum_{m=1}^{M}\left[p_{zm}\cos(\alpha_m x)\right]\sin\omega t \tag{16}$$

where,

$$p_{zm} = \frac{2}{a}\int p_z \cos(\alpha_m x)dx$$

After substituting the equations (15) and (16), the equation of motion rewritten as:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} + \omega^2 \begin{bmatrix} \overline{I_1} & 0 & \overline{I_2} \\ 0 & -\overline{I_1} & 0 \\ \overline{I_2} & 0 & \overline{I_3} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} + \begin{bmatrix} p_{xm} \\ -p_{zm} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (17)$$

where,

$$C_{11} = -A_{11}\alpha_m^2 - \frac{A_{55}}{R^2}$$

$$C_{12} = -\left(\frac{A_{11}}{R} + \frac{A_{55}}{R}\right)\alpha_m$$

$$C_{13} = \left(-B_{11}\alpha_m^2 + \frac{A_{55}}{R}\right)B_m$$

$$C_{21} = \left(\frac{A_{11}}{R} + \frac{A_{55}}{R}\right)\alpha_m$$

$$C_{22} = \frac{A_{11}}{R^2} + A_{55}\alpha_m^2$$

$$c_{23} = \left(\frac{B_{11}}{R} - A_{55}\right)\alpha_m$$

$$C_{31} = -B_{11}\alpha_m^2 + \frac{A_{55}}{R}$$

$$C_{32} = -\left(\frac{B_{11}}{R} - A_{55}\right)\alpha_m$$

$$C_{33} = -D_{11}\alpha_m^2 - A_{55}$$

The above equation (17) is valid for forced vibration problem. For finding the results of static problem, the value of ω in above equation should be zero and for free vibration, the external forces should be zero. For bimodular materials, there are two values of results. One for the positive half cycle and other is for the negative half cycle. After finding the value of transverse deflection and natural vibration for static and free vibration problem, the non-dimensional deflection and frequency is obtained as:

$$\left[\overline{w}_p, \overline{w}_n\right] = \left[w_p, w_n\right]\frac{100\,E_{22c}\,h^3}{p_z\,a^4} \quad (18)$$

$$\left[\Omega_p, \Omega_n\right] = \left[\omega_p, \omega_n\right]a^2\sqrt{\frac{12\rho}{E_{1c}h^2}} \quad (19)$$

where,

w_p, w_n is positive and negative side bending deflection

$\overline{w}_p, \overline{w}_n$ is non-dimensional positive and negative side bending deflection

ω_p , ω_n is positive and negative frequency and

Ω_p , Ω_n is non-dimensional positive and negative frequency

To get the results for VS and RC equivalent stiffness parameters the same solution methodology is applied; for VS, A_{11}, B_{11} and D_{11} will be replaced by $\bar{A}_{11}, \bar{B}_{11},$ and $\bar{D}_{11}$ respectively, and for RC method A_{11}, B_{11} and D_{11} will be replaced by $\tilde{A}_{11}, \tilde{B}_{11},$ and $\tilde{D}_{11}$ respectively.

III. RESULTS AND DISCUSSION

In this analysis we have considered a simply supported rectangular cross section curved beam having length (*a*) 1m, length to height ratio (*a/h*) 100,width (*b*) 0.025m and different value of curve length to radius of curvature ratio (*a/R*). Two different bimodular orthotropic composite materials have been considered for the present analysis. The material properties are given below.

Material 1: Aramid Rubber [1]**:**

In tension: E_{1t} = 3.58 GPa, $E_{2t} = E_{3t}$ = 0.00909 GPa, $G_{12t} = G_{13t}$ = 0.0037 GPa, G_{23t} = 0.0029 GPa, $\nu_{12t} = \nu_{23t} = \nu_{13t}$ = 0.416, $\rho = 1580\,kg/m^3$

In compression: $E_{1c} = E_{2c} = E_{3c}$ = 0.012 GPa, $G_{12c} = G_{13c}$ = 0.00267 GPa, G_{23c} = 0.00499 GPa, $\nu_{12c} = \nu_{23c} = \nu_{13c}$ = 0.205, $\rho = 1580\,kg/m^3$

Material 2: Polyester Rubber [1]:

In tension: E_{1t} = 0.617 GPa, $E_{2t} = E_{3t}$ = 0.008 GPa, $G_{12t} = G_{13t}$ = 0.00262 GPa, G_{23t} = 0.00223 GPa, $\nu_{12t} = \nu_{23t} = \nu_{13t}$= 0.475, ρ=970 kg/m^3

In compression: E_{1c} = 0.0369 GPa, $E_{2c} = E_{3c}$ = 0.0106 GPa, $G_{12c} = G_{13c}$ =0.0267 GPa, G_{23c} = 0.00475 GPa, $\nu_{12c} = \nu_{23c} = \nu_{13c}$ = 0.185, ρ=970 kg/m^3

In Table I the non-dimensional positive and negative side bending deflections (at x=0.0) and non-dimensional fundamental positive and negative half cycle frequencies for $[0^0]_4$ and $[0^0/90^0]_s$ laminated curved beam of Material 1 for different *a/R* ratios have been presented. For positive side bending the neutral surface is located at Z_p/h=0.445, and for negative side bending the neutral surface is located at Z_n/h=-0.445 for both VS and RC method. In these lamination schemes, the results are identical for VS and RC method. From Table I it is noticed that, the positive and negative side bending defections ($\overline{w}_p$, $\overline{w}_n$) increases, and the positive and negative half cycle frequencies (Ω_p , Ω_n) decreases with increase of *a/R* ratios.

The percentage difference between positive and negative half side bending deflections and positive and negative half cycle frequencies increases with increase of *a/R* ratio.

For location *Z/h* =0.5 to 0.445 of lamina 4, beam has tensile properties and rest part of lamina 4 *Z/h*=0.445 to 0.25, lamina 3 (ply-angle 90^0) *Z/h*=0.25 to 0.0, lamina 2 (ply-angle 90^0) *Z/h*=0.0 to -0.25 and lamina 1 (ply-angle 0^0) *Z/h*=-0.25 to -0.5 has the compressive properties which is shown in figure 3(a), and for negative half bending neutral axis locations are -0.445, which is located on the lamina 1 (ply-angle 0^0), from location *Z/h* =-0.5 to -0.445 of lamina 1, beam has tensile properties and rest part of lamina 1 (*Z/h*=-0.445 to -0.25), lamina 2 (ply-angle 90^0) *Z/h*=-0.25 to 0.0, lamina 3 (ply-angle 90^0) *Z/h*=0.0 to 0.25

and lamina 4 (ply-angle 0^0) Z/h=0.25 to 0.5 has the compressive properties which is shown in Figure 3(b). In the compression case of Material 1 compression modulus is same ($E_{11c}=E_{22c}=E_{33c}$). It can be inferred that the stiffness coefficients do not depend on ply-angle in compression case. Therefore the results of $[0^0]_4$ and $[0^0/90^0]_s$ laminated curved beam are same.

Table-I: The non-dimensional deflections (at x=0) and non-dimensional fundamental frequencies of Material 1 laminated curved beam (LS: $[0^0]_4$, $[0^0/90^0]_s$, Z_p/h=- Z_n/h=0.445)

a/R	VS/RC			
	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n
0.1	4.382	18.614	4.377	18.642
0.2	4.413	18.514	4.401	18.571
0.3	4.461	18.359	4.443	18.444
0.5	4.618	17.889	4.587	18.025
0.8	5.027	16.809	4.974	17.005
1.0	5.452	15.865	5.380	16.088

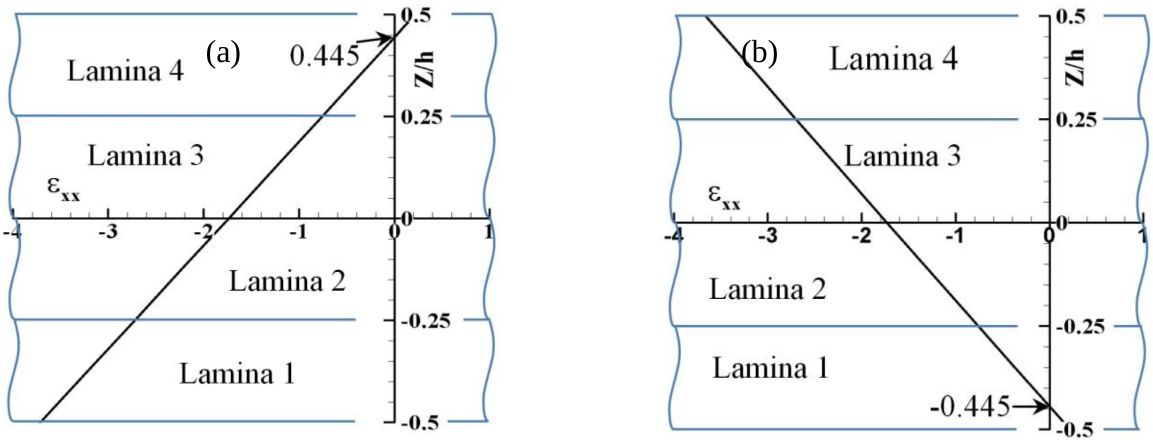

Fig. 3. Through the thickness Strain distribution of Material 1 laminated curved beam (*a/R*=1.0, LS: $[0^0]_4$ and $[0^0/90^0]_s$): a) Positive side bending, b) negative side bending

Table- II: The non-dimensional deflections (at x=0) and non-dimensional fundamental frequencies of Material 1 laminated curved beam (LS: $[0^0{}_2/90^0{}_2]$, Z_p/h=-0.011, Z_n/h=-0.445)

a/R	VS				RC			
	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n
0.1	17.916	9.211	4.377	18.642	17.542	9.309	4.377	18.642
0.2	18.030	9.168	4.401	18.571	17.641	9.269	4.401	18.571
0.3	18.215	9.099	4.443	18.444	17.822	9.199	4.444	18.444
0.5	18.826	8.880	4.587	18.025	18.419	8.978	4.587	18.025
0.8	20.452	8.361	4.974	17.005	20.010	8.453	4.974	17.005
1.0	22.149	7.901	5.380	16.088	21.669	7.988	5.380	16.088

In Table II the non-dimensional positive and negative side bending deflections (at x=0.0) and fundamental positive and negative half cycle frequencies for $[0^0{}_2/90^0{}_2]$ laminated curved beam of Material 1 have been presented. For positive side bending the neutral surface is located at Z_p/h=-0.011 for VS, and Z_p/h=-0.012 for RC method, and for negative side bending the neutral surface is located at Z_n/h=-0.445 for both VS and RC method. In this lamination scheme (LS) the negative side bending deflection and the negative half cycle frequency are same as $[0^0/90^0]_s$ laminated curved beam and the positive side bending deflection and the positive half cycle frequency are different. The difference of positive and negative half side bending deflection is about (75.56-75.71) % for VS, and (75.04-75.17) % for RC method respectively. The difference of positive and negative half cycle frequency is about 100 % for both the methods. The results of $[0^0{}_2/90^0{}_2]$ laminated curved beam are not same as $[0^0]_4$ laminated curved beam for positive side bending, but for negative side bending the results $[0^0{}_2/90^0{}_2]$ laminated curved beam are identical of the results $[0^0]_4$ laminated curved beam. These phenomena can be explained as below. For positive side bending Lamina 4 (ply-angle 90^0) Z/h=0.5 to 0.0.25, lamina 3 (ply-angle 90^0) Z/h=0.25 to 0.0, some part of lamina 2 (ply-angle 90^0) Z/h=0.0 to -0.011 has tensile properties, and rest part of lamina 2 Z/h=-0.011 to -0.25 and lamina 1 (ply-angle 0^0) Z/h=-0.25 to -0.5 has the compressive properties. In case of tension,

Material 1 has different modulus properties ($E_{11t} \neq E_{22t}=E_{33t}$) in different directions. So stiffness coefficient will depend on ply-angle in tension case. In the case of negative half bending neutral axis locations is Z_n/h= -0.445 which is located on the lamina 1 (ply-angle 0^0) as shown in figure 4(b). From location *Z/h* =-0.5 to -0.445 of lamina 1, beam has tensile properties; and rest part of lamina 1 (*Z/h*=-0.445 to -0.25), lamina 2 (ply-angle 0^0) *Z/h*=-0.25 to 0.0, lamina 3 (ply-angle 90^0) *Z/h*=0.0 to 0.25 and lamina 4 (ply-angle 90^0) *Z/h*=0.25 to 0.5 has the compressive properties. So in the case of negative side bending results of $[0^0]_4$ and $[0^0{}_2/90^0{}_2]$ are identical.

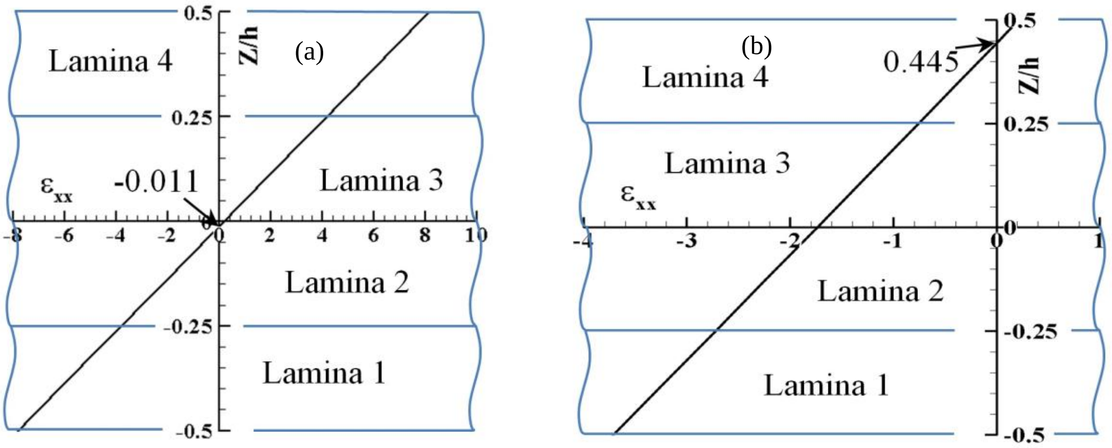

Fig. 4. Through the thickness Strain distribution of Material 1 laminated curved beam (*a/R*=1.0, LS: $[0^0{}_2/90^0{}_2]$): a) Positive side bending, b) negative side bending

Table- III: The non-dimensional deflections (at x=0) and non-dimensional fundamental frequencies of Material 1 laminated curved beam (LS: $[45^0]_4$)

a/R	VS		RC	
	$\bar{w}_p = \bar{w}_n$	$\Omega_p = \Omega_n$	$\bar{w}_p = \bar{w}_n$	$\Omega_p = \Omega_n$
0.1	18.387	9.092	17.180	9.407
0.2	18.500	9.051	17.286	9.364
0.3	18.691	8.982	17.463	9.293
0.5	19.320	8.765	18.050	9.069
0.8	20.994	8.251	19.612	8.538
1.0	22.738	7.796	21.241	8.068

Table- IV: The non-dimensional deflections (at x=0) and non-dimensional fundamental frequencies of Material 1 laminated curved beam (LS: $[30^0{}_2/60^0{}_2]$)

a/R	VS				RC			
	$\bar{w}_p$	Ω_p	$\bar{w}_n$	Ω_n	$\bar{w}_p$	Ω_p	$\bar{w}_n$	Ω_n
0.1	19.592	8.809	14.235	10.335	19.540	8.820	9.639	12.538
0.2	19.711	8.769	14.321	10.289	19.660	8.780	9.698	12.500
0.3	19.914	8.703	14.467	10.212	19.862	8.7149	9.799	12.405
0.5	20.583	8.493	14.951	9.967	20.529	8.504	10.129	12.104
0.8	22.363	7.996	16.240	9.386	22.304	8.007	11.007	11.395
1.0	24.219	7.556	17.586	8.870	24.155	7.566	11.922	10.767

Table III depicts the positive and negative half bending deflections (at x=0.0) and fundamental positive and negative half cycle frequencies for $[45^0]_4$ laminated curved beam of Material 1. The neutral surface for positive side bending is located at Z_p/h=0.007 and Z_p/h=-0.009 for VS and RC method respectively, and for negative side bending the neutral surface is located at Z_n/h=-0.007 and Z_n/h=0.009 VS and RC method respectively. For this LS the positive and negative side bending deflections, and positive and negative half cycle frequencies are equal in magnitude. The percentage difference of deflections and frequencies calculated with the help of VS and RC method is about 6.56% and 3.46% respectively. In Table IV the positive and negative side bending deflections (at x=0.0), and fundamental positive and negative half cycle frequencies are presented for $[30^0{}_2/60^0{}_2]$ laminated curved beam of Material 1. In this laminated curved beam for positive side bending the neutral surface is located at Z_p/h=-0.019 and Z_p/h=-0.012 for VS and RC method respectively, and for negative side bending the neutral surface is located at Z_n/h=-0.063 and Z_n/h=0.017. For this LS the difference between positive and negative half side bending deflection is 27.35% for VS, and 50.65% for RC.

The difference of positive and negative half cycle free vibrations frequencies are about 17.35% and 42.20% for VS and RC method respectively. The difference of results obtained from VS and RC method is 0.26% for positive side bending deflections, 32.15% for negative half side deflection, 0.13% for positive half cycle frequency, and 21.45% for negative half cycle frequency. It is observed that a/R ratio has no significant effect on the above mention percentage differences.

Table- V: The non-dimensional deflections (at x=0) and non-dimensional fundamental frequencies of Material 2 laminated curved beam (LS: $[0^0]_4$, Z_p/h=- Z_n/h=0.303)

a/R	VS/RC			
	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n
0.1	1.742	15.826	1.740	15.842
0.2	1.754	15.745	1.750	15.778
0.3	1.772	15.618	1.768	15.667
0.5	1.834	15.225	1.825	15.304
0.8	1.995	14.314	1.981	14.428
1.0	2.163	13.516	2.143	13.645

Table- VI: The non-dimensional deflections (at x=0) and non-dimensional fundamental frequencies of Material 2 laminated curved beam (LS: $[0^0/90^0]_s$, Z_p/h=- Z_n/h=0.334)

a/R	VS				RC			
	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n
0.1	2.184	14.135	2.182	14.150	2.180	14.145	2.178	14.162
0.2	2.198	14.063	2.194	14.093	2.195	14.073	2.191	14.105
0.3	2.222	13.948	2.216	13.994	2.218	13.959	2.212	14.006
0.5	2.298	13.597	2.288	13.670	2.295	13.607	2.285	13.682
0.8	2.501	12.783	2.483	12.889	2.497	12.793	2.479	12.899
1.0	2.711	12.070	2.687	12.189	2.707	12.079	2.683	12.199

Table- VII: The non-dimensional deflections (at x=0) and non-dimensional fundamental frequencies of Material 2 laminated curved beam (LS: $[0^0{}_2/90^0{}_2]$, Z_p/h=-0.079, Z_n/h=-0.364)

a/R	VS				RC			
	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n
0.1	9.004	6.965	4.293	10.088	8.962	6.981	4.288	10.094
0.2	9.057	6.934	4.317	10.049	9.014	6.951	4.312	10.055
0.3	9.148	6.883	4.359	9.979	9.104	6.899	4.354	9.985
0.5	9.450	6.719	4.500	9.750	9.405	6.736	4.495	9.756
0.8	10.260	6.330	4.881	9.196	10.211	6.345	4.875	9.201
1.0	11.106	5.984	5.279	8.699	11.053	5.998	5.273	8.704

Table V represents the positive and negative half side bending deflections (at x=0.0), and positive and negative half cycle fundamental frequencies for $[0^0]_4$ laminated curved beam of Material 2 for different *a*/*R* ratios. For positive side bending the neutral surface is located at Z_p/h=0.303 and for negative side bending the neutral surface is located at Z_n/h=-0.303 for both VS and RC method. In this laminated curved beam the results are same for VS and RC method as Material 1 and the nature of changing the all parameters are also same as $[0^0]_4$ laminated curved beam of Material 1. In Table VI the positive and negative half bending deflections (at x=0.0), and positive and negative half cycle fundamental frequencies are presented for $[0^0/90^0]_s$ laminated curved beam of Material 2. For positive side bending the neutral surface is located at Z_p/h=0.334 and negative side bending the neutral surface is located at Z_n/h=-0.334 for both VS and RC method, and the results of VS and RC are slightly different. The effect of *a/R* ratio on deflections and free vibration frequencies are identical as $[0^0]_4$ laminated curved beam. Table VII depicts positive and negative half bending deflections (at x=0.0) and fundamental positive and negative half cycle free vibration frequencies for $[0^0{}_2/90^0{}_2]$ laminated curved beam of Material 2. For positive side bending the neutral surface is located at Z_p/h=-0.079 and for negative side bending the neutral surface is located at Z_n/h=-0.364 for both VS and RC method. For material 2 the negative side bending deflection and negative half cycle frequency are not same of negative side bending deflection and negative half cycle frequency of $[0^0/90^0]_s$ laminated curved beam like material 1 laminated curved beam because the compression modulus at different directions of material 2 are not same(for material 2: $E_{1c} \neq E_{2c} = E_{3c}$)

like Material 1(for material 1: $E_{1c}=E_{2c}=E_{3c}$). The difference of positive and negative half side bending deflection is about 52.35% and 52.20% for VS and RC method respectively and the difference of positive and negative half cycle frequency is about 45.0% for both methods.

Table- VIII: The non-dimensional deflections (at x=0) and non-dimensional fundamental frequencies of Material 2 laminated curved beam (LS: $[45^0]_4$)

a/R	VS		RC	
	$\overline{w}_p = \overline{w}_n$	$\Omega_p = \Omega_n$	$\overline{w}_p = \overline{w}_n$	$\Omega_p = \Omega_n$
0.1	20.591	4.605	20.37	4.630
0.2	20.717	4.584	20.494	4.609
0.3	20.930	4.549	20.705	4.574
0.5	21.634	4.439	21.401	4.463
0.8	23.507	4.179	23.253	4.202
1.0	25.459	3.949	25.185	3.971

Table- IX: The non-dimensional deflections (at x=0) and non-dimensional fundamental frequencies of Material 2 laminated curved beam (LS: $[30^0{}_2/60^0{}_2]$)

a/R	VS				RC			
	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n	$\overline{w}_p$	Ω_p	$\overline{w}_n$	Ω_n
0.1	18.925	4.804	16.657	5.120	15.153	5.368	9.852	6.657
0.2	19.039	4.782	16.758	5.097	15.244	5.344	9.913	6.627
0.3	19.233	4.746	16.929	5.059	15.399	5.304	10.015	6.576
0.5	19.877	4.632	17.496	4.938	15.913	5.177	10.352	6.417
0.8	21.593	4.362	19.006	4.650	17.285	4.876	11.248	6.041
1.0	23.382	4.123	20.581	4.394	18.716	4.608	12.183	5.708

In Table VIII the positive and negative half bending deflections (at x=0.0) and fundamental positive and negative half cycle frequencies are presented for $[45^0]_4$ laminated curved beam of Material 2. For positive side bending the neutral surface is located at Z_p/h=-0.005 and Z_p/h=-0.007 for VS and RC method respectively, and for negative side bending the neutral surface is located at Z_n/h=0.005 and Z_n/h=0.007 for VS and RC method respectively. The positive and the negative side bending deflection and positive and negative half cycle frequency are same for both VS and RC method like material 1 laminated $[45^0]_4$ beam. The difference of deflections and frequencies determined from VS and RC method is about 1.07% and 0.5% respectively.

Table IX represents the positive and negative half side bending deflections (at x=0.0) and fundamental positive and negative half cycle frequencies for $[30^0{}_2/60^0{}_2]$ laminated curved beam of Material 2. For positive side bending the neutral surface is located at Z_p/h=-0.056 and Z_p/h=-0.045 for VS and RC method respectively, and for negative side bending the neutral surface is located at Z_n/h=-0.053 and Z_n/h=0.021 for VS and RC method respectively. The difference between positive and negative half side bending deflection is about 11.98% for VS method, and 34.95% for RC method, whereas the difference between positive and negative half cycle frequency is about 6.57% for VS method, and 24.0% for RC method. The difference between results obtained from VS and RC method is 19.95% for positive side bending deflections, 40.83% for negative half side deflection, 11.75% for positive half cycle frequency, and 30.0% for negative half cycle frequency.

IV. CONCLUSIONS

Two different equivalent stiffness methods have been implemented for the bending and free vibration analysis of simply supported curved beam. Two different bimodular materials with different bimodularity ratio have been considered. The following conclusions are drawn from the above discussions.

1) The positions of the neutral axis don't change with the increase of a/R ratio.
2) The non-dimensional positive and negative side bending deflections increase and non-dimensional positive and negative half cycle frequencies decrease as a/R increased.
3) The non-dimensional neutral surface location, positive and negative side bending deflections and positive and negative half cycle frequencies are same for $[0^0]_4$ or $[0^0/90^0]_s$ laminated curved beam of Material 1, and for Material 2 they are not same.
4) For unsymmetric cross-ply $[0^0{}_2/90^0{}_2]$ laminated curved beam, the percentage difference of positive and negative side bending deflection, positive and negative half cycle frequency is high as compare to unsymmetric angle-ply $[30^0{}_2/60^0{}_2]$ laminated curved beam.

5) For angle-ply laminated [$30^0{}_2/60^0{}_2$] beam the percentage difference between positive and negative side bending deflection, and percentage difference between positive and negative half cycle frequency is high of Material 1compare to Material 2 for both VS and RC method.

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