# Multiobjective Quadratic Fractional Programming using Iterative Parametric Function

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***Abstract***: ***The paper proposed the Model of multiobjective quadratic fractional optimisation problem with a set of quadratic constraints and a methodology for obtaining a set of solutions based on the approach of using iterative parametric functions. Firstly, each fractional objective function is transformed into non-fractional parametric objective function by assigning a vector of parameters to each objective function. In this approach, the Decision Maker(DM) predecides the desired tolerance levels of the objective functions in the form of termination constants. Then, by using ε-constraint method, a set of efficient solutions is obtained and termination conditions are checked for each parametric objective function. Also, a comparative study of the proposed method and fuzzy approach is given to reveal the validity of the method. A numerical for Multiobjective quadratic fractional programming Model (MOQFPM) is given in the end to check the applicability of the approach.***

***Keywords*** **:** ***Multiobjective quadratic fractional programming Model, parametric objective function, vector of parameters, ε-constraint method..***

## I. INTRODUCTION

From the past few decades, fractional optimization problems have gained huge importance and attracted many researchers due to their wide range of applications in health care management, corporate and financial planning, banking sector, science and engineering and in so many other fields. Multiobjective quadratic fractional programming Model (MOQFPM) are studied due to the fact that various real life conditions such as purchase/cost exist where several inter-related objectives are to be satisfied which are generally conflicting to each other. Model in which both numerator and denominator of the fractional objectives are quadratic with a set of constraints are termed as MOQFPM. The idea of tackling quadratic fractional programming goes back to Dinkelbach [12]. His approach was used by various researchers to study fractional optimization problems with the help of parametric functions. Hannu Valiaho [14] proposed unified approach to one-parametric quadratic programming. Maziar Salahi and Saeed Fallahi [13] also studied parametric approach for quadratic fractional problems. M.Borza et al. [1] proposed parametric method for absolute value LFP with interval coefficients. Zhixia and Fengqi [15] studied mixed Integer linear and Non-linear fractional programming problems. Mishra and Ghosh [7] gave Fuzzy approach to quadratic fractional problems. Heesterman [4] also studied parametric methods in quadratic programming.Hertog [5] proposed interior point approach to linear and quadratic programming. Lachhwani [6] presented FGP approach to multiobjective fractional programming problem(MOFPP). Osman et al. [11] also propounded multi-level MOFPP with fuzzy parameters. Gupta and Puri [3] also studied extreme point quadratic fractional programming problems. Ojha and Biswal [10] presented ε-constraint method for MOFPP. Nayak and Ojha [8],[9] also proposed parametric approach for fractional programming problem(FPP) in linear form. Emam [2] studied multiobjective integer bi-level quadratic fractional programming problems with the help of ε-constraint method. Multiobjective fractional programming problems usually do not have single optimal solution to satisfy all the objectives simultaneously and hence the concept of pareto optimality came into forefront developed by Vilfredo Pareto. This pareto optimal or efficient solution optimises atleast one objective without dissatisfying the remaining objectives.

Throughout the paper, we have used parametric approach proposed by Nayak and Ojha [8],[9] and extended their work to MOQFPM with the help of ε-constraint method. In this method, efficient solution is obtained by converting quadratic FPP into non-fractional problem by predefining termination constants.

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## II. NOTATIONS AND PRELIMINARIES

In the paper, we denote the space of n-dimensional real vectors by $R^n$. For a given vector $x$, $x^T$ represents the transpose of $x$. We assign a vector of parameters $\alpha^{(t)}$ to objective functions where $'t'$ denotes iteration number. $T_i$ in the paper represents termination constants defined by decision maker. $'S'$ represents a set of constraints.

*1) Pareto or Efficient Solution*

A vector $u \in S$ is called a pareto or efficient solution if $\nexists$ another feasible solution $v \in S$ such that $f_i(v) \leq f_i(u)$ for all $i$ and $f_i(v) < f_i(u)$ for atleast one $i$, otherwise $`u'$ will not remain efficient solution as $f(v)$ dominates $f(u)$. Moreover, $u \in S$ is said to be weakly efficient solution if $\nexists$ another feasible solution $v \in$ S such that $f_i(v) < f_i(u) \forall i$.

## III. MULTIOBJECTIVE QUADRATIC FRACTIONAL PROGRAMMING MODEL

In MOQFPM, we need to simultaneously optimize several inter-related objective functions which are generally conflicting to each other under a common set of constraints. In general, all the objective functions are not satisfied by only one optimal solution. Hence, a set of efficient solutions is found which satisfies atleast one objective function without dissatisfying other objectives. MOQFPM is given as follows:

$$Min f(x) = \{f_1(x), f_2(x), f_3(x), \ldots\ldots\ldots, f_m(x)\}$$

$$such\ that\ x \in S$$

$$with\ \ f_i(x) = \frac{f_{i1}(x)}{f_i(x)};\ \ i = 1,2,\ldots,m$$

$$where\ \ f_i(x) = \frac{\frac{1}{2}x^T D_{i1}x\ +\ C_{i1}x + d_{i1}}{\frac{1}{2}x^T D_{i2}x + C_{i2}x + d_{i2}}$$

And S is the set of quadratic constraints given by

$$S = \left\{ x \in R^n \,|\, \frac{1}{2}x^T A_j x + B_j x + d_j \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} 0, x \geq 0 \right\}$$

Where $D_{i1}, D_{i2}$ are $n \times n$ real matrices.
$C_{i1}, C_{i2} \in R^n; d_{i1}, d_{i2} \in R$ and $A_j$ is a $k \times n$ real matrix $B_j \in R^n; d_j \in R$ $where\ j \in \{1, 2, \ldots, k\}$

## IV. PARAMETRIC APPROACH TO QUADRATIC FRACTIONAL PROGRAMMING

In parametric approach [1], [8], [9], [12], we assign a vector of parameters $\alpha_i$ to each objective function $f_i(x)$ which transforms fractional programming Model into non-fractional parametric Model given as:

$$M_1: Min f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$$

$$with\ \ f_i(x) = \frac{f_{i1}(x)}{f_{i2}(x)},\qquad i = 1,2,\ldots,m$$

Take $f_i$ (x) = $\alpha_i$ i.e $\frac{f_{i1}(x)}{f_{i2}(x)} = \alpha_i$
and let $P_i(x) = f_{i1}(x) - \alpha_i f_{i2}(x)$

∴ Model $M_1$ is reduced to the following non-fractional Model $M_2$.

$$M_2: \underset{x \in S}{Min}\ f(x) = \min_i\{P_i(x)\}$$

$$= \{f_{i1}(x) - \alpha_i f_{i2}(x)\},\qquad i = 1,2,\ldots,m$$

By using results of Dinkelbach on parametric and Quadratic fractional programming problem, we have the following results:

• **Result 1:** A vector $u \in S$ is referred to be an optimal solution of $M_1$ iff

$$\underset{x \in S}{Min}\{f_{i1}(x) - \alpha_i' f_{i2}(x)\} = 0$$

$$where\ \ \alpha_i' = \frac{f_{i1}(u)}{f_{i2}(u)}$$

• **Result 2:** A vector $u \in S$ is referred to be an efficient solution of $M_2$ if ∀ $x \in S$,

$$f_{i1}(x) - \alpha_i' f_{i2}(x) = 0\ \ \forall i\ \ or\ \ f_{i1}(x) - \alpha_i' f_{i2}(x) > 0\ \ for\ at\ least\ one\ \ i$$

**Theorem:** A vector $u \in S$ is referred to be an efficient solution of $M_1$ iff `$u'$ is an efficient solution of $M_2$.
**Proof:** Suppose $u \in S$ is an efficient solution of $M_1$.

$$\text{Let } P_i(x) = [f_{i1}(x) - \alpha_i' f_{i2}(x)]; 1 \leq i \leq m$$

• **To prove: $u \in S$ is an efficient solution of $M_2$**
On contrary, let us assume that `$u'$ is not an efficient solution of $M_2$.
∴ By definition of efficient solution, ∃ $v \in S$ such that $P_i(v) \leq P_i(u)$ ∀ $i$ and $P_i(v) < P_i(u)$ for atleast one $i$.
i.e $f_{i1}(v) - \alpha_i' f_{i2}(v) \leq f_{i1}(u) - \alpha_i' f_{i2}(u)\ \forall i$.

$$f_{i1}(v) - \alpha_i' f_{i2}(v) < f_{i1}(u) - \alpha_i' f_{i2}(u)\ \ for\ atleast\ one\ i$$

i.e. $f_{i1}(v) - \alpha_i' f_{i2}(v) \leq 0\ \forall i. \left[as \frac{f_{i1}(u)}{f_{i2}(u)} = \alpha_i'\right]$
and $f_{i1}(v) - \alpha_i' f_{i2}(v) < 0\ \ for\ atleast\ one\ i$

i.e. $\frac{f_{i1}(v)}{f_{i2}(v)} \leq \alpha_i'\ \ \forall\ i$

and $\frac{f_{i1}(v)}{f_{i2}(v)} < \alpha_i'\ \ for\ atleast\ one\ i$
∴ $f_i(v) \leq f_i(u)\ \forall i$.
and $f_i(v) < f_i(u)\ \ for\ atleast\ one\ i$
This contradicts that `$u'$ is an efficient solution of $M_1$.
∴ Our supposition is wrong.
Hence, `$u'$ is also an efficient solution of $M_2$.
**Conversely**, Suppose that `$u'$ is an efficient solution of $M_2$.
• **To prove: `$u'$is an efficient solution $M_1$**
On contrary, suppose that $u \in S$ is not an efficient solution of $M_1$.

∴ ∃ $v \in S$ such that $f_i(v) \leq f_i(u)\ \forall i$.
and $f_i(v) < f_i(u)$ for atleast one i.

i.e. $\frac{f_{i1}(v)}{f_{i2}(v)} < \alpha_i'\ \ \forall i;\ \ where\ \ \alpha_i' = \frac{f_{i1}(u)}{f_{i2}(u)}$

and $\frac{f_{i1}(v)}{f_{i2}(v)} < \alpha_i'$ for one $i$ atleast.
∴ $f_{i1}(v) - \alpha_i' f_{i2}(v) \leq 0\ \forall i$.
and $f_{i1}(v) - \alpha_i' f_{i2}(v) < 0$ for one $i$ atleast.

∴ $P_i(v) \leq 0\ \ \forall i$.
and $P_i(v) < 0$ for atleast one $i$.

$$\therefore\ \ P_i(u) = f_{i1}(u) - \alpha_i' f_{i2}(u)$$

$$P_i(u) = f_{i1}(u) - \frac{f_{i1}(u)}{f_{i2}(u)} f_{i2}(u)$$

∴ $P_i(u) = 0$
∴ $P_i(v) \leq P_i(u)\ \forall i$.
and $P_i(v) < P_i(u)$ for atleast one i.
This contradicts that `$u'$ is an efficient solution of $M_2$.
So, our supposition is wrong.
Hence, $u \in S$is also an efficient solution of $M_1$.

## V. ε - CONSTRAINT METHOD

This method is used to obtain efficient solutions of Multiobjective problems [2], [8], [9]. In this method, one objective function is optimized to its best desired level and remaining objectives are converted into constraints with their acceptability levels maintained by the efficient solution. The ε -constraint method is expressed as follows:

$$Min\ \ P_r(x),\quad r \in \{1,2,\ldots,\ m\}$$

$$such\ that\ P_i(x) \leq \varepsilon_i\ \forall i = 1,2,\ldots,r-1,\ r+1,\ldots m\ and\ x \in S$$

where $\varepsilon_i \in [\varepsilon_i^L, \varepsilon_i^U]$ and $\varepsilon_i^L$ & $\varepsilon_i^U$ are the lowest and the greatest values of the objective function $P_i(x)$. By putting different values of $\varepsilon_i$, we can find a set of efficient solutions.

## VI. FORMULATION AND METHODOLOGY OF MODEL

$$M_1: Min\ f(x) = \{f_1(x), f_2(x), \ldots, f_m(x)\}$$

$$with\ f_i(x) = \frac{f_{i1}(x)}{f_{i2}(x)}, \quad \forall 1 \leq i \leq m$$

$$subject\ to\ x \in S$$

$$S = \left\{ x \in R^n \mid \frac{1}{2} x^T A_j\ x + B_j x + d_j \begin{pmatrix} \geq \\ \leq \\ = \end{pmatrix} 0, x \geq 0 \right\}$$

$$where\ j = 1,2,\ldots,k$$

Let us assume that each $f_i(x) = \alpha_i^{(t)}$, $i = 1,2,\ldots, m$ where `$t$' is the iteration no.

Let $\alpha^{(t)} = \left(\alpha_1^{(t)}, \alpha_2^{(t)}, \ldots, \alpha_m^{(t)}\right)$ be the vector of parameters for the objective function $f(x)$. and suppose $P_i(\alpha^{(t)}) = f_{i1}(x) - \alpha_i^{(t)} f_{i2}(x),\ i = 1,2,\ldots,m.$

So, the above Model $M_1$ is transformed to Multiobjective parametric non-fractional Model $M_2$given as follows :

$$M_2: Min\ P_i(\alpha^{(t)}) = \left\{ f_{i1}(x) - \alpha_i^{(t)} f_{i2}(x) \right\}$$

$$subject\ to\ x \in S$$

Now, by ε-constraint method, we will optimize one objective function depending upon the priorities decided by the Decision Maker (DM) and convert other objective functions as constraints.

Thus, we can convert Model $M_2$ into Model $M_3$ as follows:

$$M_3: Min\ P_r(\alpha^{(t)}) = f_{r1}(x) - \alpha_r^{(t)} f_{r2}(x)$$

$$subject\ to$$

$$P_i\left(\alpha^{(t)}\right) = f_{i1}(x) - \alpha_i^{(t)} f_{i2}(x) \leq \varepsilon_i$$
$$\forall\ i = 1,2,\ldots,r-1,r+1,\ldots,m$$
$$and\ x \in S\ where\ \varepsilon_i \in [\varepsilon_i^L, \varepsilon_i^U]$$

Let $X_i$ $(i = 1,2,\ldots,m)$ be the individual optimal solutions of $f_i(x)$ subject to $x \in S$.

Table I is constructed to find the values of $f_i(X_i)$ $\forall$ $i = 1,2,\ldots, m$ as follows:

**Table I: objective function values of $M_1$**

| $X_i$ | $f_1(X_i)$ | $f_2(X_i)$ | $f_3(X_i)$ | ...... | $f_m(X_i)$ |
|---|---|---|---|---|---|
| $X_1$ | $f_1(X_1)$ | $f_2(X_1)$ | $f_3(X_1)$ | ...... | $f_m(X_1)$ |
| $X_2$ | $f_1(X_2)$ | $f_2(X_2)$ | $f_3(X_2)$ | ...... | $f_m(X_2)$ |
| | | | | | |
| $X_m$ | $f_1(X_m)$ | $f_2(X_m)$ | $f_3(X_m)$ | ...... | $f_m(X_m)$ |

Define $\varepsilon_i^L$ $and$ $\varepsilon_i^U$ as follows :

$$\varepsilon_i^L = \min\{P_i(X_i) | 1 \leq i \leq m\}$$
$$\varepsilon_i^U = \max\{P_i(X_i) | 1 \leq i \leq m\}$$

Then, calculate initial feasible solution $X^{(0)}$ to $M_3$ as follows:

$$X^{(0)} = \sum_{i=1}^{m} w_i X_i$$

Where $\sum_{i=1}^{m} w_i = 1\ and\ w_i \succ 0$ and $X_i$ are the individual optimal solutions of $f_i(x)$ $\forall$ $i = 1,2,\ldots, m$. Nearly equal weights are considered for each $X_i$. Next, we obtain the initial vector of parameters as

$$\alpha^{(1)} = \left(\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots \alpha_m^{(1)}\right)$$
$$= \left\{f_1(X^{(0)}), f_2(X^{(0)}), \ldots, f_m(X^{(0)})\right\}$$

We then substitute $\alpha^{(1)}$ in each $P_i(\alpha^{(t)})$ and check termination conditions and continue the process till the termination conditions are satisfied.

## VII. TERMINATION CONSTANTS AND CONDITIONS

Terminations constants $(T_i)$ are basically the tolerance values of the objective functions $f_i(x)$ which are acceptable by the DM. These values are predetermined by DM considering the priority of the objective function and are generally taken nearer to zero. So, Termination conditions are defined as :

$$|P_i(\alpha^{(t)})| \leq T_i, i = 1,2,\ldots,m$$

Where each $T_i > 0$.

## VIII. ASSUMPTIONS

- Equal weightage is given to individual solutions of each fractional function in the initial solution.
- Termination constants are decided by the Decision Maker for every objective function and generally taken close to zero.
- Initial feasible solution to the problem is given by

$$X^{(0)} = \sum_{i=1}^{m} w_i X_i$$

Where $\sum_{i=1}^{m} w_i = 1\ and\ w_i \succ 0$.

## IX. ALGORITHM

1. Take the Initial value of the vector of parameters as $\alpha^{(1)} = (\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots \alpha_m^{(1)})$

$$= \left\{f_1(X^{(0)}), f_2(X^{(0)}), \ldots, f_m(X^{(0)})\right\}$$

2. Obtain non-fractional parametric functions $P_i(\alpha^{(1)})$ by substituting $(\alpha^{(1)})$.

3. Select $P_r(\alpha^{(1)})$ as the objective function with least value of $T_r$.

4. Select different values of $\varepsilon_i \in [\varepsilon_i^L, \varepsilon_i^U]$; where i = 1, ..., r – 1, r + 1, ... m as follows:

(a) If $[-T_i, T_i] \cap [\varepsilon_i^L, \varepsilon_i^U] = \phi$, then, select$\varepsilon_i \in [\varepsilon_i^L, \varepsilon_i^U]$

(b) Otherwise select $\varepsilon_i \in [-T_i, T_i]$

5. Find a set of efficient solutions for Model $M_3$ by substituting different values of $\varepsilon_i$. Software lingo 15 is used for this purpose.

6. Check the Termination conditions $|P_i(\alpha^{(1)})| \leq T_i \;\forall\; i = 1,2,\dots, m$

7. If termination conditions are satisfied, then we end up our process. Otherwise, go to step 8.
8. Determine $Min \sum_i \left(|P_i(\alpha^{(1)})| - T_i\right)$ for $i \in \{1, 2, ..., m\}$ at which conditions are not satisfied for every set of efficient solution.
9. Suppose $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \dots, X_m^{(1)})$ be the compromised solution at which
$\sum \left(|P_i(\alpha^{(1)})| - T_i\right)$ is minimum.
10. Compute $\alpha^{(2)} = (\alpha_1^{(2)}, \alpha_2^{(2)}, \dots, \alpha_m^{(2)})$
i.e. $\alpha^{(2)} = (f_i(X^{(1)}), f_2(X^{(1)}), \dots, f_m(X^{(1)}))$
11. Find another set of efficient solution of $M_3$ and test termination conditions for them.
12. Repeat the method until we obtain a set of efficient solution which satisfies $\left|P_i\left(\alpha^{(t)}\right)\right| \leq T_i \forall \; i = 1,2,\dots, m.$ Otherwise, redefine the termination constants.
13. Once, efficient solution set is obtained, then decision maker can choose any one value out of them as the efficient solution.

## X. ILLUSTRATIVE NUMERICAL

Consider the MOQFPM given below :-

$$Min f(x) = \left\{ f_1(x) = \frac{2x_1^2 + x_3}{x_2^2 + 3}, f_2(x) = \frac{2x_3^2 + x_2}{x_1^2 + 3} \right\}$$

$$subject\ to$$

$$S = \left\{ \begin{array}{l} 2x_1^2 + x_2^2 + x_3 \leq 4 \\ 2x_1^2 + x_3^2 + x_1 \leq 5 \\ x_3^2 + x_1^2 + x_2 \geq 3 \\ x_2^2 + x_1^2 + x_3^2 \leq 6 \\ x_1, x_2, x_3 \geq 0 \end{array} \right\}$$

**Solution with the help of parametric method:**

Individual initial optimal solutions of the functions $f_1(x)$ and $f_2(x)$ are obtained with the help of software lingo 15 and they come out to be

$$X_1 = (x_1^1, x_2^1, x_3^1) = (0,1.281,1.31)$$

$$X_2 = (x_1^2, x_2^2, x_3^2) = (1.13,0.619,1.05)$$

**Table II: Objective functions at initial solution**

| $X_i$ | $f_1(X_i)$ | $f_2(X_i)$ |
|---|---|---|
| $X_1$ | 0.2823 | 1.571 |
| $X_2$ | 1.065 | 0.66 |

From the Table II, we can see that

$$0.2823 \leq f_1(x) \leq 1.065$$

and $0.66 \leq f_2(x) \leq 1.571$

Let equal weights be assigned to each solution
i.e. $w_1 = w_2 = 0.5$
Initial optimal solution is given by :

$$X^{(0)} = w_1 X_1 + w_2 X_2$$

$$X^{(0)} = 0.5(0,1.281,1.31) + 0.5(1.13,0.619,1.05)$$

$$X^{(0)} = (0.565,0.95,1.18)$$

So, initial value of the vector of parameters is

$$\alpha^{(1)} = (\alpha_1^{(1)}, \alpha_2^{(1)})$$

$$\alpha^{(1)} = (f_1(X^{(0)}), f_2(X^{(0)}))$$

$$\alpha^{(1)} = (0.466,1.125)$$

For converting fractional objectives into non-fractional parametric functions, suppose

$$P_1(\alpha^{(t)}) = (2x_1^2 + x_3) - \alpha_1^{(t)}(x_2^2 + 3)$$

Where `$t'$ represents iteration number

$$P_2(\alpha^{(t)}) = (2x_3^2 + x_2) - \alpha_2^{(t)}(x_1^2 + 3)$$

Thus, $P_1(\alpha^{(1)}) = (2x_1^2 + x_3) - 0.466(x_2^2 + 3)$

$$P_1(\alpha^{(1)}) = 2x_1^2 - 0.466x_2^2 + x_3 - 1.398$$

$$P_2(\alpha^{(1)}) = (2x_3^2 + x_2) - 1.125(x_1^2 + 3)$$

$$= -1.125x_1^2 + 2x_3^2 + x_2 - 3.375$$

Thus, non-fractional parametric programming problem is given by :

$$Min\ f(x) = Min\ \{P_1(\alpha^{(1)}), P_2(\alpha^{(1)})\}$$

$$subject\ to\ x \in S$$

Defining termination constants as

$$T_1 = 0.02 \quad and \quad T_2 = 0.03$$

Initial solutions of $P_1(\alpha^{(1)})$ and $P_2(\alpha^{(1)})$ comes out to be $X_1 = (0,1.28,1.31)$ and $X_2 = (1.16,0.48,1.09)$respectively
Because $T_1 < T_2$.

Therefore, by using ε-constraint method, non-fractional parametric problem is considered as following:

$$Min\ P_1(\alpha^{(1)}) = 2x_1^2 - 0.466x_2^2 + x_3 - 1.398$$

$$subject\ to$$

$$2x_3^2 - 1.125x_1^2 + x_2 - 3.375 \leq \varepsilon_2$$

$$and\ x \in S$$

$$where\ \varepsilon_2 \in [\varepsilon_2^L, \varepsilon_2^U]$$

and $\varepsilon_2^L = Min\ \{P_2(X_i); i = 1,2\}$
i.e. $\varepsilon_2^L = Min\ \{P_2(X_1), P_2(X_2)\} = -2.033$
and $\varepsilon_2^U = Max\ \{P_2(X_1), P_2(X_2)\} = 1.337$

$$\therefore \quad [\varepsilon_2^L, \varepsilon_2^U] = [-2.033,1.337]$$

Thus, $[-T_2, T_2] \subseteq [\varepsilon_2^L, \varepsilon_2^U]$.
So, we choose $\varepsilon_2 \in [-T_2, T_2]$
i.e. $\varepsilon_2 \in [-0.03,0.03]$
So, by substituting different values of $\varepsilon_2$, we get a set of pareto optimal solutions which are shown in Table III.

**Table III: Values of Efficient solutions**

| $\varepsilon_2$ | $x_1$ | $x_2$ | $x_3$ | $P_1(\alpha^{(1)})$ | $\lvert P_1(\alpha^{(1)})\rvert - T_1$ |
|---|---|---|---|---|---|
| – 0.03 | 0.6762 | 1.226 | 1.147 | – 0.036858 | 0.016858 |
| – 0.024 | 0.6749 | 1.226 | 1.148 | – 0.0396018 | 0.0196 |
| – 0.018 | 0.6734 | 1.226 | 1.149 | – 0.04235 | 0.0223 |
| – 0.012 | 0.67199 | 1.226 | 1.15 | – 0.04509 | 0.02509 |
| – 0.006 | 0.67059 | 1.226 | 1.15 | – 0.04785 | 0.02785 |
| 0.006 | 0.6678 | 1.225 | 1.15 | – 0.05335 | 0.03335 |
| 0.012 | 0.6663 | 1.225 | 1.15 | – 0.05611 | 0.03611 |
| 0.018 | 0.6649 | 1.225 | 1.15 | – 0.05887 | 0.03887 |
| 0.024 | 0.6635 | 1.225 | 1.16 | – 0.0616 | 0.0416 |
| 0.03 | 0.6621 | 1.225 | 1.16 | – 0.06439 | 0.04439 |

So, the above set of efficient solution is obtained using Lingo 15 software.

Thus, at each efficient solution obtained above, $|P_2(\alpha^{(1)})| \leq T_2$. But we can see that $|P_1(\alpha^{(1)})| > T_1$. So, termination condition is not satisfied by $P_1(\alpha^{(1)})$.

Now,

$$Min \sum_i |(P_i(\alpha^{(1)})| - T_i) = Min\ (|P_1(\alpha^{(1)})| - T_1)$$

$$= 0.016858(\text{fromTableIII})$$

It occurs at $X^1 = (0.6762, 1.226, 1.147)$.

Thus, $X^1$ is the compromised solution.

So, the next iterated vector of parameters is given by

$$\alpha^{(2)} = (\alpha_1^{(2)}, \alpha_2^{(2)})$$

$$\alpha^{(2)} = (f_1(X^{(1)}), f_2(X^{(1)}))$$

$$\alpha^{(2)} = (0.4578, 1.116)$$

Thus, New iterated parametric programming Model is:

$$P_1(\alpha^{(2)}) = (2{x_1}^2 + x_3) - 0.4578({x_2}^2 + 3)$$

$$= 2x_1^2 - 0.4578x_2^2 + x_3 - 1.3734$$

$$P_2(\alpha^{(2)}) = (2x_3^2 + x_2) - 1.116(x_1^2 + 3)$$

$$= 2x_3^2 - 1.116x_1^2 + x_2 - 3.348$$

Model: $Min\ f\ (x) = Min\ \{P_1(\alpha^{(2)}), P_2(\alpha^{(2)})\}$

$$subject\ to\ x \in S$$

With the help of software Lingo 15, initial individual optimal solutions of $P_1(\alpha^{(2)})$ and $P_2(\alpha^{(2)})$ comes to be

$$X_1 = (0,1.281,1.311), X_2 = (1.159,0.4772,1.086)$$

Since $T_1 < T_2$.

So, above Model is transformed to following Model:

$$Min\ P_1(\alpha^{(2)}) = 2x_1^2 - 0.4578x_2^2 + x_3 - 1.3734$$

$$subject\ to$$

$$2x_3^2 - 1.116x_1^2 + x_2 - 3.348 \ \leq\ \varepsilon_2$$

$$and\ x\ \in\ S$$

$$where\ \varepsilon_2 \in\ [\varepsilon_2^L, \varepsilon_2^U]$$

and $\varepsilon_2^L = \min\{P_2(X_1), P_2(X_2)\} = -2.01$

an$\varepsilon_2^U = Max\ \{P_2(X_1), P_2(X_2)\} = 1.37$.

Thus $[\varepsilon_2^L, \varepsilon_2^U] = [-2.01, 1.37]$ and $[-T_2, T_2] \ \subseteq\ [\varepsilon_2^L, \varepsilon_2^U]$.

So, we choose $\varepsilon_2\ \in\ [-T_2, T_2]$

**Table IV: values of efficient solutions**

| $\varepsilon_2$ | $x_1$ | $x_2$ | $x_3$ | $P_1(\alpha^{(2)})$ |
|---|---|---|---|---|
| – 0.03 | 0.6834 | 1.226 | 1.143 | 0.0143 |
| – 0.024 | 0.6819 | 1.227 | 1.144 | 0.0116 |
| – 0.018 | 0.6806 | 1.227 | 1.145 | 0.0088 |
| – 0.012 | 0.6792 | 1.227 | 1.146 | 0.0061 |
| – 0.006 | 0.6778 | 1.226 | 1.146 | 0.0033 |
| 0.006 | 0.6750 | 1.226 | 1.148 | – 0.0022 |
| 0.012 | 0.6736 | 1.226 | 1.149 | – 0.0049 |
| 0.018 | 0.6722 | 1.226 | 1.15 | –0.00768 |
| 0.024 | 0.6708 | 1.226 | 1.151 | – 0.0104 |
| 0.03 | 0.6694 | 1.226 | 1.151 | – 0.0132 |

From the Table IV, it is clear that $|P_1(\alpha^{(2)})| < T_1$ at each efficient solution. Moreover, we can easily check that $|P_2(\alpha^{(2)})| \ \leq\ T_2$ at each solution. Thus, Termination condition is satisfied by both $P_1(\alpha^{(2)})$ and $P_2(\alpha^{(2)})$. So, the decision maker can choose any of the above find efficient solution as the best solution to the problem. The values of $f_1(x)\ and\ f_2(x)$ which are evaluated at each above found solution are :

**Table V: values of objective functions**

| $x_1$ | $x_2$ | $x_3$ | $f_1(x)$ | $f_2(x)$ |
|---|---|---|---|---|
| 0.6834 | 1.226 | 1.143 | 0.4613 | 0.7304 |
| 0.6819 | 1.227 | 1.144 | 0.4603 | 0.73181 |
| 0.6806 | 1.227 | 1.145 | 0.3569 | 0.7328 |
| 0.6792 | 1.227 | 1.146 | 0.3567 | 0.7339 |
| 0.6778 | 1.226 | 1.146 | 0.3565 | 0.7340 |
| 0.6750 | 1.226 | 1.148 | 0.3561 | 0.7372 |
| 0.6736 | 1.226 | 1.149 | 0.3559 | 0.7372 |
| 0.6722 | 1.226 | 1.15 | 0.3557 | 0.7383 |
| 0.6708 | 1.226 | 1.151 | 0.3555 | 0.7394 |
| 0.6694 | 1.226 | 1.151 | 0.3551 | 0.7398 |

**Comparative study of parametric method and fuzzy programming method:**

Solution with the help of fuzzy goal programming is given by:

We know $(x_1^1, x_2^1, x_3^1) = (0,1.281,1.31)$ and $(x_1^2, x_2^2, x_3^2) = (1.13,0.619,1.05)$ are the individual optimal solutions of $f_1(x)\ and\ f_2(x)$.

Also, $(f_1)_{\min} = 0.2823 \ \leq\ f_1(x) \ \leq\ 1.065 = (f_1)_{\max}$

$(f_2)_{\min} - 0.66 \ \leq\ f_2(x) \ \leq\ 1.511 = (f_2)_{\max}$

By solving the considered numerical example with fuzzy goal programming, best optimal solution comes out to be

$$(f_1(x), f_2(x)) = (0.2842, 0.9072)$$

Thus, we can see that the values of $(f_1(x), f_2(x))$ calculated with the help of the proposed method and the fuzzy approach are comparable to each other and this validates the proposed method of parametric functions.

## XI. CONCLUSION

This paper proposed an approach of solving MOQFPM by converting it into single objective non- fractional parametric programming Model with the help ofε-constraint method and this method can be extended for solving bi-level and multilevel fractional programming Models.The parametric approach used in the study makes it very easy to transform fractional objectives into non-fractional functions for which efficient solutions can be obtained easily. In the numerical example illustrated in the paper, the set of solutions obtained with the proposed method are comparable to those obtained with fuzzy approach, which validates the feasibility of our approach.

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