Results on Matrices

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Abstract: In the present research paper some techniques are obtained to form nth order square matrices from a given matrix of order m x n. The obtained results can be generalized for any order of matrix. The examples are given to support the results.

I. INTRODUCTION:
In a Matrix it is difficult to find how many square matrices can be obtain of order n directly. To find this some calculations are required. The present paper contains four theorems and four corollaries with proofs to explain desired results. Some examples are given to make easy to understand all the results. This paper will help its readers to solve many tedious problems mainly asked in all engineering entrance examination and in various exams. Let the set $D_{(m,n,Y)}$ denotes the set of all fixed m by n matrices (or matrices of fixed order $m \times n$ with known information of number of rows m and column n respectively such that $m,n \in \mathbb{N}$ and the elements of each matrix $A \in D_{(m,n,Y)}$ belong to the selection set $Y_r$ of finite order $r$, $r \in \mathbb{N}$.

1. Main Results

**Theorem 3.1.1.** The total numbers of matrices in the set $D_{(m,n,Y)}$ are $r^{mn}$.

**Theorem 3.1.2.** The numbers of matrices in the set $D_{(m,n,Y)}$ such that each matrix $A \in D_{(m,n,Y)}$ contains distinct element are $P \cdot r \cdot m \cdot n \cdot \square$, where the notation $P \cdot r \cdot m \cdot n \cdot \square$ stands for permutation defined as

$$P(m,n) = \begin{cases} 0, & \text{if } m < n \\ m!, & \text{if } m = n \\ \frac{m!}{(m-n)!}, & \text{if } m > n \\ \end{cases}$$

**Theorem 3.1.3.** The numbers of matrices in the set $D_{(m,n,Y)}$ such that each matrix $A \in D_{(m,n,Y)}$ contains all the elements of the selection set $Y_r$ are $(r!) \times S (mn, r)$, where the notation $S (mn, r)$ stands for Stirling number of second kind defined as

$$S(m,n) = \begin{cases} 0, & \text{if } m < n \\ 1, & \text{if } m = n \\ \sum_{k=1}^{n} (-1)^{n-k} \frac{k^{m-1}}{(k-1)! (n-k)!}, & \text{if } m > n \\ \end{cases}$$

**Theorem 3.1.4.** The numbers of matrices in the set $D_{(m,n,Y)}$ such that each matrix $A \in D_{(m,n,Y)}$ contains all the elements of the selection set $Y_r$ and also distinct, are

$$n \cdot (r! \cdot S(n,m))$$

Proof: Suppose that the matrix $A \in D_{(m,n,Y)}$ is given as $A = [a_{ij}]_{m \times n}$, where $1 \leq i \leq m$, $1 \leq j \leq n$, $m,n \in \mathbb{N}$ and order of the matrix $m \times n$ is fixed with known number of rows m and column n respectively. Suppose the set $X_A$ denotes the set of all the elements of the matrix $A$ i.e., $X_A = \{a_{ij} : 1 \leq i \leq m, 1 \leq j \leq n, \text{ and } m,n \in \mathbb{N}\}$ such that all the elements of the matrix $A$ belongs to the selection set $Y_r$ of finite order $r$. Then, $r \in \mathbb{N}$ containing real or complex scalars, i.e., $Y_r = \{a_i \in C | 1 \leq i \leq r, r \in \mathbb{N}\}$ where $a_i$ are scalars.

Clearly, the numbers of elements in the set $X_A$ are $mn$ and the numbers of elements in the set $Y_r$ are $r$.

3.1.1: Choose mn elements of the set $X_A$ from the selection set $Y_r$ having r elements satisfying the required condition. It is equivalent to map mn elements of the set $X_A$ to r elements of the selection set $Y_r$ such that the elements of the set $X_A$ can have any unique image in the set $Y_r$. It can be described as:

From the mapping diagram, we get the element a11 has r choices to map the element a12 has r choices to map...
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the element aij has r choices to map

the element amn has r choices to map.

Thus, total number of possible mappings from the set XA to the selection set

\[ Y_r = \prod_{i=1}^{r} r \times r \times \ldots = r^{mn} \times \frac{r^{mn}}{mn \text{times}} \]

Hence, the total numbers of matrices in \( A \in D_{(m,n,Y_r)} \) are rmn. (See Figure 3.1.1)

Example 3.1.1 (i)

Let \( Y_r = \{0, 1\} \), here \( r = 2 \)

Number of matrices in the set \( D_{(m,n,Y_r)} \) of order 3 x 2 is = 26 = 64

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{bmatrix}_{3x2}
\]

Example 3.1.1 (ii)

Let \( Y_r = \{-1, 0, 1\} \).

Number of matrices in the set \( D_{(m,n,Y_r)} \) of order 2 x 2 is = 34 = 81

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}_{2x2}
\]

Proof 3.1.2.

Choose mn elements of the set XA from the selection set Yr having r elements such that all elements of the set XA takes distinct values of the selection set Yr. It is equivalent to map mn elements of the set XA to r elements of the selection set Yr such that no any two elements of the set XA have same image in the selection set Yr. Let us consider some possible cases listed below:

Case-1.

If \( mn > r \), there is no any possible mapping that can fulfill the required criterion.

Thus, number of possible mappings = 0.

Case-2.

If \( mn > r \), the possible mappings can be describes as:

the element a11 has r choices to map

the element a12 has (r - 1) choices to map

the element a13 has (r - 2) choices to map

\[ \ldots \]

the element aij has \( \frac{r!}{(r-ij)!} \) choices to map

the element amn has \( \frac{r!}{(r-mn)!} \) choices to map

Thus, total number of possible mappings

\[ r \times (r - 1) \times (r - 2) \times \ldots \times (r - ij + 1) \times \ldots \times (r - mn + 1) = \frac{r!}{(r - mn)!} \]

Summing up all three cases discussed above, the numbers of matrices in the set \( D_{(m,n,Y_r)} \) such that each matrix \( A \in D_{(m,n,Y_r)} \) contains distinct elements

\[ = \begin{cases} 
0, & \text{if } mn > r \\
r!, & \text{if } mn = r \\
\frac{r!}{(mn - r)!}, & \text{if } mn < r 
\end{cases} = P(r,mn) \]

Where, the notation \( P \) stands for permutation.

Example 3.1.2. (i)

Let \( Y_r = \{1, 2, 3, 4\} \), then no of matrices in the set \( D_{(m,n,Y_r)} \)

(i) of order 2 x 2 having distinct entries = 4p4 = 4! = 24

(ii) of order 3 x 2 having distinct entries = 4p6 = 0

(iii) of order 2 x 3 having distinct entries = 4p6 = 0

Example 3.1.2. (ii)

Let \( Y_r = \{-2, -1, 0, 1, 2\} \), the no of matrices in the set \( D_{(m,n,Y_r)} \)

(i) of order 2 x 2 having distinct entries = \( \frac{5!}{4!} = 5 \times 4 \times 3 \times 2 \times 1 = 120 \)

(ii) of order 5 x 1 having distinct entries = \( \frac{5!}{5!} = 1 \)

(iii) of order 2 x 3 having distinct entries = \( \frac{6!}{2!} = 360 \)

Proof 3.1.3.

Choose mn elements of the set XA from the selection set Yr having r elements such that the set XA contains all the elements of the selection set Yr.

It is equivalent to map mn elements of the set XA to r elements of the selection set Yr such that each element of the selection set Yr is image of some element of the set XA or each element of the selection set Yr has some pre-image in the set XA. Let us consider some possible cases listed below:

Case-1.

If \( mn \geq r \), there is no any possible mapping that can fulfill the required criterion.

Thus, number of possible mappings = 0.

Case-2.

If \( mn < r \), the possible mappings can be describes as:

the element a11 has r choices to map

the element a12 has (r - 1) choices to map

the element a13 has (r - 2) choices to map

\[ \ldots \]

the element aij has \( \frac{r!}{(r-ij)!} \) choices to map

the element amn has \( \frac{r!}{(r-mn)!} \) choices to map

Thus, total number of possible mappings
\[ r \times (r-1) \times (r-2) \times \ldots \times (r-ij+1) \times \ldots \times 2 \times 1 = r! \]

**Case-3.**

If \( mn \in Y_r \), it is possible to map \( mn \) elements the set \( X_A \) to \( r \) elements of the selection set \( Y_r \) such that each element of the selection set \( Y_r \) has some pre-image in the set \( X_A \) but to do it we should create partition of the set \( X_A \) into \( r \) blocks and consider each block as a single element in the set \( X_A \). After such partition and consideration, the problem would be simple to map \( r \) blocks of the set \( X_A \) to \( r \) elements of the selection set \( Y_r \) such that all elements in a single block will be assigned the same element from the selection set \( Y_r \).

With the help of Case-2, we get

Total number of possible mappings

\[
= (r!) \times \left( \text{total number of ways of partitioning} \right)
\]

of \( X_A \) into \( r \) non-empty blocks

By using Stirling number of second kind, the total number of ways of partitioning of the set \( X_A \) with \( mn \) elements into \( r \) non-empty blocks are given by \( S(mn, r) \), where \( S(mn, r) \) is the Stirling number of second kind such that

\[
S(mn, r) = \sum_{k=1}^{r} (-1)^{r-k} \frac{k^{mn-1}}{(k-1)! (r-k)!}
\]

Thus, total number of possible mappings

\[
= (r!) \times \sum_{k=1}^{r} \frac{(-1)^{r-k} k^{mn-1}}{(k-1)! (r-k)!}
\]

Summing up all three cases discussed above, The numbers of matrices in the set \( D_{(mn, Y_r)} \) such that each matrix \( A \in D_{(mn, Y_r)} \) contains all the elements of the selection set \( Y_r \)

\[
= \begin{cases} 
0, & \text{if } mn < r \\
1, & \text{if } mn = r \\
(r!) \sum_{k=1}^{r} \frac{(-1)^{r-k} k^{mn-1}}{(k-1)! (r-k)!}, & \text{if } mn > r
\end{cases}
\]

\( = r! \sum_{k=1}^{r} \frac{(-1)^{r-k} k^{mn-1}}{(k-1)! (r-k)!}, \) where \( S \) stands for Stirling number of second kind.

Example 3.1.3. (i)

Let \( Y_r = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \), then no of matrices in the set \( D_{(mn, Y_r)} \) which contains all the elements of selection set \( Y_r \),

(i) of order \( 3 \times 3 \) is equal to \( 11! \times S(9, 11) = 0 \) (\( \because 9 < 11 \))

(ii) of order \( 3 \times 4 \) is equal to \( 11! \times S(12, 11) \)

\[
= 11! \times \sum_{k=1}^{11} (-1)^{11-k} \frac{k^{11}}{(k-1)! (11-k)!} \quad (\because 12 < 11)
\]

(iii) of order \( 5 \times 2 \) is equal to \( 11! \times S(9, 10) = 0 \)

Proof 3.1.4.

Choose \( mn \) elements of the set \( X_A \) from the selection set \( Y_r \) having \( r \) elements such that the set \( X_A \) contains all the elements of the selection set \( Y_r \) and are distinct. It is equivalent to map \( mn \) elements of the set \( X_A \) to \( r \) elements of the selection set \( Y_r \) such that no two elements of the set \( X_A \) have same image in the selection set \( Y_r \) and each element of the selection set \( Y_r \) has some pre-image in the set \( X_A \).

Let us consider some possible cases listed below:

Case-1.

If \( mn \in Y_r \), there is no any possible mapping to map \( mn \) elements of the set \( X_A \) to \( r \) elements of the selection set \( Y_r \) such that no two elements of the set \( X_A \) have same image in the selection set \( Y_r \). Thus, number of possible mappings = 0.

Case-2.

If \( mn \notin Y_r \), the possible mappings can be described as:

- The element \( a11 \) has \( r \) choices to map
- The element \( a12 \) has \( r \times 1 \) choices to map
- The element \( a13 \) has \( r \times r \times 2 \) choices to map

the element \( aij \) has \( r \times r \times i \) choices to map

\[
\sum_{r=1}^{r} \sum_{i=1}^{r} \sum_{j=1}^{r} \text{choices to map}
\]

the element \( a11 \) has \( r \times r \times i \) choices to map

\[
\sum_{r=1}^{r} \sum_{i=1}^{r} \sum_{j=1}^{r} \text{choices to map}
\]

the element \( aij \) has \( r \times r \times i \) choices to map

\[
\sum_{r=1}^{r} \sum_{i=1}^{r} \sum_{j=1}^{r} \text{choices to map}
\]

Thus, number of possible mappings = 0.

Summing up all three cases discussed above, The numbers of matrices in the set \( D_{(mn, Y_r)} \) such that each matrix \( A \in D_{(mn, Y_r)} \) contains all the elements of the selection set \( Y_r \) and are also distinct, are

\[
\begin{cases} 
0, & \text{if } mn < r \\
1, & \text{if } mn = r \\
(r!) \sum_{k=1}^{r} \frac{(-1)^{r-k} k^{mn-1}}{(k-1)! (r-k)!}, & \text{if } mn > r
\end{cases}
\]

This completes the proof.

Example 3.1.4. (i)

Let \( Y_r = \{1, 2, 3, 4, 5, 6, 7, 8\} \), the number of matrices in the set \( D_{(mn, Y_r)} \) of order \( 3 \times 3 \) having distinct entries and having all elements of selection set = 9!

Example 3.1.4. (ii)

Let \( Y_r = \{1, 2, 3\} \), the number of matrices in the set \( D_{(mn, Y_r)} \) of order \( 2 \times 2 \) having all entries of \( Y_r \) but all entries distinct is same as the no of injection \( Y_r \to Y_r \) and is equal to \( 4! = 24 \)

Corollary 3.1.

The set \( D_{(mn, Y_r)} \) is a non-empty set having at least one matrix.

Proof: Since the selection set \( Y_r \) denotes a well-defined set of finite order \( r, r \in \mathbb{N} \) such that its elements are scalars may be real or complex, thus least number of elements in \( Y_r \) are one i.e., \( r \in \mathbb{N} \) and since \( m, n \in \mathbb{N} \), thus any matrix

\[ \text{Example 3.1.5.} \]
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A ∈ D_{(m,n,Y)} is not an empty matrix.

Therefore, least number of elements in the set $D_{(m,n,Y)}$ is 1mn = 1 for r ▷ 1. Hence, the set $D_{(m,n,Y)}$ has at least one element (matrix). This completes the proof.

Corollary 3.2.

The numbers of matrices in the set $D_{(m,n,Y)}$ such that each matrix $A \in D_{(m,n,Y)}$ does not contain distinct elements are $r^{mn} \cdot P\{r, mn \}$, where, the notation P stands for permutation.

Proof : The number of matrices in the set $D_{(m,n,Y)}$ such that each matrix $A \in D_{(m,n,Y)}$ does not contain distinct elements = Total number of matrices in the set $D_{(m,n,Y)}$ – Total number of matrices containing distinct elements in the set $D_{(m,n,Y)}$ for $mn \neq P\{r, mn \}$. Where, the notation P stands for permutation. This completes the proof.

Corollary 3.3. : The numbers of matrices in the set $D_{(m,n,Y)}$ such that each matrix $A \in D_{(m,n,Y)}$ does not contain all element of the selection set $Y_r$ are $r^{mn} \cdot S\{mn, r\}$, where, the notation S stands for Stirling number of second kind.

Proof : The numbers of matrices in the set $D_{(m,n,Y)}$ such that each matrix $A \in D_{(m,n,Y)}$ does not contain all element of the selection set $Y_r$ = Total number of matrices in $D_{(m,n,Y)}$ – Total number of matrices containing all the elements of the selection set $Y_r$ for $mn \neq S\{mn, r\}$. Where, the notation S stands for Stirling number of second kind. This completes the proof.

Corollary 3.4. : The numbers of matrices in the set $D_{(m,n,Y)}$ such that each matrix $A \in D_{(m,n,Y)}$ neither contains all element of the selection set $Y_r$ nor all the elements are distinct are

$$r^{mn} - \begin{cases} r^{|r|}, & \text{if } mn \neq r \\ r^{|r|} - |r|, & \text{if } mn = r \end{cases}$$

Proof : The numbers of matrices in $D_{(m,n,Y)}$ such that $A \in D_{(m,n,Y)}$ neither contains all element of the selection set $Y_r$ nor all the elements are distinct = (The total number of matrices in $D_{(m,n,Y)}$) – (The number of matrices in $D_{(m,n,Y)}$ such that $A \in D_{(m,n,Y)}$ contains all the elements of the selection set $r Y$ and is also distinct) =

$$r^{mn} - \begin{cases} 0, & \text{if } mn \neq r \\ |r|, & \text{if } mn = r \end{cases}$$

$$r^{mn} - |r|, \quad \text{if } mn = r$$

This completes the proof.

Conclusion: At the basis of given examples and proof it is concluded that number of square matrices can be obtained directly from different type matrix.

REFERENCES:

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