Closed Form Solution for Outage and Average Bit Error Rate of Asymmetric Relaying System

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Abstract: In this paper, closed form solution of outage and bit error rate (BER) is evaluated for the purpose of performance analysis of the amplify and forward relaying scheme under the asymmetric fading environment. In this dual hop system, we have used Rayleigh fading along source S to relay R and Mixture Gamma fading along relay R to destination D. First, we have derived closed form solution of outage and then we have used this derived expression for getting closed form solution of bit error rate (BER) for different kinds of modulation. Since Mixture Gamma fading channel represents many fading channels as its special case, so proposed closed form solution of outage and bit error rate may be used for analysis of outage and bit error rate under various fading scenarios. Specifically, in this paper, for analysis of outage and bit error rate (BER), we have taken Nakagami-m fading as a particular case of Mixture Gamma fading and analyzed the performance of proposed system after observing effect of fading severity factor and signal to noise ratio (SNR) on outage probability and bit error rate (BER).

Keywords : Asymmetric Fading, Bit Error Rate, Mixture Gamma fading, Outage Probability, Rayleigh Fading, Signal to Noise Ratio.

I. INTRODUCTION

Wireless communication has continued to make a profound impact on our daily lives. To increase effectiveness of wireless communication systems, use of relay in a network has been proposed. This transmission strategy of using relay as intermediate nodes between transmitter and receiver is known as a relay-assisted communication.

A huge number of literatures present the modeling and analysis of relay based wireless communication system. For enhancing the performance of wireless communication system, use of full duplex relay was suggested in [1]. Performance of partial relay selection scheme and optimal relay selection scheme was analyzed in [2]. Cooperative communication has been used in [3] for getting full cooperative diversity gains along with preserving spectral and energy efficiency. For energy harvesting and information decoding, optimization of the power splitting protocol under various relaying schemes has been done in [4] and efficient algorithms has been proposed for finding the optimal solutions. In [5], for Nakagami-m fading channel, a lower reduced limit of outage probability has been evaluated. In [6], performance has been investigated for N-hop relaying system having amplify and forward scheme and asymptotic expression of outage probability has also been derived. In [7], a generalized fading model based on Mixture of Gaussians (MoG) distribution was suggested. In [8], it was evaluated that during wireless transmission, communication system can have better energy efficiency when there is relay at some particular distance from source. In [9], a new channel model for relay named as Gaussian half duplex relay channel was proposed and it was found that system for this scheme is less complicated in comparison of other conventional relaying schemes.

For sharing of bandwidth using relay in cognitive radio networks, a new protocol named as two phase protocol was suggested in [10]. In this protocol, whenever transmitter and receiver of secondary user are not directly connected then in this case, relay supports both, primary user and secondary user. Using this protocol, outage probability improves in comparison of outage we get without using this protocol. Interference produced by secondary transmission is taken care by compensation obtained by use of Relay. For maximal ratio combining having dual branch, probability of link success was evaluated in [11]. It was recognized from this study that maximal ratio combining and minimum mean square error combining shows comparable results. To meet conditions of hardware requirement of practical relay-based system, this observation provide basic guidelines [12].

For Relay based system in which channel state information in known, expression of bit error rate with reduced limits was deduced in [13]. Closed form solution of CDF and PDF for signal to noise ratio, in case of cooperative dual-hop diversity schemes having non-regenerative relays, were derived in [14].

In our proposed system, we have taken Mixture Gamma fading channel along relay to destination. Here, It is worthy to mention that Mixture Gamma distribution represents almost all the fading channels till the date as its special case. So, the proposed analysis presents a generic closed form solution for outage and bit error rate (BER) of the asymmetric relaying system.

The rest of the paper is structured as given below. Section II presents the system model. For proposed system, method to derive closed form solution of outage and BER are presented in section III and section IV. Numerical analysis for the outage probability and bit error rate (BER) has been discussed in section V. Finally, in section VI, we have drawn our conclusion.

II. SYSTEM MODEL

Figure 1 shows relaying system which consist of relay along with source and destination. Here, first of all, message is transmitted by source. Now, relay receives message sent by
source and then retransmit it to destination. Such system is
known as a dual hop transmission system. Furthermore, we
have considered amplify and forward type of relaying system
which is placed in asymmetric fading channel.

In AF scheme, the relays act as analog repeaters. Relay R
receives the signal transmitted by source S and then amplify it
with some fixed gain and then retransmit it to the destination
(D). Furthermore, we have considered asymmetric relaying
system. In this system, fading channel along S-R and R-D are
Rayleigh fading channel and Mixture Gamma fading channel
respectively.

Fading channel having Rayleigh distribution is described by
probability density function (PDF)

\[ f(\gamma) = \frac{1}{\gamma} e^{-\frac{\gamma}{\gamma_1}} \]  

(1)

where \( \gamma_1 \) and \( \theta_1 \) are instantaneous and average SNR for
S-R link. Fading channel having Mixture Gamma distribution is
described by following PDF given in equation 1 of [15].

\[ f_i(x) = \sum_{i=1}^{N} w_i f_i(x) = \sum_{i=1}^{N} \alpha_i x^{\theta_i-1} e^{-\gamma x} \]  

(2)

Where \( f_i(x) = \frac{\Gamma(\beta_i)}{\Gamma(\bar{\beta})} x^{\theta_i-1} e^{-\gamma x} \) represents standard Gamma
distribution, \( \Gamma(\beta) \) is the Gamma function, \( w_i = \frac{\alpha_i \Gamma(\bar{\beta})}{\tau_i^{\beta_i}} \).

N represents number of terms within the summation and \( \alpha_i, \beta_i \) and \( \tau_i \) represent the parameters corresponding to \( i \)th
Gamma factor. Here, \( \sum w_i = 1 \).

Mixture Gamma represents many fading channels as its
particular case. When value of N is equal to 1, it gives
Rayleigh fading distribution and Nakagami-\( m \) fading distribution.

III. CLOSED FORM EXPRESSION OF OUTAGE
PROBABILITY

Probability corresponding to outage is described as the
probability for which instantaneous overall signal to noise
ratio (SNR) \( \gamma_i \) declines to a value lower than a particular
threshold value \( \gamma_{th} \). Mathematically, probability corresponding
to outage for any particular link is given by

\[ P_{outage} = P(\gamma < \gamma_{th}) = \int_{0}^{\gamma_{th}} f(\gamma) d\gamma \]  

(3)

For the proposed scenario, outage probability can be given as

\[ P_{outage} = P(\min(\gamma_1, \gamma_2 < \gamma_{th})) \]  

(4)

Here \( \gamma_1 \) and \( \gamma_2 \) are instantaneous SNR along source to relay
link and relay to destination link and \( \gamma_{th} \) is threshold value.
Since link of S-R and link of R-D are not dependent, hence
outage for the proposed system can be rewritten as

\[ P_{outage} = P(\gamma_1 < \gamma_{th})P(\gamma_2 < \gamma_{th}) \]  

(5)

Now, we will evaluate the outage probability along source S
to relay R link and relay R to destination D link separately.
Finally, we will substitute the individual outage of each link in
Equation (5).

Using equation (1), outage for S-R link can be given as

\[ P(\gamma_1 < \gamma_{th}) = \int_{0}^{\gamma_{th}} \frac{1}{\gamma_1} e^{-\frac{\gamma_1}{\gamma_1}} d\gamma_1 \]  

(6)

This gives outage between S-R link as

\[ P(\gamma_1 < \gamma_{th}) = 1 - e^{-\frac{\gamma_{th}}{\gamma_1}} \]  

(7)

Further, using Mixture Gamma fading distribution along relay
R to destination D, outage for relay to destination link may be
given as

\[ P(\gamma_2 < \gamma_{th}) = \int_{0}^{\gamma_{th}} \prod_{i=1}^{N} \alpha_i \left( \frac{\gamma_2}{\bar{\gamma}_2} \right)^{\theta_i-1} e^{-\tau_i \gamma_i} d\gamma_2 \]  

(8)

Here, \( \gamma_2 \) and \( \bar{\gamma}_2 \) are instantaneous and average SNR along
relay R to destination D link.

Let \( \gamma_2 = q \Rightarrow d\gamma_2 = \bar{\gamma}_2 dq \)

After using above substitution in Equation (8), we get

\[ P(\gamma_2 < \gamma_{th}) = \int_{0}^{\gamma_{th}} \prod_{i=1}^{N} \alpha_i \left( \frac{q}{\bar{\gamma}_2} \right)^{\theta_i-1} e^{-\tau_i (q)} dq \]  

(9)

Using [16] (Table of integrals, series, and products, 3.381,
equation 1), we get

\[ P(\gamma_2 < \gamma_{th}) = \prod_{i=1}^{N} \Delta_i \left( \frac{\gamma_{th}}{\bar{\gamma}_2} \right)^{\theta_i} \gamma_2 \]  

(10)

where \( \Delta_i \) is the incomplete gamma function.

Now, after using equation (7) and equation (10) into equation
(5), we get the outage probability between source to
destination for dual hop AF relaying system as given below.

\[ P_{outage} = \sum_{i=1}^{N} \left( 1 - e^{-\frac{\gamma_{th}}{\gamma_i}} \right) \alpha_i \frac{\gamma_{th}}{\bar{\gamma}_2} \gamma_2 \]  

(11)

Now, equation (11) can also be represented as
The expression of above outage probability between S-D for dual hop AF relaying system may represent the outage under different fading scenarios because Mixture Gamma fading taken between R-D represents many fading channels as its special cases. Specifically, for

\[ N = 1, \quad \alpha_i = \frac{m^m}{\Gamma(m)(x_2)^m}, \quad \beta_i = m \quad \text{and} \quad \tau_i = \frac{m}{x_2} \]

it gives Nakagami-\( m \) fading. Thus, in case of Nakagami-\( m \) fading between R-D, outage probability between S-D for dual hop AF relaying system is given as

\[
P_{\text{outage}}(x_1, x_2) = \sum_{i=1}^{N} \left( \frac{m^m}{\Gamma(m)(x_2)^m} \right) \frac{x^m}{\Gamma(m)(x_2)^m} \gamma^{m-1}(m, \frac{m\gamma_{th}}{x_2}) - \right)
\]

Further, we will consider two special cases for asymmetric relaying system as given below.

**Case 1:** if \( x_1 = x_2 = x \) then

\[
P_{\text{outage}}(x) = \sum_{i=1}^{N} \left( \frac{m^m}{\Gamma(m)(x)^m} \right) x^m \gamma^{m-1}(m, x) - \right)
\]

**Case 2:** if \( x_1 = 2x_2 \) and \( x_1 = x \) then \( x_2 = x/2 \) and

\[
P_{\text{outage}}(x) = \sum_{i=1}^{N} \left( \frac{m^m}{\Gamma(m)(x/2)^m} \right) x^{2m} \gamma^{m-1}(m, \frac{x}{2}) - \right)
\]

### IV. CLOSED FORM EXPRESSION FOR BIT ERROR RATE (BER)

For performance analysis of this asymmetric relaying system (Rayleigh fading channel for S-R link and Mixture Gamma fading channel R-D link), we have also evaluated closed form expression of average bit error rate (ABER). Using equation 20 of [17], for various types of modulation schemes, average bit error rate (ABER) is given as

\[
P_b = \frac{r_p}{2\Gamma(p)} \int_{0}^{\infty} e^{(-\gamma)} \gamma^{p-1} F_{\gamma}(\gamma) d\gamma
\]

Here, \( F_{\gamma}(\gamma) \) is cumulative distribution function.

In equation (15), values of parameters \( r \) and \( p \) for binary phase shift keying (BPSK) are 1 and 0.5 respectively. For \( r = 0.5 \) and \( p = 0.5 \), equation (15) represent expression of average bit error rate (ABER) of binary frequency shift keying (BFSK).

In equation (11), when we replace \( \gamma_{th} \) by \( \gamma \), we get value of cumulative distribution function \( F_{\gamma}(\gamma) \) as given below.

\[
F_{\gamma}(\gamma) = \sum_{i=1}^{N} \left( 1 - e^{-\frac{\gamma}{\gamma_i}} \right) \alpha_i \frac{\gamma}{\gamma_i} \gamma^{m-1}(m, \frac{m\gamma_{th}}{x_2})
\]

Here, \( \gamma(-, -) \) is the incomplete gamma function.

Using equation (15) and (16), we get

\[
P_b = \frac{r_p}{2\Gamma(p)} \int_{0}^{\infty} e^{(-\gamma)} \gamma^{p-1} \sum_{i=1}^{N} \left( 1 - e^{-\frac{\gamma}{\gamma_i}} \right) \alpha_i \frac{\gamma}{\gamma_i} \gamma^{m-1}(m, \frac{m\gamma_{th}}{x_2}) d\gamma
\]

Here, \( \gamma(-, -) \) is the incomplete gamma function.

Now, from equation (17), we get

\[
P_b = I_1 - I_2
\]

where value of \( I_1 \) and \( I_2 \) are expressed by equation (19) and (20).
\[ I_1 = \frac{r^p}{2\Gamma(p)} \sum_{i=1}^{N} \alpha_i \beta_2(\tau_i)^{\beta_1} \left( \int_0^\infty \gamma^{p-1} e^{(-\gamma)} \frac{d\gamma}{\beta_1, \frac{\tau_i \gamma}{\beta_2}} \right) \]

(19)

and

\[ I_2 = \frac{r^p}{2\Gamma(p)} \sum_{i=1}^{N} \alpha_i \beta_2(\tau_i)^{\beta_1} \left( \int_0^\infty \gamma^{p-1} e^{\left(-\frac{x}{\tau_i}\right)^p} \frac{d\gamma}{\beta_1, \frac{x}{\tau_i}} \right) \]

(20)

After putting \( \frac{\tau_i \gamma}{\beta_2} = x \) i.e. \( \gamma = \frac{\beta_2}{\tau_i} x \) and \( d\gamma = \frac{\beta_2}{\tau_i} dx \) in equation (19) and equation (20), we get following two expressions in the form of equation (21) and equation (22).

\[ I_1 = \frac{r^p}{2\Gamma(p)} \left( \sum_{i=1}^{N} \left( \frac{\beta_2}{\tau_i} \right)^p \alpha_i \beta_2(\tau_i)^{\beta_1} \left( \int_0^\infty x^{p-1} e^{-\frac{x}{\tau_i}} \frac{dx}{\beta_1, x} \right) \right) \]

(21)

and

\[ I_2 = \frac{r^p}{2\Gamma(p)} \left( \sum_{i=1}^{N} \left( \frac{\beta_2}{\tau_i} \right)^p \alpha_i \beta_2(\tau_i)^{\beta_1} \left( \int_0^\infty x^{p-1} e^{-\frac{x}{\tau_i}} \frac{dx}{\beta_1, x} \right) \right) \]

(22)

In equation (21) and equation (22), \( \gamma^{p-1} \) is the incomplete gamma function.

From [16] (Table of integrals, series, and products, 6.455, equation 2), we know that

\[ \int_0^\infty x^{p-1} \gamma(b, x) \frac{dx}{b(1+s)^{a+b}} = \left( \frac{\Gamma(a + b)}{(b(1+s)^{a+b})} \right) \]

\[ \left( \sum_{i=1}^{N} \left( \frac{\beta_2}{\tau_i} \right)^p \alpha_i \beta_2(\tau_i)^{\beta_1} \right) \left( \int_0^\infty x^{p-1} e^{-\frac{x}{\tau_i}} \frac{dx}{\beta_1, x} \right) \]

(23)

After using equation (23) into equation (21) and equation (22), we get following two expressions in the form of equation (24) and (25).

\[ I_1 = \frac{r^p}{2\Gamma(p)} \left( \sum_{i=1}^{N} \left( \frac{\beta_2}{\tau_i} \right)^p \alpha_i \beta_2(\tau_i)^{\beta_1} \left( \frac{\Gamma(p + \beta_i)}{\beta_i \left(1 + \frac{r \beta_2}{\tau_i} \right)^{p+\beta_i}} \right) \right) \]

\[ \left( \sum_{i=1}^{N} \left( \frac{\beta_2}{\tau_i} \right)^p \alpha_i \beta_2(\tau_i)^{\beta_1} \left( \frac{\Gamma(p + \beta_i)}{\beta_i \left(1 + \frac{r \beta_2}{\tau_i} \right)^{p+\beta_i}} \right) \right) \]

(24)

and

\[ I_2 = \frac{r^p}{2\Gamma(p)} \left( \sum_{i=1}^{N} \left( \frac{\beta_2}{\tau_i} \right)^p \alpha_i \beta_2(\tau_i)^{\beta_1} \left( \frac{\Gamma(p + \beta_i)}{\beta_i \left(1 + \frac{r \beta_2}{\tau_i} \right)^{p+\beta_i}} \right) \right) \]

(25)

Now, after putting values of \( I_1 \) and \( I_2 \) from equation (24) and equation (25) into equation (18), we get expression of bit error rate (BER) in the form of equation (26).
Here, figure 2 shows outage probability for the case 1 for which $\tilde{P}_1 = \tilde{P}_2$ has been taken i.e. signal to noise ratio along S-R link and R-D link are same. We have taken threshold value $\gamma_{th} = 10dB$. For Nakagami-m fading channel as the special case of Mixture Gamma fading channel, different values of severity factor m have been taken. The severity factor m varies in the range of $0.5 \leq m < \infty$ 0 and amount of fading is represented by1/m. Fig. 2 shows that as the values of severity factor m is increasing, the outage probability is decreasing. Fig. 2 also describes that as the value of average SNR increases, outage probability decreases. These two observations made from Fig. 2 show perfect agreement with theory.

So, for relay based wireless communication system having asymmetric Rayleigh and Mixture Gamma fading channel, equation (11) and equation (26) represents expression of outage probability and BER respectively.

V. NUMERICAL ANALYSIS

Numerical analysis for outage and BER is provided in this section. MAPLE has been used for plotting closed form solution of outage and BER evaluated in section III and section IV respectively. For performance analysis of this proposed system, Rayleigh distribution in S-R link and Mixture Gamma distribution in R-D link have been taken. Mixture Gamma fading shows utility of our model because it includes various fading channel as its special cases. We have taken Nakagami-m fading as the special case of Mixture Gamma fading and for this, we have put $N = 1$,

$$\alpha_i = \frac{m^\alpha}{\Gamma(m)\tilde{P}_2}, \quad \beta_i = m \quad \text{and} \quad \tau_i = \frac{m}{\tilde{P}_2}$$

For performance analysis of this asymmetric relaying system, bit error rate (BER) has also been plotted under the condition of $\tilde{P}_1 = \tilde{P}_2$ and $\tilde{P}_1 = 2\tilde{P}_2$ for BPSK and BFSK modulation schemes. In figure 4, bit error rate (BER) has been plotted with respect to signal to noise ratio (SNR) for BPSK modulation scheme after taking different values of parameter $\beta$ and under the condition of $\tilde{P}_1 = \tilde{P}_2$. It can be observed from figure 4 that as the value of signal to noise ratio (SNR) increases, bit error rate (BER) decreases. It is also clear from the figure 4 that bit error rate (BER) decreases as there is rise in the amount of parameter $\beta$. These two observations corresponding to figure 4 show perfect agreement with theory.

Fig. 2. Outage probability under asymmetric Rayleigh and Mixture Gamma fading channel with $\tilde{P}_1 = \tilde{P}_2 = x$

Fig. 3. Outage probability under asymmetric Rayleigh and Mixture Gamma fading channel with $\tilde{P}_1 = 2\tilde{P}_2$
In figure 6 and figure 7, bit error rate (BER) has been plotted for BFSK modulation scheme under the condition of \( \overline{\gamma}_1 = 2\overline{\gamma}_2 \) and \( \overline{\gamma}_1 = 2\overline{\gamma}_2 \). After observing figure 6 and figure 7, we can conclude that there is decline in the value of bit error rate (BER) if we increase value of parameter \( \beta \) and signal to noise ratio (SNR) and this observation also shows perfect agreement with theory. After comparing figure 6 and figure 7, we can observe that in case of BFSK modulation schemes also, if we have \( \overline{\gamma}_1 = 2\overline{\gamma}_2 \) then value of bit error rate (BER) is lesser in comparison of bit error rate (BER) under the condition of \( \overline{\gamma}_1 = \overline{\gamma}_2 \).

VI. CONCLUSION

This paper analyses outage probability and bit error rate in case of cooperative communication system. Since Mixture Gamma fading channel taken along relay to destination can represent the many fading scenarios, so we have taken Nakagami-\( m \) fading as a particular case of Mixture Gamma fading and plotted outage and BER for performance analysis of proposed system. From the plots of outage, it is clear that as the value of SNR increases, outage probability decrease. Outage probability also declines as we increase the value of fading severity factor. We have also plotted bit error rate (BER) of proposed system for binary phase shift keying (BPSK) and binary frequency shift keying (BFSK). It can be observed from the plots of BER that for both, BPSK and BFSK, when we have high value of SNR and fading severity factor, we get lower value of bit error rate (BER). Analysis presented here may also be very useful for evaluation of outage and BER when many other fading scenarios will be considered from relay to destination along with Rayleigh fading channel from source to relay. In future, using analysis of outage and BER, capacity can also be analyzed for performance analysis of this asymmetric cooperative communication system.
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