International Journal of Innovative Technology and Exploring Engineering (IJITEE)
ISSN: 2278-3075 (Online), Volume-8 Issue-11, September 2019

Pascal Triple Entire Sequence Space of Fibonacci Binomial Matrix on Rough Statistical Convergence And Its Rate

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***Abstract:** This paper initially discusses the definition of new rough statistical convergence with Pascal Fibonacci binomial matrix. Some general properties of rough statistical convergence are inspected. Further, approximation theory worked as a rate of the rough statistical convergence has been presented.*

***Keywords—**rough statistical convergence, natural density, triple entire sequences, Korovkin type approximation theorems, Pascal Fibonacci matrix, positive linear operator.*

2010 Mathematics Subject Classification: 40F05, 40J05, 40G05.

I. INTRODUCTION

The triple Pascal matrix is an infinite matrix comprising the binomial coefficients as the elements. This can be achieved asany of the below three types of matrices. The 4×4 truncation of these are demonstrated below.

Triple upper triangular

$$U_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 27 & 96 \\ 0 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

Triple lower triangular

$$L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 27 & 1 & 0 \\ 1 & 96 & 500 & 1 \end{pmatrix};$$

Symmetric

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 27 & 500 & 8575 \\ 1 & 96 & 3375 & 87808 \\ 1 & 250 & 15435 & 592704 \end{pmatrix}$$

These matrices have the pleasing relationship $A_n = L_n U_n$. It can be perceived that all 3 matrices poccess determinant 1. The symmetric triple Pascal matrix has its elements as the binomial coefficients. (i.e.)

$$A_{ijk} = \binom{r}{m}\binom{s}{n}\binom{t}{k} = \frac{r!}{m!\,(r-m)!}\frac{s!}{n!\,(n-s)!}\frac{t!}{k!\,(k-t)!},$$

where $r, s, t = i + j + k$ and $m = i, n = j, k = t$.

In other words

$$A_{ijk} = {}_{i+j+k}C_{ijk} = \frac{(i+j+k)!}{i!\,j!\,k!}.$$

Thus the trace of A_n is given by

$$tr(A_n) = \sum_{m=0}^{r-1}\sum_{n=0}^{s-1}\sum_{k=0}^{t-1}\frac{(2m)!}{(m!)^2}\frac{(2n)!}{(n!)^2}\frac{(2k)!}{(k!)^2}$$

with the first few terms given by the sequence $1,27,729,24389,\ldots$ Let A_n be $n \times n \times n$ matrix whose skew diagonals are successively the rows (truncated where necessary) of pascals triangle. In general, $A_n = (a_{ijk})$, where

$$a_{ijk} = \binom{i+j+k}{i}\binom{i+j+k}{j}\binom{i+j+k}{k}$$

for $i, j, k = 0,1,2,\ldots,n-1$.

A_n possesses the factorization

$$A_n = L_n L_n^T \qquad (1)$$

where L_n^T denotes the transpose of L_n. For the $[ijk]^{th}$ section of element of this product is the coefficient of x^{ijk} in $(1+x)^i (1+x)^j (1+x)^k$. That is.,

$$a_{ijk} = \binom{i+j+k}{i}\binom{i+j+k}{j}\binom{i+j+k}{k}$$

clearly

$$|L_n| = 1 \qquad (2)$$

so that

$$|A_n| = |L_n L_n^T| = |L_n|^2 = 1$$

We observe that L_n^{-1} is simply related to L_n. For example

$$L_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -27 & 1 & 0 \\ 1 & 96 & -500 & 1 \end{pmatrix};$$

and in general

$$L_n^{-1} = (-1)^{i+j-2k} I_{ijk} \qquad (3)$$

Additionally, 1 is an eigen value of A_n when n is odd and that if λ is an eigen value of A_n then so is λ^{-1}. These conjectures are readily verified for small values of n. In general, let

$$P_n(\lambda) = |\lambda I_n - A_n|$$

where I_n is the $n \times n \times n$ identity matrix. Then by (1), (2) and (3)

Manuscript published on 30 September 2019.
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$$P_n(\lambda) = |\lambda L_n L_n^{-1} - L_n L_n^T|$$
$$= |L_n||\lambda L_n^{-1} - L_n^T|$$
$$= \left|\left((-1)^{i+j-2k}\lambda l_{ijk} - l_{kji}\right)\right|$$
$$= (-\lambda)^n\left|\left(\lambda^{-1} l_{kji} - (-1)^{i+j-2k} l_{ijk}\right)\right|.$$

Multiplying odd numbered rows and columns of the matrix by -1 and transposing, we get

$$P_n(\lambda) = (-\lambda)^n\left|\left((-1)^{i+j-2k}\lambda^{-1} l_{ijk} - l_{kji}\right)\right|$$

$$P_n(\lambda) = (-\lambda)^n P_n\left(\frac{1}{\lambda}\right) \qquad (4)$$

But eigen values of A_n are the roots of $P_n(\lambda) = 0$ and thus it follows from (4) that if λ is an eigen value of A_n then so is λ^{-1}.

II. THE TRIPLE PASCAL MATRIX OF INVERSE AND TRIPLE PASCAL SEQUENCE SPACES

Let P denote the Pascal means defined by the Pascal matrix as is defined by

$$P = [P_{mnk}^{rst}] = \begin{cases} \binom{r}{m}\binom{s}{n}\binom{t}{k} & , \quad \text{if}\left(0 \le (m,n,k) \le (r,s,t)\right) \\ 0 & , \quad \text{if}\left((m,n,k) > (r,s,t), r,s,t,m,n,k \in \mathbb{N}\right) \end{cases}$$

and the inverse of Pascal's matrix

$$P = [P_{mnk}^{rst}]^{-1} = \begin{cases} (-1)^{(r-m)+(s-n)+(t-k)}\binom{r}{m}\binom{s}{n}\binom{t}{k} & \text{if} \quad \left(\begin{matrix} 0 \le (m,n,k) \le \\ (r,s,t) \end{matrix}\right) \\ 0 & \text{if} \quad \left(\begin{matrix} (m,n,k) > (r,s,t) \\ ,r,s,t,m,n,k \in \mathbb{N} \end{matrix}\right) \end{cases}$$

(5)

There is some interesting properties of Pascal matrix. For example, we can form three types of matrix; symmetric, lower triangular and upper triangular; for any integer $i,j,k > 0$. The symmetric Pascal matrix of order $n \times n \times n$ is defined by

$$A_{ijk} = a_{ijk} = \binom{i+j+k}{i}\binom{i+j+k}{j}\binom{i+j+k}{k} \text{ for } i,j,k = 0,1,2,\dots,n. \qquad (6)$$

We can define the lower triangular Pascal matrix of order $n \times n \times n$ by

$$L_{ijk} = \left(L_{ijk}\right) = \frac{1}{(-1)^{i+j-2k} I_{ijk}}; i,j,k = 1,2,\dots n. \qquad (7)$$

and the upper triangular Pascal matrix of order $n \times n \times n$ is defined by

$$U_{ijk} = \left(U_{ijk}\right) = \frac{1}{(-1)^{k-(i+j)} I_{ijk}}; i,j,k = 1,2,\dots n. \qquad (8)$$

We know that $U_{ijk} = \left(L_{ijk}\right)^T$ for any positive integer i,j,k.

(i). Let A_{ijk} be the symmetric Pascal matrix of order $n \times n \times n$ defined by (5), L_{ijk} be the lower triangular Pascal matrix of order $n \times n \times n$ defined by (7), then $A_{ijk} = L_{ijk}U_{ijk}$ and $\det\left(A_{ijk}\right) = 1$.

(ii) Let A and B be $n \times n \times n$ matrices.A is treated similar to Bwhen invertible $n \times n \times n$ matrix P occursso that $P^{-1}AP = B$.

(iii) Let A_{ijk} be the symmetric Pascal matrix of order $n \times n \times n$ defined by (6), then A_{ijk} is similar to its inverse A_{ijk}^{-1}.

(iv) LetL_{ijk} be the lower triangular Pascal matrix of order $n \times n \times n$ defined by (7), then $L_n^{-1} = L_{ijk}^{-1} = (-1)^{i+j-2k} I_{ijk}$.

We wish to introduce the Pascal sequence spaces P_{Λ^3} and P_{χ^3} as the set of all sequences such that $P-$ transforms of them are in the spaces Λ^3 and χ^3, respectively, that is

$$\Lambda_P^3 = \eta_{mnk} = \left\{x = (x_{mnk}) : \sup_{rst} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}|x_{mnk}|^{\frac{1}{m+n+k}} < \infty\right\},$$

and

$$\Gamma_P^3 = \mu_{mnk} = \left\{x = (x_{mnk}) : \lim_{rst\to\infty} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}|x_{mnk}|^{\frac{1}{m+n+k}} = 0\right\}.$$

We may redefine the spaces Λ_P^3, Γ_P^3 as follows: $\Lambda_P^3 = P_\Lambda^3$, $\Gamma_P^3 = P_\Gamma^3$. If λ is an normed or paranormed sequence space; then matrix domain λ_P is called an Pascal triple sequence space. We define the triple sequence $y = (y_{rst})$ as the $P-$ transform of a triple sequence $x = (x_{rst})$ i.e.,

$$y_{rst} = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}x_{mnk}, (r,s,t \in \mathbb{N}). \qquad (9)$$

Pascal sequence spaces P_{Λ^3} and P_{Γ^3} as the set of all sequences such that $P-$ transforms of them are in the spaces Λ^3 and Γ^3, respectively, that is it can be shown easily that P_{Γ^3} are linear and metric space by the following metric:

$$d(x,y)_{P_{\Gamma^3}} = d(Px, Py) = \sup_{mnk}\left\{|x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m,n,k = 1,2,3,\dots\right\}.$$

A. Densities and rough statistical convergence

Several definitions of densityexists related to the theory of numbers and the most prevalent is asymptotic density.

However, asymptotic density cannot be estimatedfor severalsequences. Soit is necessary to define densities to cover such areas.

It is known intuitively that the positive integers are much more compared to the perfect squares. The set of perfect squares as well as the set of positive integers are infinite, countable and mayhave one-to-one correspondence. The squares become increasingly rare when natural numbers are considered. Natural density aids us in this instance and makes this intuition accurate.

Let α and β be the subsets of set of positive integers $\mathbb{Z}^+$.Consider the interval $[1,n]$. Choose an integer in this interval. The ratio of the count of elements of $\alpha \in [1,n]$ to overallcount of elements in $[1,n]$ belongs to α, probably. For $n \to \infty$, if this probability occurs, i.e.,it inclines to certain limit, it is considered as the asymptotic density of the set α. It should be noted thatα is asymptotically equal to β ($\alpha \approx \beta$) if the symmetric difference $\alpha\Delta\beta$ is finite. Freedman and Sember presented the notion of a lower asymptotic density.

Definition 2.1

Let f be a function which defined for all sets of natural numbers and take values in the interval [0,1]. Then, the function f is said to a lower asymptotic density, if the following conditions hold:

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(i) $f(\alpha) = f(\beta)$, if $\alpha \approx \beta$,
(ii) $f(\alpha) + f(\beta) \leq f(\alpha \cup \beta)$, if $\alpha \cap \beta = \phi$,
(iii) $f(\alpha) + f(\beta) \leq 1 + f(\alpha \cap \beta)$, for all α,
(iv) $f(\mathbb{Z}^+) = 1$.

The upper density can be stated based on the statement related to lower density as below:

Let f be some density. The function $\bar{f}$ is upper density related with f, if $\bar{f}(\alpha) = 1 - f(\mathbb{Z}^+ \backslash \alpha)$for any natural number set.

Consider the case of set $\alpha \subset \mathbb{Z}^+$. If $f(\alpha) = \bar{f}(\alpha)$, then the set α has natural density with respect to α. The term asymptotic density if frequently used for the function

$$d(\alpha) = \lim_{u,v,w \to \infty} \inf \frac{\alpha(u,v,w)}{u,v,w},$$

where $\alpha \subset \mathbb{N}$ and $\alpha(u,v,w) = \sum_{(a,b,c) \leq (u,v,w),(a,b,c) \in \alpha} 1$.

And the natural density of α is

$$d(\alpha) = \lim_{u,v,w} \frac{1}{uvw} |\alpha(u,v,w)|$$

where $|\alpha(u,v,w)|$represents the number of elements in $\alpha(u,v,w)$.

Steinhaus [26] and Fast [13]brought up the concept of statistical convergence for real/complex sequences. A triple sequence can be expressed as $x: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$represent the sets comprising natural numbers, real numbers and complex numbers respectively. Various kinds of concepts of triple sequence was discussed by Bipan Hazarika et al. [2], Sahiner et al. [17, 18], Esi et al. [3, 4, 5, 6, 7, 8, 9, 10], Dutta et al. [11], Subramanian et al. [19, 20, 21, 22, 23, 24], Velmurugan [25], Debnath et al. [12] and many others.

ConsiderK, a subset of the set $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Let'srepresent the set $\{(m,n,k) \in K: m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Now the natural density of K is $\delta(K) = \lim_{uvw \to \infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$symbolizes the number of elements in K_{uvw}. Noticeably, a finite subset has natural density zero, and therefore$\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} \backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Let$x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}$, $m,n,k \in \mathbb{N}$.

A triple sequence $x = (x_{mnk})$ is regarded statistically convergent to $0 \in \mathbb{R}$, written as $st - \lim \ x = 0$, if the set

$$\{(m,n,k) \in \mathbb{N}^3: |x_{mnk} - 0| \geq \varepsilon\}$$

contains natural density zero for any $\varepsilon > 0$. Zero is known as the statistical limit of the triple sequence xin this situation.

Consider a triple sequence that is statistically convergent. For every $\varepsilon > 0$, infinitely several terms of the sequence may stay outside the $\varepsilon-$ neighbourhood of the statistical limit, provided the natural density of the set involving the indices of these terms is zero. This identity differentiates statistical and ordinary convergence. It can be concluded that every ordinary convergent sequence is statistically convergent as the natural density is zero for finite set.

Assume a triple sequence $x = (x_{mnk})$fulfillscertain property P for all m,n,kexcluding a set having natural density zero. Then the triple sequence xfulfillsP for "almost every(m,n,k)" and we representit by "a.a. (m,n,k)".

Let $\left(x_{m_i n_j k_\ell}\right)$ be a sub sequence of $x = (x_{mnk})$. If the natural density of the set $K = \left\{\left(m_i, n_j, k_\ell\right) \in \mathbb{N}^3: (i,j,\ell) \in \mathbb{N}^3\right\}$ is other than zero, then $\left(x_{m_i n_j k_\ell}\right)$ is known as a non thin sub sequence of x.

$c \in \mathbb{R}$ is known as statistical cluster point of $x = (x_{mnk})$if the natural density for the below set

$$\{(m,n,k) \in \mathbb{N}^3: |x_{mnk} - c| < \varepsilon\}$$

is dissimilar from zero for each$\varepsilon > 0$. The set of entire statistical cluster points of the sequence xis represented by Γ_x.

A triple sequence $x = (x_{mnk})$ is statistically analytic if there occurs a positive number M such that

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3: |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0$$

In present work, we define the Pascal Fibonacci binomial matrix $F = \left(f_{ij\ell}^{mnk}\right)_{m,n,k=1}^{\infty}$, which is different from existing Pascal Fibonacci binomial matrix by employing Fibonacci numbers $f_{ij\ell}$ and presentcertain new triple sequence space of P_{Γ^3} and P_{Λ^3}. We define the Pascal Fibonacci binomial matrix $Ab^{rs} = Ab^{rs}_{uvw,mnk}$, where

$$= \begin{cases} \frac{f_{sr}}{f_{(s+r)^{u+v+w}}} \binom{u}{m} \binom{v}{n} \binom{w}{k} s^{(u-m)+(v-n)+(w-k)} & , \quad \text{if} m \leq u, n \leq v, k \leq w \\ 0 & , \quad \text{if} m > u, n > v, k > w \end{cases}$$

Phu [16]came up with the concept of rough convergence. This idea has remarkable applications.This idea was extended by Aytar [1] into rough statistical convergence. Pal et al. [15] elaborated the view of rough convergence employing the notion of ideals. In this paper, we present the concept of rough statistical convergence of triple sequences. Pascal Fibonacci binomial matrix criteria associated with this set of rough statistical convergence has been obtained. All through this paper ris taken as nonnegative real number.

Definition 2.2

A Pascal triple sequence $\mu = (\mu_{mnk})$ is said to be rough convergent $(r-\text{convergent})$ to l (Pringsheim's sense), denoted as $\mu_{mnk} \to^r l$, provided that

$$\forall \varepsilon > 0, \exists \ i_\varepsilon \in \mathbb{N}: m,n,k \geq i_\varepsilon \Rightarrow |\mu_{mnk} - l| < r + \varepsilon, \tag{10}$$

or equivalently, if

$$\limsup |\mu_{mnk} - l| \leq r. \tag{11}$$

The symbolr is known as the roughness degree. The ordinary convergence of a Pascal triple sequence will be attained if $r = 0$.

Definition 2.3

It is obvious that the $r-$ limit set of a Pascal triple sequence is not unique. The $r-$ limit set of the Pascal triple sequence $\mu = (\mu_{mnk})$ is defined as $\text{LIM}^r \mu_{mnk} := \{l \in \mathbb{R}: \mu_{mnk} \to^r l\}$.

Definition 2.4

A Pascal triple sequence $\mu = (\mu_{mnk})$ is said to be $r-$ convergent if $\text{LIM}^r \mu \neq \phi$. In this case, r is known as the convergence degree of the Pascal triple sequence $\mu = (\mu_{mnk})$. For $r = 0$, we obtain the ordinary convergence.

Definition 2.5

A Pascal triple sequence (μ_{mnk}) is said to be $r-$ statistically convergent to l, denoted by $\mu_{mnk} \to^{rst} l$, provided that the set

$$\{(m,n,k) \in \mathbb{N}^3 : |\mu_{mnk} - l| \geq r + \varepsilon\}$$

has natural density zero for every $\varepsilon > 0$, or equivalently, if the condition

$$st - \limsup \; |\mu_{mnk} - l| \leq r$$

is satisfied.

Additionally, $\mu_{mnk} \to^{rst} l$ if and only if the inequality

$$|\mu_{mnk} - l| < r + \varepsilon$$

holds for all $\varepsilon > 0$ and almost all (m,n,k).

Definition 2.6

A Pascal triple sequence $\mu = (\mu_{mnk})$ is said to be rough statistically Cauchy sequence if for every $\varepsilon > 0$ and r be a positive number there is positive integer $N = N(r+\varepsilon)$ such that $d\left(\{(m,n,k) \in \mathbb{N} : |\mu_{mnk} - \mu_{N(r+\varepsilon)}| \geq r + \varepsilon\}\right) = 0$.

Assuming that a Pascal triple sequence $\gamma = (\gamma_{mnk})$ is statistically convergent and cannot be estimated accurately. an approximated triple sequence $\mu = (\mu_{mnk})$has to be used fulfilling$|\mu_{mnk} - \gamma_{mnk}| \leq r$ for all m,n,k (or for almost every(m,n,k), i.e.,

$$\delta(\{(m,n,k) \in \mathbb{N}^3 : |\mu_{mnk} - \gamma_{mnk}| > r\}) = 0.$$

Then the Pascal triple sequence μ is not statistically convergent no longer, but because the inclusion

$$\{(m,n,k) \in \mathbb{N}^3 : |\gamma_{mnk} - l| \geq \varepsilon\} \supseteq \{(m,n,k) \in \mathbb{N}^3 : |\mu_{mnk} - l| \geq r + \varepsilon\} \quad (12)$$

holds and we have

$$\delta(\{(m,n,k) \in \mathbb{N}^3 : |\gamma_{mnk} - l| \geq \varepsilon\}) = 0,$$

i.e., we get

$$\delta(\{(m,n,k) \in \mathbb{N}^3 : |\gamma_{mnk} - l| \geq r + \varepsilon\}) = 0,$$

i.e., the Pascal triple sequence spaces μ is $r-$ statistically convergent.

B. Approximation theory

Korovkin type approximation theorems can be used to verify a specified Pascal triple sequence $(\alpha_{mnk})_{mnk \geq 1}$ of positive linear operators on $C[a,b]$ of all continuous functions on the real interval $[a,b]$ is an approximation process. Theyintroduced a variety of test functions. These functionswill provide the approximation property that is true on the whole space. Such an identity was presented by Korovkin for the functions $1, x, x^2$ in the space $C[a,b]$. Also he discussed on the functions 1, cosx,sinx in the space of all continuous 2π periodic functions on the real line.

III PASCAL FIBONACCI BINOMIAL OF ROUGH STATISTICAL CONVERGENCE

A Pascal sequence $\eta = (\eta_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |\eta_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The Pascal triple sequence space P_{Λ^3} is a metric space with the metric

$$d(x,y) = \sup_{m,n,k} \left\{ |\mu_{mnk} - \gamma_{mnk}|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,\dots \right\}, \quad (13)$$

for all$\mu = \{\mu_{mnk}\}$and$\gamma = \{\gamma_{mnk}\}$ in P_{Λ^3}. Then,

$$P_{\Gamma^3}\left(Ab^{rs}_{uvw,mnk}\right) = \left\{\mu = (\mu_{mnk}) \in w : \left(Ab^{rs}_{uvw,mnk}\mu_{mnk}\right) \in P_{\Gamma^3}\right\}.$$

If P_{Γ^3} is a linear space then $P_{\Gamma^3}\left(Ab^{rs}_{uvw,mnk}\right)$ is also a linear space.

If P_{Γ^3} is a complete metric space, then, $P_{\Gamma^3}\left(Ab^{rs}_{uvw,mnk}\right)$ is also a complete metric space with the metric

$$d(x,y) = \sup\{|Ab^{rs}\mu - Ab^{rs}\gamma| : m,n,k = 1,2,3,\dots\}_{P_{\Gamma^3}}$$

Lemma 3.1

If $P_{\Gamma^3_\mu} \subset P_{\Gamma^3_\gamma}$ then $P_{\Gamma^3}(Ab^{rs}\mu) \subset P_{\Gamma^3}(Ab^{rs}\gamma)$.

Proof. It is trivial.

Theorem 3.1

Consider that P_{Γ^3} is a complete metric space and α is closed subset of P_{Γ^3}. Then $\alpha(Ab^{rs})$ is also closed in $P_{\Gamma^3(Ab^{rs})}$.

Proof.Becauseα is a closed subset of P_{Γ^3} from Lemma 3.1,

$$\alpha(Ab^{rs}) \subset P_{\Gamma^3}(Ab^{rs}).$$

$\overline{\alpha(Ab^{rs})}$, $\overline{\alpha}$ denote the closure of $\alpha(Ab^{rs})$ and α respectively. It is enough to prove that $\overline{\alpha(Ab^{rs})} = \overline{\alpha}(Ab^{rs})$.

Firstly, we take $\mu \in \overline{\alpha(Ab^{rs})}$, there exists a sequence $(\mu^{uvw}) \in \alpha(Ab^{rs})$ such that $|\mu^{uvw} - x|_{Ab^{rs}} \to 0$ in $\alpha(Ab^{rs})$ for $u,v,w \to \infty$. Thus, $|\mu^{uvw}_{mnk} - \mu_{mnk}|_{Ab^{rs}} \to 0$ as $u,v,w \to \infty$ in $\mu \in \alpha(Ab^{rs})$ so that

$$\sum_{r=1}^{i}\sum_{s=1}^{j}\sum_{t=1}^{\ell} |\mu^{uvw}_{rst} - \mu_{rst}| + |Ab^{rs}\mu^{uvw}_{mnk} - Ab^{rs}\mu_{mnk}| \to 0$$

for $(u,v,w) \to \infty$, in α. Therefore, $Ab^{rs}\mu \in \overline{\alpha}$ and so $\mu \in \overline{\alpha}(Ab^{rs})$.

Conversely, if we take $\mu \in \overline{\alpha(Ab^{rs})}$, then $\mu \in \alpha(Ab^{rs})$. Since α is closed. Then $\overline{\alpha(Ab^{rs})} = \overline{\alpha}(Ab^{rs})$. Hence $\alpha(Ab^{rs})$ is a closed subset of $P_{\Gamma^3}(Ab^{rs})$.

Corollary 3.1

If P_{Γ^3} is a separable space, then $P_{\Gamma^3}(Ab^{rs})$ is also a separable space.

Definition 3.1

A Pascal triple sequence $\mu = (\mu_{mnk})$ is said to be Pascal Fibonacci binomial matrix on rough statistically convergence if there is a number l such that for every $\varepsilon > 0$ and r be a positive number the set

$$K_{r+\varepsilon}(Ab^{rs}) := \{(m,n,k) \leq (u,v,w) : |Ab^{rs}\mu_{mnk} - l| \geq r + \varepsilon\}$$

has natural density zero, i.e.; $d\left(K_{r+\varepsilon}(Ab^{rs})\right) = 0$. That is $\lim_{uvw \to \infty} \frac{1}{uvw} |\{(m,n,k) \leq (u,v,w) : |Ab^{rs}\mu_{mnk} - l| \geq r + \varepsilon\}| = 0$.

Here we write $d(Ab^{rs}) - \lim \mu_{mnk} = l$ or $\mu_{mnk} \to l\left(rs(Ab^{rs})\right)$. The set of $Ab^{rs}-$ rough statistically convergent Pascal triple sequence space will be denoted by $rs(Ab^{rs})$. Here$l = 0$, we will write $rs_0(Ab^{rs})$.

Definition 3.2

A Pascal triple sequence $\mu = (\mu_{mnk})$ is said to be Pascal Fibonacci binomial matrix on rough statistically Cauchy if there exists a number $N = N(r+\varepsilon)$ such that for every $\varepsilon > 0$ and r be a positive number the set

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DOI: 10.35940/ijitee.K2495.0981119
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$$\lim_{uvw\to\infty}\frac{1}{uvw}|\{(m,n,k)\le(u,v,w):|Ab^{rs}\mu_{mnk}-Ab^{rs}\mu_N|\ge r+\varepsilon\}|=0.$$

Theorem 3.2

If a Pascal triple sequence space μ is a Pascal Fibonacci binomial matrix on rough statistically convergent sequence then μ is a Pascal Fibonacci binomial matrix on rough statistically Cauchy sequence.

Proof. Let $\varepsilon>0$ and r be a positive real number. Assume that $(\mu_{mnk})\to l(rs(Ab^{rs}))$. Then

$$|Ab^{rs}\mu_{mnk}-l|<\frac{r+\varepsilon}{2}\text{ for almost all } m,n,k.$$

If N is chosen so that

$$|Ab^{rs}\mu_N-l|<\frac{r+\varepsilon}{2}$$

then we have

$$|Ab^{rs}\mu_{mnk}-Ab^{rs}\mu_N|<|Ab^{rs}\mu_{mnk}-l|+|Ab^{rs}\mu_N-l|<\left(\frac{r+\varepsilon}{2}\right)+\left(\frac{r+\varepsilon}{2}\right)=r+\varepsilon$$

for almost all m,n,k.

$\Rightarrow\mu$ is Pascal Fibonacci binomial matrix on rough statistically Cauchy sequence.

Theorem 3.3

If μ is Pascal triple sequence for which there is a Pascal Fibonacci binomial matrix on rough statistically convergent sequence $\gamma=(\gamma_{mnk})$ such that $Ab^{rs}\mu_{mnk}=Ab^{rs}\gamma_{mnk}$ for almost all m,n,k, then μ is Pascal Fibonacci binomial matrix on rough statistically convergent sequence.

Proof. Suppose that $Ab^{rs}\mu_{mnk}=Ab^{rs}\gamma_{mnk}$ for almost all m,n,k, and $(\gamma_{mnk})\to l(rs(Ab^{rs}))$. Then, $\varepsilon>0$ and r be a positive real number and for each u,v,w,

$$\{(m,n,k)\le(u,v,w):|Ab^{rs}\mu_{mnk}-l|\ge r+\varepsilon\}\subseteq\{(m,n,k)\le(u,v,w):|Ab^{rs}\mu_{mnk}\ne Ab^{rs}\gamma_{mnk}|\ge r+\varepsilon\}\cup\{(m,n,k)\le(u,v,w):|Ab^{rs}\mu_{mnk}-l|\le r+\varepsilon\}.$$

Since $(\gamma_{mnk})\to l(rs(Ab^{rs}))$, the latter set contains a fixed number of integers, say $g=g(r+\varepsilon)$. Therefore, for $Ab^{rs}\mu_{mnk}=Ab^{rs}\gamma_{mnk}$ for almost all m,n,k,

$$\lim_{uvw\to\infty}\frac{1}{uvw}|\{(m,n,k)\le(u,v,w):|Ab^{rs}\mu_{mnk}-l|\ge r+\varepsilon\}|\le\lim_{uvw\to\infty}\frac{1}{uvw}|\{(m,n,k)\le(u,v,w):|Ab^{rs}\mu_{mnk}\ne Ab^{rs}\gamma_{mnk}|\}|+\lim_{uvw}\frac{g(r+\varepsilon)}{uvw}=0.$$

Hence $(\mu_{mnk})\to l(rs(Ab^{rs}))$.

Definition 3.3

A Pascal triple sequence $\mu=(\mu_{mnk})$ is said to be rough statistically analytic if there exists some $l\ge 0$ such that

$$d\left(\{(m,n,k):|\mu_{mnk}|^{1/m+n+k}>l\}\right)=0,$$

i.e., $|\mu_{mnk}|^{1/m+n+k}\le l a.a.k.$

Analytic sequences are clearly rough statistically analytic because of the zero natural density of empty set. But the converse is not correct.

For example, consider the Pascal triple sequence

$$\mu=(\mu_{uvw})=\begin{cases}(uvw)^{u+v+w}, & if\ (m,n,k)\ is\ a\ square\\ 0, & if\ (m,n,k)\ is\ not\ a\ square\end{cases}$$

clearly the Pascal triple sequence (μ_{mnk}) is not a analytic sequence. However,

$$d\left(\left\{(m,n,k):|\mu_{mnk}|^{1/m+n+k}>\frac{1}{6}\right\}\right)=0,$$

because the of squares has zero natural density and therefore the Pascal triple sequence (μ_{mnk}) is rough statistically analytic.

Corollary 3.2

Every convergent sequence is rough statistically triple Pascal analytic.

Corollary 3.3

Every rough statistical convergent sequence is rough statistically triple Pascal analytic.

Corollary 3.4

Every Pascal Fibonacci binomial matrix of rough statistical convergent sequence is Pascal Fibonacci binomial matrix of rough statistically triple Pascal analytic.

IV.RATE OF PASCAL FİBONACCİ BİNOMİAL MATRİX ON ROUGH STATİSTİCAL CONVERGENCE & RESULTS

Let $F(\mathbb{R})$represent the linear space of real value function on $\mathbb{R}$. Let $C(\mathbb{R})$ be space of all real-valued continuous functions f on $\mathbb{R}$. $C(\mathbb{R})$ with the metric given as follows:

$$d((f,\mu),(f,\gamma))=\sup_{\mu\in\mathbb{R}}|(f,\mu)-(f,\gamma)|^{1/m+n+k}, f\in C(\mathbb{R})$$

and we denote $C_{2\pi}(\mathbb{R})$ the space of all $2\pi-$ periodic functions $f\in C(\mathbb{R})$ with the metric is given by

$$d((f,\mu),(f,\gamma))_{2\pi}=\sup_{t\in\mathbb{R}}|(f,\mu(t))-(f,\gamma(t))|^{1/m+n+k}, f\in C(\mathbb{R}).$$

We calculate rate of Pascal Fibonacci binomial matrix on rough statistical convergence of a triple Pascal sequence of positive linear operators defined $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

Definition 4.1

Let (a_{uvw}) be a positive non-increasing sequence. The triple Pascal sequence $\mu=(\mu_{mnk})$ is rate of Pascal Fibonacci binomial matrix on rough statistical convergence to l with the rate $o(a_{uvw})$ if for every $\varepsilon>0$ and r be a real number such that

$$\lim_{uvw\to\infty}\frac{1}{h_{uvw}}|\{(m,n,k)\le(u,v,w):|Ab^{rs}\mu_{mnk}-l|\ge r+\varepsilon\}|=0.$$

We can write $(\mu_{mnk})-l=d(Ab^{rs})-o(a_{uvw})$.

Lemma 4.1

Let (a_{uvw}) and (b_{uvw}) be two positive non-increasing sequences. Let $\mu=(\mu_{mnk})$ and $\gamma=(\gamma_{mnk})$ be two triple Pascal sequences such that $(\mu_{mnk})-l_1=d(Ab^{rs})-o(a_{uvw})$ and $(\gamma_{mnk})-l_2=d(Ab^{rs})-o(b_{uvw})$. Then we have

(i) $\alpha(\mu_{mnk}-l_1)=d(Ab^{rs})-o(a_{uvw})$ for any scalar α,

(ii) $(\mu_{mnk}-l_1)\pm(\gamma_{mnk}-l_2)=d(Ab^{rs})-o(c_{uvw})$,

(iii) $(\mu_{mnk}-l_1)\cdot(\gamma_{mnk}-l_2)=d(Ab^{rs})-o(a_{uvw}b_{uvw})$, where $c_{uvw}=max\{a_{uvw},b_{uvw}\}$.

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DOI: 10.35940/ijitee.K2495.0981119
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For any $\delta > 0$, the modulus of continuity of f, $w(f,\delta)$ is defined by

$$w(f,\delta) = \sup_{|\mu-\gamma|<\delta} |(f,\mu) - (f,\gamma)|.$$

A function $f \in C[a,b]$, $\lim_{uvw\to 0^+} w(f,\delta) = 0$. For any $\delta > 0$

$$|(f,\mu) - (f,\gamma)| \leq w(f,\delta)\left(\frac{|\mu-\gamma|}{\delta} + 1\right). \quad (14)$$

Theorem 4.1

Let (l_{mnk}) be triple Pascal sequence of positive linear operator from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Assume that

(i) $d\left(l_{mnk}\left((1,\mu)-\mu\right),0\right)_{2\pi} = d(Ab^{rs}) - o(h_{uvw})$

(ii) $w(f,\theta_{mnk}) = d(Ab^{rs}) - o(g_{uvw})$ where $\theta_{mnk} = \sqrt{L_{mnk}\left[\sin^2\left(\frac{t-\mu}{2}\right),\mu\right]}$. Then for all $f \in C_{2\pi}(\mathbb{R})$, we get $d\left(l_{mnk}\left((f,\mu)-f(\mu)\right),0\right)_{2\pi} = d(Ab^{rs}) - o(e_{uvw})$ where $e_{uvw} = \max\{h_{uvw}, g_{uvw}\}$.

Proof. Let $f \in C_{2\pi}(\mathbb{R})$ and $\mu \in [-\pi,\pi]$, we can write

$$\begin{aligned}
\left|l_{mnk}\left((f,\mu)-f(\mu)\right)\right| &\leq l_{mnk}\left((f,t)-f(\mu),\mu\right) + |f(\mu)|\left|l_{mnk}\left((1,\mu)-f(1)\right)\right| \\
&\leq l_{mnk}\left(\frac{|\mu-\gamma|}{\delta}+1,\mu\right)w(f,\delta) + |f(\mu)|l_{mnk}\left((1,\mu)-f(1)\right) \\
&\leq l_{mnk}\left(\frac{\pi^2}{\delta^2}\sin^2\left(\frac{\gamma-\mu}{2}\right)+1,\mu\right)w(f,\delta) + |f(\mu)|l_{mnk}\left((1,\mu)-f(1)\right) \\
&\leq \left\{l_{mnk}(1,\mu) + \frac{\pi^2}{\delta^2}l_{mnk}\left(\sin^2\left(\frac{\gamma-\mu}{2}\right),x\right)\right\}w(f,\delta) + |f(\mu)|l_{mnk}\left((1,\mu)-f(1)\right) \\
&= \left\{l_{mnk}(1,\mu) + \frac{\pi^2}{\delta^2}l_{mnk}\left(\sin^2\left(\frac{\gamma-\mu}{2}\right),\mu\right)\right\}w(f,\delta) + |f(\mu)|l_{mnk}\left((1,\mu)-f(1)\right).
\end{aligned}$$

By choosing $\sqrt{\theta_{mnk}} = \delta$, we get

$$\begin{aligned}
d\left(l_{mnk}\left((f,\mu)-f(\mu)\right),0\right)_{2\pi} &\leq d\left((f,\mu),(f,\gamma)\right)_{2\pi} \\
d&\left(l_{mnk}\left((f,\mu)-f(\mu)\right) + 2w(f,\theta_{mnk}) + w(f,\theta_{mnk})\right) d\left(l_{mnk}\left((1,\mu) - f(\mu)\right),0\right)_{2\pi} \\
&\leq K\left\{d\left(l_{mnk}\left((1,\mu)-f(\mu)\right),0\right)_{2\pi} + w(f,\theta_{mnk}) + w(f,\theta_{mnk})l_{mnk}\left((1,\mu)-f(\mu)\right)_{2\pi}\right\},
\end{aligned}$$

where $K = \max\left\{2, d\left((f,\mu),(f,\gamma)\right)_{2\pi}\right\}$.

V. ACKNOWLEDGMENT

Then authors gratefully acknowledge TATA Realty-SASTRA Srinivasa Ramanujan Research chair for supporting this research.

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DOI: 10.35940/ijitee.K2495.0981119
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