Multiobjective Quadratic Fractional Programming using Iterative Parametric Function


Throughout the paper, we have used parametric approach proposed by Nayak and Ojha [8,9] and extended their work to MOQFPM with the help of ε-constraint method. In this method, efficient solution is obtained by converting quadratic FPP into non-fractional problem by predefining termination constants.

II. NOTATIONS AND PRELIMINARIES

In the paper, we denote the space of n-dimensional real vectors by $R^n$. For a given vector $x$, $x^T$ represents the transpose of $x$. We assign a vector of parameters $\alpha (^{(t)} \beta)$ to objective functions where $^{(t)} \beta$ denotes iteration number. $T_i$ in the paper represents termination constants defined by decision maker. $S'$ represents a set of constraints.

1) Pareto or Efficient Solution

A vector $u \in S'$ is called a pareto or efficient solution if there is no another feasible solution $v \in S'$ such that $f_i(v) \leq f_i(u)$ for all $i$ and $f_j(v) < f_j(u)$ for atleast one $i$, otherwise ‘$u$’ will not remain efficient solution as $f(v)$ dominates $f(u)$. Moreover, $u \in S'$ is said to be weakly efficient solution if $f_i(v) < f_i(u) \forall i$.

III. MULTIOBJECTIVE QUADRATIC FRACTIONAL PROGRAMMING MODEL

In MOQFPM, we need to simultaneously optimize several inter-related objective functions which are generally conflicting to each other under
Multiobjective Quadratic Fractional Programming using Iterative Parametric Function

In general, all the objective functions are not satisfied by only one optimal solution. Hence, a set of efficient solutions is found which satisfies at least one objective function without dissatisfying other objectives. MOQFPM is given as follows:

\[
\text{Min} f(x) = \{f_1(x), f_2(x), \ldots, f_m(x)\}
\]

such that \( x \in S \)

\[
\text{with } f_i(x) = \frac{f_i(x)}{x}; \quad i = 1, 2, \ldots, m
\]

where \( f_i(x) = \frac{1}{2} x^T D_i x + C_i x + d_i \)

And S is the set of quadratic constraints given by

\[
S = \left\{ x \in \mathbb{R}^n \mid \frac{1}{2} x^T A_i x + B_j x + d_j \leq 0, x \geq 0 \right\}
\]

Where \( D_{i1}, D_{i2} \) are \( n \times n \) real matrices, \( C_{i1}, C_{i2} \in \mathbb{R}^n; d_{i1}, d_{i2} \in \mathbb{R} \) and \( A_i \) is a \( k \times n \) real matrix \( B_j \in \mathbb{R}^n; d_j \in \mathbb{R} \) where \( j \in \{1, 2, \ldots, k\} \)

IV. PARAMETRIC APPROACH TO QUADRATIC FRACTIONAL PROGRAMMING

In parametric approach [1], [8], [9], [12], we assign a vector of parameters \( \alpha \) to each objective function \( f_i(x) \) which transforms fractional programming Model into non-fractional parametric Model given as:

\[
M_1: \text{Min} f(x) = \{f_1(x), f_2(x), \ldots, f_m(x)\}
\]

\[
\text{with } f_i(x) = \frac{f_i(x)}{x}; \quad i = 1, 2, \ldots, m
\]

Take \( f_i(x) = \alpha \) i.e \( f_i(x) = \alpha \)

and let \( P_i(x) = f_i(x) - \alpha f_i(x) \)

\[
\text{• Model}\ M_1 \text{is reduced to the following non-fractional Model}\ M_2:
\]

\[
M_2: \text{Min} \quad \frac{P_i(x)}{x} = \min \{P_i(x)\}\]

\[
\text{such that } \frac{P_i(x)}{x} > 0 \text{ and } \frac{P_i(x)}{x} > 0 \text{ for at least one } i
\]

By using results of Dinkelbach on parametric and Quadratic fractional programming problem, we have the following results:

• Result 1: A vector \( u \in S \) is referred to be an optimal solution of \( M_1 \) iff

\[
\text{Min} \left( f_i(x) - \alpha f_i(x) \right) > 0; \quad i = 1, 2, \ldots, m
\]

• Result 2: A vector \( u \in S \) is referred to be an efficient solution of \( M_2 \) if \( \forall \ x \in S, \ f_i(x) - \alpha f_i(x) > 0 \text{ for at least one } i
\]

Theorem: A vector \( u \in S \) is referred to be an efficient solution of \( M_1 \) iff ‘\( u \)’ is an efficient solution of \( M_2 \).

Proof: Suppose \( u \in S \) is an efficient solution of \( M_1 \). Let \( P_i(x) = f_i(x) - \alpha f_i(x); \quad 1 \leq i \leq m
\]

• To prove: \( u \in S \) is an efficient solution of \( M_2 \)

On contrary, let us assume that \( u \) is not an efficient solution of \( M_2 \).

\[
\text{By definition of efficient solution, } \exists \ v \in S \text{ such that } P_i(u) \leq P_i(v) \forall i \text{ and } f_i(u) < f_i(v) \text{ for at least one } i
\]

\[
i.e. \quad f_i(u) - \alpha f_i(x) \leq f_i(v) - \alpha f_i(x) \forall i
\]

\[
\text{and } f_i(u) - \alpha f_i(x) < 0 \text{ for at least one } i
\]

\[
i.e. \quad f_i(u) - \alpha f_i(x) < 0 \forall i
\]

This contradicts that ‘\( u \)’ is an efficient solution of \( M_1 \).

∴ Our supposition is wrong.

Hence, ‘\( u \)’ is also an efficient solution of \( M_2 \).

Conversely. Suppose that ‘\( u \)’ is an efficient solution of \( M_1 \).

∴ \( \exists \ v \in S \text{ such that } P_i(v) \leq P_i(u) \forall i \text{ and } f_i(v) < f_i(u) \text{ for at least one } i
\]

\[
i.e. \quad f_i(v) - \alpha f_i(x) < 0 \forall i
\]

This contradicts that ‘\( u \)’ is an efficient solution of \( M_2 \).

∴ Our supposition is wrong.

Hence, \( u \in S \) is also an efficient solution of \( M_1 \).

V. \( \varepsilon \)-CONSTRAINT METHOD

This method is used to obtain efficient solutions of Multiobjective problems [2], [8], [9]. In this method, one objective function is optimized to its best desired level and remaining objectives are converted into constraints with their acceptability levels maintained by the efficient solution. The \( \varepsilon \)-constraint method is expressed as follows:

\[
\text{Min } P_r(x), \quad r \in \{1, 2, \ldots, m\}
\]
such that $P_i(x) \leq \epsilon_i \ \forall i = 1, 2, \ldots, r - 1, r + 1, \ldots, m$ and $x \in S$

where $\epsilon_i \in [\epsilon^L_i, \epsilon^U_i]$ and $\epsilon^L_i$ & $\epsilon^U_i$ are the lowest and the greatest values of the objective function $P_i(x)$. By putting different values of $\epsilon_i$, we can find a set of efficient solutions.

VI. FORMULATION AND METHODOLOGY OF MODEL

$M_1$: Min $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$

with $f_i(x) = \frac{f_{i1}(x)}{f_{i2}(x)}$, $\forall 1 \leq i \leq m$

subject to $x \in S$

$S = \{ x \in R^n | \frac{1}{2} x^T A_j x + B_j x + d_j \leq 0, x \geq 0 \}$

where $j = 1, 2, \ldots, k$

Let us assume that each $f_i(x) = \alpha_i^t$, $i = 1, 2, \ldots, m$ where 't' is the iteration no.

Let $\alpha^{(t)} = (\alpha^{(t)}_1, \alpha^{(t)}_2, \ldots, \alpha^{(t)}_m)$ be the vector of parameters for the objective function $f(x)$, and suppose $P_i(\alpha^{(t)}) = f_{i1}(x) - \alpha^{(t)}_i f_{i2}(x), i = 1, 2, \ldots, m$.

So, the above Model $M_1$ is transformed to Multiobjective parametric non-fractional Model $M_2$ given as follows:

$M_2$: Min $P_i(\alpha^{(t)}) = f_{i1}(x) - \alpha^{(t)}_i f_{i2}(x)$

subject to $x \in S$

Now, by $\epsilon$-constraint method, we will optimize one objective function depending upon the priorities decided by the Decision Maker (DM) and convert other objective functions as constraints.

Thus, we can convert Model $M_2$ into Model $M_3$ as follows:

$M_3$: Min $P_i(\alpha^{(t)}) = f_{i1}(x) - \alpha^{(t)}_i f_{i2}(x)$

subject to

$P_i(\alpha^{(t)}) = f_{i1}(x) - \alpha^{(t)}_i f_{i2}(x) \leq \epsilon_i$

$\forall i = 1, 2, \ldots, r - 1, r + 1, \ldots, m$

and $x \in S$ where $\epsilon_i \in [\epsilon^L_i, \epsilon^U_i]$

Let $X_i (i = 1, 2, \ldots, m)$ be the individual optimal solutions of $f_i(x)$ subject to $x \in S$. Table 1 is constructed to find the values of $f_i(X_i)$ $\forall i = 1, 2, \ldots, m$ as follows:

| $X_i$ | $f_1(X_i)$ | $f_2(X_i)$ | $f_3(X_i)$ | $\ldots$ | $f_m(X_i)$ |
|---|---|---|---|\ldots|---|
| $x_1$ | $f_1(x_1)$ | $f_2(x_1)$ | $f_3(x_1)$ | $\ldots$ | $f_m(x_1)$ |
| $x_2$ | $f_1(x_2)$ | $f_2(x_2)$ | $f_3(x_2)$ | $\ldots$ | $f_m(x_2)$ |
| $x_m$ | $f_1(x_m)$ | $f_2(x_m)$ | $f_3(x_m)$ | $\ldots$ | $f_m(x_m)$ |

Define $\epsilon^L_i$ and $\epsilon^U_i$ as follows:

$\epsilon^L_i = \min\{P_i(X_i) | 1 \leq i \leq m\}$

$\epsilon^U_i = \max\{P_i(X_i) | 1 \leq i \leq m\}$

Then, calculate initial feasible solution $X^{(0)}$ to $M_3$ as follows:

$X^{(0)} = \sum_{i=1}^{m} w_i X_i$

Where $\sum_{i=1}^{m} w_i = 1$ and $w_i > 0$ and $X_i$ are the individual optimal solutions of $f_i(x)$ $\forall i = 1, 2, \ldots, m$. Nearly equal weights are considered for each $X_i$. Next, we obtain the initial vector of parameters as

$\alpha^{(1)} = (\alpha^{(1)}_1, \alpha^{(1)}_2, \ldots, \alpha^{(1)}_m) = \{f_1(X^{(0)}), f_2(X^{(0)}), \ldots, f_m(X^{(0)})\}$

We then substitute $\alpha^{(1)}$ in each $P_i(\alpha^{(1)})$ and check termination conditions and continue the process till the termination conditions are satisfied.

VII. TERMINATION CONSTANTS AND CONDITIONS

Terminations constants ($T_i$) are basically the tolerance values of the objective functions $f_i(x)$ which are acceptable by the DM. These values are predetermined by DM considering the priority of the objective function and are generally taken nearer to zero. So, Termination conditions are defined as:

$|P_i(\alpha^{(t)})| \leq T_i, i = 1, 2, \ldots, m$

Where each $T_i > 0$.

VIII. ASSUMPTIONS

• Equal weightage is given to individual solutions of each fractional function in the initial solution.

• Termination constants are decided by the Decision Maker for every objective function and generally taken close to zero.

• Initial feasible solution to the problem is given by

$X^{(0)} = \sum_{i=1}^{m} w_i X_i$

Where $\sum_{i=1}^{m} w_i = 1$ and $w_i > 0$.

IX. ALGORITHM

1. Take the Initial value of the vector of parameters as

$\alpha^{(1)} = (\alpha^{(1)}_1, \alpha^{(1)}_2, \ldots, \alpha^{(1)}_m) = \{f_1(X^{(0)}), f_2(X^{(0)}), \ldots, f_m(X^{(0)})\}$

2. Obtain non-fractional parametric functions $P_i(\alpha^{(1)})$ by substituting $\alpha^{(1)}$.

3. Select $P_i(\alpha^{(1)})$ as the objective function with least value of $T_i$.

4. Select different values of $\epsilon_i \in [\epsilon^L_i, \epsilon^U_i]$; where $i = 1, \ldots, r - 1, r + 1, \ldots, m$ as follows:

(a) If $[-T_i, T_i] \cap [\epsilon^L_i, \epsilon^U_i] = \phi$, then, select $\epsilon_i \in [\epsilon^L_i, \epsilon^U_i]$

(b) Otherwise select $\epsilon_i \in [-T_i, T_i]$

5. Find a set of efficient solutions for Model $M_3$ by substituting different values of $\epsilon_i$. Software lingo 15 is used for this purpose.
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6. Check the Termination conditions $|P_i(α^{(1)})| ≤ T_i \forall i = 1, 2, ..., m$

7. If termination conditions are satisfied, then we end up our process. Otherwise, go to step 8.

8. Determine $\text{Min } \sum (P_i(α^{(2)}) - T_i)$ for $i \in \{1, 2, ..., m\}$ at which conditions are not satisfied for every set of efficient solution.

9. Suppose $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, ..., X_m^{(1)})$ be the compromised solution at which

$$\sum (P_i(α^{(1)})) - T_i \text{ is minimum.}$$

10. Compute $α^{(2)} = (α_1^{(2)}, α_2^{(2)}, ..., α_m^{(2)})$

i.e. $α^{(2)} = (f_1(X^{(1)}), f_2(X^{(1)}), ..., f_m(X^{(1)}))$

11. Find another set of efficient solution of $M_3$ and test termination conditions for them.

12. Repeat the method until we obtain a set of efficient solution which satisfies $|P_i(α^{(1)})| ≤ T_i \forall i = 1, 2, ..., m$.

Otherwise, redefine the termination constants.

13. Once, efficient solution set is obtained, then decision maker can choose any one value out of them as the efficient solution.

X. ILLUSTRATIVE NUMERICAL

Consider the MOOFPM given below :-

$$\text{Min } f(x) = \left\{ f_1(x) = \frac{2x_1^2 + x_3}{x_1^2 + 3}, f_2(x) = \frac{2x_3^2 + x_2}{x_1^2 + 3} \right\}$$

subject to

$$2x_1^2 + x_3 + x_3 ≤ 4$$

$S = \left\{ 2x_1^2 + x_3 + x_3 ≤ 4, 2x_1^2 + x_2 + x_1 ≤ 5, x_2^2 + x_1^2 + x_3 ≥ 3, x_1^2 + x_2^2 + x_3^2 ≥ 6, x_1, x_2, x_3 ≥ 0 \right\}$

Solution with the help of parametric method:

Individual initial optimal solutions of the functions $f_1(x)$ and $f_2(x)$ are obtained with the help of software lingo 15 and they come out to be

$X_1 = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) = (0.1281, 1.31)$

$X_2 = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}) = (1.13, 0.619, 1.05)$

Table II: Objective functions at initial solution

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$f_1(X_1)$</th>
<th>$f_2(X_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0.2823</td>
<td>1.571</td>
</tr>
<tr>
<td>$X_2$</td>
<td>1.065</td>
<td>0.66</td>
</tr>
</tbody>
</table>

From the Table II, we can see that

$0.2823 ≤ f_1(x) ≤ 1.065$

and $0.66 ≤ f_2(x) ≤ 1.571$

Let equal weights be assigned to each solution

i.e. $w_1 = w_2 = 0.5$

Initial optimal solution is given by :

$X^{(0)} = w_1X_1 + w_2X_2$

$X^{(0)} = 0.5(0.1281, 1.31) + 0.5(1.13, 0.619, 1.05)$

$X^{(0)} = (0.565, 0.95, 1.18)$

So, initial value of the vector of parameters is $α^{(1)} = (a_1^{(1)}, a_2^{(1)})$

$α^{(1)} = (f_1(X^{(0)}), f_2(X^{(0)}))$

$α^{(1)} = (0.4661, 1.125)$

For converting fractional objectives into non-fractional parametric functions, suppose

$P_i(α^{(2)}) = (2x_1^2 + x_3) - a_1^{(1)}(x_1^2 + 3)$

Where ‘$t’$ represents iteration number

$P_2(α^{(2)}) = (2x_3^2 + x_2) - a_2^{(1)}(x_1^2 + 3)$

Thus, $P_1(α^{(2)}) = (2x_1^2 + x_3) - 0.466(x_1^2 + 3)$

$P_2(α^{(2)}) = (2x_3^2 + x_2) - 1.125(x_1^2 + 3)$

Thus, non-fractional parametric programming problem is given by :

$$\text{Min } f(x) = \text{Min } \{ P_1(α^{(1)}), P_2(α^{(1)}) \},$$

subject to $x ∈ S$

Defining termination constants as

$T_1 = 0.02$ and $T_2 = 0.03$

Initial solutions of $P_1(α^{(1)})$ and $P_2(α^{(1)})$ comes out to be

$X_1 = (0.1281, 1.31)$ and $X_2 = (1.16, 0.48, 1.09)$ respectively

Therefore, by using $ε$-constraint method, non-fractional parametric problem is considered as following:

$$\text{Min } P_1(α^{(1)}) = 2x_1^2 - 0.466x_1^2 + x_3 - 1.398$$

subject to

$$2x_3^2 - 1.125x_3^2 + x_3 - 3.375 ≤ ε_2$$

and $x ∈ S$

where $ε_2 ∈ [ε_2^L, ε_2^U]$ and $ε_2^L = \text{Min } \{ P_2(X_1), i = 1, 2 \}$

i.e. $ε_2^L = \text{Min } \{ P_2(X_1), P_2(X_2) \} = -2.033$

and $ε_2^U = \text{Max } \{ P_2(X_1), P_2(X_2) \} = 1.337$

Thus, $[-T_2, T_2] ⊆ [-ε_2^L, ε_2^U]$. So, we choose $ε_2 ∈ [-ε_2^L, ε_2^U]$ i.e. $ε_2 ∈ [-0.03, 0.03]$.

So, by substituting different values of $ε_2$, we get a set of pareto optimal solutions which are shown in Table III.

Table III: Values of Efficient solutions

| $ε_2$ | $x_1$ | $x_2$ | $x_3$ | $P_1(α^{(1)})$ | $|P_1(α^{(1)}) - T_1|$ |
|-------|-------|-------|-------|----------------|------------------------|
| 0.03  | 0.6762| 1.226 | 1.147 | -0.056858      | 0.056858               |
| 0.024 | 0.6749| 1.226 | 1.148 | -0.056018      | 0.056018               |
| 0.018 | 0.6734| 1.226 | 1.149 | -0.04235       | 0.04235                |
| 0.012 | 0.67199|1.226 | 1.15  | -0.04509       | 0.04509                |
| 0.006 | 0.67059|1.226 | 1.15  | -0.04785       | 0.04785                |
| 0.006 | 0.6678 | 1.225 | 1.15  | -0.05335       | 0.05335                |
| 0.012 | 0.6663 | 1.225 | 1.15  | -0.05611       | 0.05611                |
| 0.018 | 0.6649 | 1.225 | 1.15  | -0.05887       | 0.05887                |
| 0.024 | 0.6635 | 1.225 | 1.16  | -0.0616       | 0.0616                 |
| 0.03  | 0.6621 | 1.225 | 1.16  | -0.06439       | 0.06439                |

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So, the above set of efficient solution is obtained using Lingo 15 software. Thus, at each efficient solution obtained above, \(|P_2(a^{(1)})| \leq T_2\). But we can see that \(|P_1(a^{(1)})| > T_1\). So, termination condition is not satisfied by \(P_1(a^{(1)})\). Now,

\[
\min \sum_i \left| P_i(a^{(1)}) - T_i \right| = \text{Min} \left( P_i(a^{(1)}) - T_i \right) = 0.016858 \text{(from Table II)}
\]

It occurs at \(X^1 = (0.6762, 1.226, 1.147)\). Thus, \(X^1\) is the compromised solution.

So, the next iterated vector of parameters is given by

\[
\begin{align*}
\alpha^{(2)} &= (\alpha^{(1)}, \alpha^{(2)}) \\
\alpha^{(2)} &= (f_1(X^1), f_2(X^1)) \\
\alpha^{(2)} &= (0.4578, 1.116) \\
\end{align*}
\]

Thus, New iterated parametric programming Model is:

\[
P_1(a^{(2)}) = (2x_1^2 - 0.4578x_2^2 + 3) \\
P_2(a^{(2)}) = (2x_1^2 + x_3 - 1.116(x_1^2 + 3)) \\
\]

Model: \(\min f(x) = \text{Min} \left( P_1(a^{(2)}), P_2(a^{(2)}) \right) \) subject to \(x \in S\)

With the help of software Lingo 15, initial individual optimal solutions of \(P_1(a^{(2)})\) and \(P_2(a^{(2)})\) comes to be

\[
X_1 = (0.1281, 1.311), X_2 = (1.159, 0.4772, 1.086)
\]

Since \(T_1 < T_2\).

So, above Model is transformed to following Model:

\[
\min P_1(a^{(2)}) = 2x_1^2 - 0.4578x_2^2 + 3 = 1.3734 \\
\min P_2(a^{(2)}) = 2x_1^2 + x_3 - 1.116(x_1^2 + 3) = 2x_3 + 1.116x_1^2 + x_2 - 3.348
\]

So, \(X_1 = (0.6722, 1.226, 1.151)\) and \(X_2 = (0.6708, 1.226, 1.151)\).

Comparative study of parametric method and fuzzy programming method:

Solution with the help of fuzzy goal programming is given by:

\[
(\begin{array}{c}
0.6778 \\
0.6750 \\
0.6736 \\
0.6722 \\
0.6708 \\
0.6694
\end{array}) = (\begin{array}{c}
1.226 \\
1.226 \\
1.226 \\
1.226 \\
1.226 \\
1.226
\end{array}) \begin{array}{c}
1.146 \\
1.148 \\
1.149 \\
1.15 \\
1.151 \\
1.151
\end{array}
\]

Thus, we can see that the values of \(f_1(x), f_2(x)\) calculated with the help of the proposed method and the fuzzy approach are comparable to each other and this validates the proposed method of parametric functions.

XI. CONCLUSION

This paper proposed an approach of solving MOQFPFM by converting it into single objective non-fractional parametric programming Model with the help of constraint method and this method can be extended for solving bi-level and multilevel fractional programming Models. The parametric approach used in the study makes it very easy to transform fractional objectives into non-fractional functions for which efficient solutions can be obtained easily. In the numerical example illustrated in the paper, the set of solutions obtained with the proposed method are comparable to those obtained with fuzzy approach, which validates the feasibility of our approach.

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Multiobjective Quadratic Fractional Programming using Iterative Parametric Function


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