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Results on Matrices

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***Abstract:** In the present research paper some techniques are obtained to form n^{th} order square matrices from a given matrix of order m x n. The obtained results can be generalized for any order of matrix. The examples are given to support the results.*

I. INTRODUCTION:

In a Matrix it is difficult to find how many square matrices can be obtain of order n directly. To find this some calculations are required. The present paper contains four theorems and four corollaries with proofs to explain desired results. Some examples are given to make easy to understand all the results. This paper will help its readers to solve many tedious problems mainly asked in all engineering entrance examination and in various exams. Let the set $D_{(m\times n, Y_r)}$ denotes the set of all fixed m by n matrices (or matrices of fixed order $m\times n$ with known information of number of rows m and column n respectively such that $m,n\in N$ and the elements of each matrix $A\in D_{(m\times n, Y_r)}$ belong to the selection set Y_r of finite order $r, r\in N$.

1. Main Results

Theorem 3.1.1. : The total numbers of matrices in the set $D_{(m\times n, Y_r)}$ are r^{mn}.

Theorem 3.1.2.: The numbers of matrices in the set $D_{(m\times n, Y_r)}$ such that each matrix $A\in D_{(m\times n, Y_r)}$ contains distinct element are P□r,mn□, where the notation P□m,n□□stands for permutation defined as

$$P(m,n)=\begin{cases}0, & \text{if } m<n\\ m!, & \text{if } m=n\\ \dfrac{m!}{(m-n)!}, & \text{if } m>n\end{cases}$$

Theorem 3.1.3.: The numbers of matrices in the set $D_{(m\times n, Y_r)}$ such that each matrix□ $A\in D_{(m\times n, Y_r)}$ contains all the elements of the selection set Y_r are (r!) x S (mn, r), where the notation S□m,n□□stands for Stirling number of second kind defined as

$$S(m,n)=\begin{cases}0, & \text{if } m<n\\ 1, & \text{if } m=n\\ \displaystyle\sum_{k=1}^{n}(-1)^{n-k}\frac{k^{m-1}}{(k-1)!(n-k)!}, & \text{if } m>n\end{cases}$$

Theorem 3.1.4. : The numbers of matrices in the set $D_{(m\times n, Y_r)}$ such that each matrix $A\in D_{(m\times n, Y_r)}$ contains all the elements of the selection set Y_r and also distinct, are

$$\begin{cases}0, & \text{if } mn\neq r\\ r!, & \text{if } mn=r\end{cases}$$

Proof:

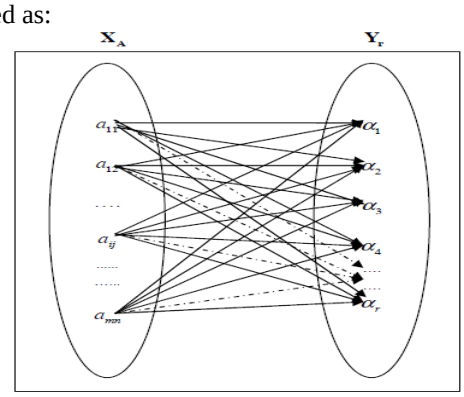
Suppose that the matrix $A\in D_{(m\times n, Y_r)}$ is given as $A=[a_{ij}]_{m\times n}$, where $1\le i\le m,\ 1\le j\le n,\ m,n\in N$ and order of the matrix $m\times n$ is fixed with known number of rows m and column n respectively. Suppose the set X_A denotes the set of all the elements of the matrix A i.e., $X_A=\{a_{ij}: 1\le i\le m, 1\le j\le n, \text{ and } m,n\in N\}$

such that all the elements of the matrix A belongs to the selection set Y_r of finite order $r, r\in N$ containing real or complex scalars, i.e.,

$$Y_r=\{\alpha_i\in C \mid 1\le i\le r, r\in N \text{ where } \alpha_i \text{ are scalars}\}.$$

Clearly, the numbers of elements in the set X_A are mn and the numbers of elements in the set Y_r are r.

3.1.1: : Choose mn elements of the set X_A from the selection set Y_r having r elements satisfying the required condition. It is equivalent to map mn elements of the set X_A to r elements of the selection set Y_r such that the elements of the set X_A can have any unique image in the set Y_r. It can be described as:



From the mapping diagram, we get the element a_{11} has r choices to map the element
a12 has r choices to map

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……
the element aij has r choices to map
……
the element amn has r choices to map.
Thus, total number of possible mappings from the set XA to the selection set

$$Y_r = \underbrace{r \times r \times r \times \ldots\ldots \times r}_{\text{mn times}} = r^{mn}$$

Hence, the total numbers of matrices in $A \in D_{(m\times n, Y_r)}$ are rmn. (See Figure 3.1.1)
Example 3.1.1 (i)

Let Yr = {0, 1}, here r = 2

Number of matrices in the set $D_{(m\times n, Y_r)}$ of order 3 x 2 is = 26 = 64

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3x2}$$

Example 3.1.1 (ii)

Let Yr = {-1, 0, 1}, $D_{(m\times n, Y_r)}$ of order 2 x 2 is = 34 = 81

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2x2}$$

Proof 3.1.2.
Choose mn elements of the set XA from the selection set Yr having r elements such that all elements of the set XA takes distinct values of the selection set Yr. It is equivalent to map mn elements of the set XA to r elements of the selection set Yr such that no any two elements of the set XA have same image in the selection set Yr. Let us consider some possible cases listed below:
Case-1.
If mn > r, there is no any possible mapping that can fulfil the required criterion.
Thus, number of possible mappings = 0.
Case-2.
If mn > r, the possible mappings can be describes as:
the element a11 has r choices to map
the element a12 has (r - 1) choices to map
the element a13 has (r - 2)□choices to map
……
the element aij has □r □□ij □1□□choices to map
……
the element am(n-1) □has 2 choices to map
the element amn has only 1choice to map
Thus, total number of possible mappings

$$r \times (r-1) \times (r-2) \times \ldots \times (r - ij + 1) \times \ldots \times 2 \times 1 = r!$$

Case-3.
If mn □□r, the possible mappings can be describes as:
the element a11 has r choices to map
the element a12 has □r □1□□choices to map
the element a13 has □r □□2□□choices to map
……
the element aij has □r □□ij □1□□choices to map
……
the element amn has□r□mn□1□□choices to map

Thus, total number of possible mappings

$$r \times (r-1) \times (r-2) \times \ldots \times (r - ij + 1) \times \ldots \times (r - mn + 1)$$

$$= \frac{r!}{(r-mn)!}$$

Summing up all three cases discussed above, the numbers of matrices in the set□ $D_{(m\times n, Y_r)}$ □such that each matrix□ $A \in D_{(m\times n, Y_r)}$ contains distinct element

$$= \begin{cases} 0, & \text{if } mn > r \\ r!, & \text{if } mn = r \\ \dfrac{r!}{(mn-r)!}, & \text{if } mn < r \end{cases} = P(r, mn)$$

Where, the notation P stands for permutation.
Example 3.1.2. (i)

Let Yr = {1, 2, 3, 4}, then no of matrices in the set $D_{(m\times n, Y_r)}$
(i) of order 2 x 2 having distinct entries = 4p4 = 4! = 24
(ii) of order 3 x 2 having distinct entries = 4p6 = 0
(iii) of order 2 x 3 having distinct entries = 4p6 = 0
Example 3.1.2. (ii)
Let Y_r = {-2, -1, 0, 1, 2}, the no of matrices in the set $D_{(m\times n, Y_r)}$

(i) of order 2 x 2 having distinct entries = $^5p_4$ = 120
(ii) of order 5 x 1 having distinct entries = $^5p_5$ = 120
(iii) of order 2 x 3 having distinct entries = $^5p_6$ = 0
Proof 3.1.3.
Choose mn elements of the set XA from the selection set Y_r having r elements such that the set XA contains all the elements of the selection set Yr .
It is equivalent to map mn elements of the set XA to r elements of the selection set Yr such that each element of the selection set Y r is image of some element of the set XA or each element of the selection set Yr has some pre-image in the set XA. Let us consider some possible cases listed **below**:
Case-1.
If mn □□r, there is no any possible mapping that can fulfil the **required** criterion.
Thus, **number** of possible mappings = 0.
Case-2.
If mn □□r, the **possible** mappings can be describes as:
the element a11 has r choices to map
the element a12 has □r □1□□choices to map
the element a13 has □r □□2□□choices to map
……
the element aij has □r □□ij □1□choices to map
……
the element am(n-1) □has 2 choices to map
the element amn has only 1 choice to map
Thus, total number of possible mappings

$$= r \times (r-1) \times (r-2) \times \ldots \times (r-ij+1) \times \ldots \times 2 \times 1 = r!$$

Case-3.

If mn □□r, it is possible to map mn elements the set XA to r elements of the selection set Yr such that each element of the **selection** set Yr has some pre-image in the set XA but to do it we should create partition of the set XA into r blocks and consider each block as a single element in the set XA. After such partition and consideration, the problem would be simple to map r blocks of the set XA to r elements of the selection set Yr such that all elements in a single block will be assigned the same element from the selection set Yr.

With the help of **Case-2,** we get

Total number of possible mappings

$$= (r!) \times \begin{pmatrix} \text{total number of ways of partitioning} \\ \text{of } X_A \text{ into r non-empty blocks} \end{pmatrix}$$

By using Stirling number of second kind, the total number of ways of partitioning of the set XA with mn elements into r non-empty blocks are given by S(mn,r), where S(mn,r) is the Stirling number of second kind such that

$$S(mn,r) = \sum_{k=1}^{r} (-1)^{r-k} \frac{k^{mn-1}}{(k-1)!(r-k)!}$$

Thus, Total number of possible mappings =

$$(r!) \times \sum_{k=1}^{r} (-1)^{r-k} \frac{k^{mn-1}}{(k-1)!(r-k)!}$$

Summing up all three cases discussed above, The numbers of matrices in the set $D_{(m\times n, Y_r)}$ □□such that each matrix $A \in D_{(m\times n, Y_r)}$ □contains all the elements of the selection set Yr

$$= \begin{cases} 0, & \text{if } mn < r \\ r!, & \text{if } mn = r \\ (r!) \times \sum_{k=1}^{r} (-1)^{r-k} \frac{k^{mn-1}}{(k-1)!(r-k)!}, & \text{if } mn > r \end{cases}$$

$$= r! = \begin{cases} 0, & \text{if } mn < r \\ 1, & \text{if } mn = r \\ \sum_{k=1}^{r} (-1)^{r-k} \frac{k^{mn-1}}{(k-1)!(r-k)!}, & \text{if } mn > r \end{cases}$$

=□r!□□□□S□mn, r□, where S stands for Stirling number of second kind.

Example 3.1.3. (i)

Let Yr = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, then no of matrices in the set $D_{(m\times n, Y_r)}$ which contains all the elements of selection set Yr,

(i) of order 3 x 3 is equal to = 11! x S (9, 11) = 0 ($\because 9 < 11$)

(ii) of order 3 x 4 is equal to = 11! x S (12, 11)

$$= 11! \times \sum_{k=1}^{11} (-1)^{11-k} \frac{k^{11}}{(k-1)!(11-k)!} \quad (\because 12 < 11)$$

(iii) of order 5 x 2 is equal to = 11! x S (9, 10) = 0

Proof 3.1.4.

Choose mn elements of the set XA from the selection set Yr having r elements such that the set XA contains all the elements of the selection set Yr and are distinct. It is equivalent to map mn elements of the set XA to r elements of the selection set Yr such that no two elements of the set XA have same image in the selection set Yr and each element of the selection set Yr has some pre-image in the set XA. Let us consider some possible cases listed below:

Case-1.

If mn □□r, there is no any possible mapping to map mn elements of the set XA to r elements of the set Yr such that no two elements of the set XA have same image in the selection set Yr. Thus, number of possible mappings = 0.

Case-2.

If mn □□r, the possible mappings can be describes as:

the element a11 has r choices to map

the element a12 has □r □1□□choices to map

the element a13 has □r □□2□□choices to map

……

the element aij has □r □□ij □1□choices to map

……

the element am(n- 1) has 2 choices to map

the element amn has only 1 choice to map

Thus, total number of possible mappings

$$r \times (r-1) \times (r-2) \times \ldots \times (r-ij+1) \times \ldots \times 2 \times 1 = r!$$

Case-3.

If mn □□r, there is no any possible mapping to map mn elements of the set XA to r elements of the selection set Yr such that each element of the selection set Yr has some pre-image in the set XA. Thus, number of possible mappings = 0.

Summing up all three cases discussed above, The numbers of matrices in the set $D_{(m\times n, Y_r)}$ such that each matrix $A \in D_{(m\times n, Y_r)}$ contains all the elements of the selection set Yr and are also distinct, are $= \begin{cases} 0, & \text{if } mn \neq r \\ r!, & \text{if } mn = r \end{cases}$.

This completes the proof.

Example 3.1.4. (i)

Let Yr = {1, 2, 3, 4, 5, 6, 7, 8}, the number of matrices in the set $D_{(m\times n, Y_r)}$ of order 3 x 3 having distinct entries and having all elements of selection set = 9!

Example 3.1.4. (ii)

Let Yr = {1, 2, 3}, the number of matrices in the set $D_{(m\times n, Y_r)}$ of order 2 x 2 having all entries of Yr but all entries distinct is same as the no of injection $Y_r \rightarrow Y_r$ and is equal to 4! = 24

Corollary 3.1.

The set $D_{(m\times n, Y_r)}$ □is a non-empty set having at least one matrix.

Proof : Since the selection set Yr denotes a well-defined set of finite order $r, r \in N$ such that its elements are scalars may be real or complex, thus least number of elements in Yr are

one i.e., r □1 and since $m,n \in N$, thus any matrix $A \in D_{(m\times n, Y_r)}$ is not an empty matrix.

Therefore, least number of elements in the set□ $D_{(m\times n, Y_r)}$ □is 1mn = 1 for r □1. Hence, the set $D_{(m\times n, Y_r)}$ has at least one element (matrix). This completes the proof.

Corollary 3.2.

The numbers of matrices in the set $D_{(m\times n, Y_r)}$ □such that each matrix $A \in D_{(m\times n, Y_r)}$ does not contain distinct elements are r^{mn} □□*P*□r, mn□, where, the notation P stands for permutation.

Proof : The number of matrices in the set $D_{(m\times n, Y_r)}$ □such that each matrix $A \in D_{(m\times n, Y_r)}$ does not contains distinct elements = Total number of matrices in the set $D_{(m\times n, Y_r)}$ – Total number of matrices containing distinct elements in the set $D_{(m\times n, Y_r)}$ □□rmn □□P□r, mn□□Where, the notation P stands for permutation.

This completes the proof.

Corollary 3.3. :The numbers of matrices in the set $D_{(m\times n, Y_r)}$ □such that each matrix $A \in D_{(m\times n, Y_r)}$ does not contain all element of the selection set Yr are rmn □□□r!□□□□S(mn, r), where, the notation S stands for Stirling number of second kind.

Proof :The numbers of matrices in the set $D_{(m\times n, Y_r)}$ □such that each matrix $A \in D_{(m\times n, Y_r)}$ does not contain all element of the selection set Yr = Total number of matrices in $D_{(m\times n, Y_r)}$ – Total number of matrices containing all the elements of the selection set Yr

□□rmn □□□r!□□□□S□mn, r□

Where, the notation S stands for Stirling number of second kind. This completes the proof.

Corollary 3.4. : The numbers of matrices in the set $D_{(m\times n, Y_r)}$ such that each matrix $A \in D_{(m\times n, Y_r)}$ neither contains all element of the selection set Yr nor all the elements are distinct are

$$= \begin{cases} r^{mn}, & \text{if } mn \neq r \\ r^{mn} - r!, & \text{if } mn = r \end{cases}$$

Proof : The numbers of matrices in $D_{(m\times n, Y_r)}$ such that $A \in D_{(m\times n, Y_r)}$ □neither contains all element of the selection set Yr nor all the elements are distinct = (The total number of matrices in $D_{(m\times n, Y_r)}$) – (The number of matrices in $D_{(m\times n, Y_r)}$ □such that $A \in D_{(m\times n, Y_r)}$ contains all the elements of the selection set *r Y* and is also distinct) =

$$r^{mn} - \begin{cases} 0, & \text{if } mn \neq r \\ r!, & \text{if } mn = r \end{cases}$$

$$= \begin{cases} r^{mn}, & \text{if } mn \neq r \\ r^{mn} - r!, & \text{if } mn = r \end{cases}$$

This completes the proof.

Conclusion: At the basis of given examples and proof it is concluded that number of square matrices can be obtained directly from different type matrix.

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