

Z-number CCR using Trapezoidal Fuzzy Numbers

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Abstract— Data envelopment analysis (DEA) is a powerful tool for measuring efficiency of multiple inputs and outputs of a set of decision making units (DMUs). There are several models in DEA such as the Banker, Charnes and Cooper (BCC) model, Andersen and Peterson (AP) model and many more. The data used are normally crisps but in real life, data are usually vague or imprecise such as in real problems that are characterized by linguistic information given by experts. In such cases, the Z-number has been used as it takes into account expert's reliability on the information given. Currently, the triangular membership function is used in the Z-number CCR model. However, in linguistic assessment the trapezoidal membership function is better suited to capture the vagueness of the assessment. The Z-number CCR model using the trapezoidal membership function for inputs and outputs of a set of DMUs is proposed in this paper. In the present study, the Z-number CCR using trapezoidal membership function is converted into a linear programming model and a crisp linear programming model is obtained by employing α -cut approach. A numerical example on portfolio selection in Information Systems/Information Technology (IS/IT) is presented to demonstrate the proposed method and to find the best portfolio by ranking them according to their efficiency score.

Keywords: Z-number, Data envelopment analysis (DEA), fuzzy data envelopment analysis (FDEA), reliability

I. INTRODUCTION

Data envelopment analysis (DEA) has been widely used in finding the efficiency for decision making units (DMUs) that involves multiple inputs and outputs. The principal of DEA is pioneered by M. J. Farrell that developed a technique on measuring a production efficiency of an organization [12]. Recent series of discussions on this topic started with [6] (CCR model) that evaluate the efficiency of DMU by taking the maximum ratio of weighted outputs to weighted inputs subject to the condition that similar ratios for every DMU must be less than or equal to 1. Later [3] developed a new model, the Banker, Cooper and Rhodes (BCC) model which is a modification of the CCR model. By referring to the efficient boundary, the model is able to estimate the efficiency of DMUs. Other researchers made improvements or extensions to the CCR and BCC methods which can be classified into many streams such as the cross-efficiency, super-efficiency and many more [1].

Like any other methods, DEA also has its limitations. DEA usually involves missing data and lack of information. Moreover, the nature of data in real environments, for instance, machine performance, customer satisfaction, and employee performance is vague and linguistic [2]. Most of the data collected is in qualitative form. In some cases, the data evaluation is in linguistic form which is given by expert's opinion. Therefore, the conventional DEA model is no longer efficient to tackle this issue as it only focuses on frontiers and boundaries which leads to a variation in the obtained efficiencies. In 2011, Zadeh introduced the Z-number that takes into account expert's reliability on the data. There are two components in the Z-number, $Z = (A, B)$ to estimate the variable Y . The A represents the uncertain information in evaluation which Y can take, while B represents a measurement of reliability or confidence in truth or probability that Y is A [28].

Currently, [2] introduced an integrated Z-number CCR model involving triangular membership functions. However, according Herrera (2000) the trapezoidal membership functions is better suited to capture the vagueness of linguistic assessment. Hence, we propose the new Z-number CCR model using trapezoidal membership functions for inputs and outputs set of DMUs. Table 4 shows the features of the proposed model compared to previous studies. This paper is structured as follows. In Section 2 we present the proposed methodology. Result and discussion are presented in Section 3. Section 4 is the conclusion on this paper.

II. PROPOSED METHOD

The proposed method to find the efficiency score for Z-number DEA using trapezoidal membership function consists of six steps.

Step 1. Assume that n DMUs is consist of m inputs and s outputs. Inputs and outputs are in Z-number form that comprised of pairs of fuzzy number related to the possibility of its reliability values [2]. Let $\tilde{Z}x_{j_i} = (\tilde{A}x_{j_i}, \tilde{B}x_{j_i})$ where $j = 1, 2, \dots, m$ and $\tilde{Z}y_{r_i} = (\tilde{A}y_{r_i}, \tilde{B}y_{r_i})$ where $r = 1, 2, \dots, s$ represents the Z-number input and output for DMU $_i$, respectively. Further, let $\tilde{A}x_{j_i}$ be a trapezoidal fuzzy number for the j^{th} input of DMU $_i$ with $\tilde{B}x_{j_i}$ a trapezoidal fuzzy number which contain the restriction of certainty (reliability) on $\tilde{A}x_{j_i}$. Likewise, let $\tilde{A}y_{r_i}$ be a trapezoidal fuzzy number for the r^{th} output of DMU $_i$ with $\tilde{B}y_{r_i}$ the restriction on certainty (reliability) of $\tilde{A}y_{r_i}$. Thus, the new Z-number CCR model can be written as:

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$$\begin{aligned} & \text{Max} \quad \sum_{r=1}^s u_r \widetilde{Zy}_{r0} \\ & \text{subject to} \quad \sum_{j=1}^m v_j \widetilde{Zx}_{j0} = 1 \\ & \sum_{r=1}^s u_r \widetilde{Zy}_{r0} - \sum_{j=1}^m v_j \widetilde{Zx}_{j0} \leq 0, \quad i = 1, 2, \dots, n \end{aligned} \quad (1)$$

$$u_r, v_j \geq 0, \quad r = 1, 2, \dots, s, \quad j = 1, 2, \dots, m$$

Equation (1) is the structure for the new Z-number CCR model and the model is not linear.

Step 2. To linearize the model, the second part of the Z-number is added to its first number. In order to do this, the center of gravity method (defuzzification method) is used to convert the second part of the Z-number to a crisp number by using the following equation:

$$\alpha = \frac{\int x \mu_{\widetilde{B}}(x) dx}{\int \mu_{\widetilde{B}}(x) dx} \quad (2)$$

Step 3. The Z-number model is converted to weighted fuzzy data envelopment model by multiplying α with each element in the trapezoidal membership functions in the first part which is \widetilde{Ax}_{ji} . Figure 1 shows the Z-number after multiplying with reliability value.

Step 4. The weighted fuzzy numbers are then converted to the regular fuzzy numbers while remaining the properties of reliabilities [2] by assuming that the slope of its line is equal to the weighted Z-number. The impact of the slope of the lines on the weighted Z-numbers can be seen in Figure 1. Suppose the weighted Z-number, $\widetilde{Z}^\alpha \sim \text{TrFN}(a, b, c, d)$, and its relevant normal fuzzy number has a trapezoidal membership function with $\widetilde{N} \sim \text{TrFN}(a', b', c', d')$. Further assume that $b = b'$ and $c = c'$ and the slope of the lines are equal. Figure 2 shows the converted weighted Z-number and weighted Z-number.

For $x \leq b$, the linear equation for the normal fuzzy number is given by $\mu_{\widetilde{N}}(x) = \frac{\alpha}{b-a}x + h$. For finding the value of h, the point of (b,1) is inserted in this equation. Thus :

$$1 = \frac{\alpha}{b-a}a' + h \rightarrow h = 1 - \frac{\alpha b}{b-a} \quad (3)$$

$$\mu_{\widetilde{N}}(x) = \frac{\alpha}{b-a}x + 1 - \frac{\alpha b}{b-a}, \quad x \leq b$$

If $\mu_{\widetilde{N}}(x) = 0$ in Equation (3), then the value of a' is identified by Equation (4).

$$0 = \frac{\alpha}{b-a}a' + 1 - \frac{\alpha b}{b-a} \rightarrow \frac{\alpha}{b-a}a' = \frac{\alpha b - b + a}{b-a} \quad (4)$$

$$a' = \frac{\alpha b - b + a}{b-a}$$

For $x \geq c$, the linear equation for the normal fuzzy number is given by $\mu_{\widetilde{N}}(x) = \frac{\alpha}{c-d}x + h$. For finding the value of h, the point of (c,1) is inserted in this equation. Thus :

$$1 = \frac{\alpha}{c-d}d' + h \rightarrow h = 1 - \frac{\alpha c}{c-d} \quad (5)$$

$$\mu_{\widetilde{N}}(x) = \frac{\alpha}{c-d}x + 1 - \frac{\alpha c}{c-d}, \quad x \geq c$$

If $\mu_{\widetilde{N}}(x) = 0$ in Equation (5), then the value of a' is identified by Equation (6).

$$0 = \frac{\alpha}{c-d}d' + 1 - \frac{\alpha c}{c-d} \rightarrow \frac{\alpha}{c-d}d' = \frac{\alpha c - c + d}{c-d} \quad (6)$$

$$d' = \frac{\alpha c - c + d}{c-d}$$

Step 5. In the Z-number CCR model, the z-number inputs value and outputs value for each DMUs are given by experts. For example, let $\widetilde{Zx}_{ji} = (\widetilde{Ax}_{ji}, \widetilde{Bx}_{ji})$ where $\widetilde{Ax}_{ji} = \text{TrFN}(a_{ji}, b_{ji}, c_{ji}, d_{ji})$ and $\widetilde{Bx}_{ji} = \text{TrFN}(e_{ji}, f_{ji}, g_{ji}, h_{ji})$ be inputs of DMU_j. The center of gravity method is then used to defuzzify it to a crisp value, α . Further, the weighted Z-number is obtained by multiplying α with trapezoidal membership functions. The weighted Z-number is then transformed to the normal fuzzy number $\widetilde{Nx}_{ji} = \text{TrFN}(a'_{ji}, b'_{ji}, c'_{ji}, d'_{ji})$ with

$$\begin{aligned} \alpha_{ji} &= \frac{e_{ji} + f_{ji} + g_{ji} + h_{ji}}{4}, \quad b'_{ji} = b_{ji}, \quad c'_{ji} = c_{ji}, \quad a'_{ji} = \frac{\alpha_j b_{ji} - b_{ji} + a_{ji}}{\alpha_j} \\ d'_{ji} &= \frac{\alpha_j c_{ji} - c_{ji} + d_{ji}}{\alpha_j} \end{aligned} \quad (7)$$

Step 6. The stated conversion method is employed to obtain the possibilistic linear programming of Z-number DEA model. Thus, if $\widetilde{Zx}_{ji} = (\widetilde{Ax}_{ji}, \widetilde{Bx}_{ji})$ is input j of DMU_i with

$$\begin{aligned} \widetilde{Ax}_{ji} &= \text{TrFN}(ax_{ji}, bx_{ji}, cx_{ji}, dx_{ji}) \\ \widetilde{Bx}_{ji} &= \text{TrFN}(ex_{ji}, fx_{ji}, gx_{ji}, hx_{ji}) \end{aligned}$$

By using equation (2), the value of αx_{ji} is obtained. Then, the weighted fuzzy numbers can be obtained by multiplying αx_{ji} with the first part of Z-number, Ax_{ji} .

$$\alpha x_{ji} = \frac{e_{ji} + f_{ji} + g_{ji} + h_{ji}}{4} \quad (8)$$

Then, equation (7) can be used to obtain the regular fuzzy numbers as given below:

$$\begin{aligned} \beta x_{ji} &= \frac{ex_{ji} + fx_{ji} + gx_{ji} + hx_{ji}}{4} \\ x_{ji}^b &= \alpha b x_{ji} \\ x_{ji}^c &= \alpha c x_{ji} \end{aligned} \quad (9)$$

$$\begin{aligned} x_{ji}^a &= \frac{\beta x_{ji} \alpha b x_{ji} - \alpha b x_{ji} + \alpha x_{ji}}{\beta x_{ji}} \\ x_{ji}^d &= \frac{\beta x_{ji} \alpha c x_{ji} - \alpha c x_{ji} + \alpha x_{ji}}{\beta x_{ji}} \end{aligned}$$

The same calculations can be made for outputs of each DMUs. Thus, the fuzzy programming for

Z-number CCR model of trapezoidal membership function is obtained and presented in Equation (9).

$$\begin{aligned} & \text{Max} \quad \sum_{r=1}^s u_r (y_{rp}^a, y_{rp}^b, y_{rp}^c, y_{rp}^d) \\ & \text{subject to} \quad \sum_{j=1}^m v_j (x_{jp}^a, x_{jp}^b, x_{jp}^c, x_{jp}^d) = 1 \\ & \sum_{r=1}^s u_r (y_{rp}^a, y_{rp}^b, y_{rp}^c, y_{rp}^d) - \sum_{j=1}^m v_j (x_{jp}^a, x_{jp}^b, x_{jp}^c, x_{jp}^d) \leq 0, \quad i = 1, 2, \dots, n \end{aligned} \quad (9)$$

$$u_r, v_j \geq 0, \quad r = 1, 2, \dots, s, \quad j = 1, 2, \dots, m$$

$$\begin{aligned} & \text{Max} \quad \sum_{r=1}^s \tilde{y}_{rp} \\ & \text{standard to} \quad \sum_{j=1}^m \tilde{x}_{jp} = 1 \\ & \sum_{r=1}^s \tilde{y}_{ri} - \sum_{j=1}^m \tilde{x}_{ji} \leq 0, \quad i = 1, \dots, n \\ & v_j (\alpha x_{ji}^b + (1 - \alpha)x_{ji}^a) \leq \tilde{x}_{ji} \leq v_j (\alpha x_{ji}^c + (1 - \alpha)x_{ji}^d), \quad i=1, \dots, n, \quad j=1, \dots, m \end{aligned} \quad (10)$$



$$u_r(\alpha y_{ri}^b + (1 - \alpha)y_{ri}^a) \leq \tilde{y}_{ri} \leq u_r(\alpha y_{ri}^c + (1 - \alpha)y_{ri}^d), i = 1, \dots, n, r = 1, \dots, s$$

$$u_r, v_j \geq 0$$

The value of α between $[0, 1]$ is chosen by the decision makers. The optimal solutions obtained will depend on the chosen α .

III. RESULT AND DISCUSSION

Data is taken from [2] on portfolio selection in IS/IT in one of the national governmental organization. This study focuses on finding the efficiency of 16 projects. The efficiency score is measured using Z-number CCR using trapezoidal membership function method. The score that is near or equal to 1 is considered as a good portfolio. Each project is labelled as a DMU with the values of input and output evaluated by the experts. By conserving the mean value for each criteria, several changes were done to obtain the trapezoidal fuzzy numbers from the triangular fuzzy numbers in the given data (see Table 1) while the membership function parameters for the reliability value is assumed (see Table 2).

Table 1 : Trapezoidal fuzzy numbers assigns to each criteria.

Project number	Cost of the project (\$ million) Input	No. Of potential subsequent investments Output 1	Contribution to the workflow improvement Output 2	Percentage of contribution to electronic readiness Output 3
1	(412, 427.3, 442.6, 458)	(128, 130.7, 133.7, 136)	(0.73, 0.82, 0.91, 1)	(42, 44.7, 47.4, 50)
2	(174, 176.7, 179.4, 182)	(69, 73, 77, 81)	(0.05, 0.13, 0.21, 0.29)	(6, 8, 10, 12)
3	(225, 236.3, 247.6, 259)	(27, 27.7, 28.4, 29)	(0.68, 0.76, 0.84, 0.91)	(36, 39.3, 42.6, 46)
4	(308, 318, 328, 338)	(85, 85.3, 88.6, 95)	(0.55, 0.65, 0.75, 0.85)	(87, 89, 91, 93)
5	(175, 184.3, 193.6, 203)	(73, 74.3, 75.6, 77)	(0.37, 0.47, 0.57, 0.68)	(71, 73.7, 76.4, 79)
6	(84, 90, 96, 102)	(66, 68.7, 71.4, 74)	(0.07, 0.15, 0.23, 0.31)	(45, 46.3, 47.6, 49)
7	(349, 363, 377, 391)	(123, 127.7, 132.4, 137)	(0.95, 0.99, 0.99, 0.99)	(39, 42.3, 45.6, 49)
8	(245, 249, 273, 297)	(41, 42.3, 43.6, 45)	(0.31, 0.40, 0.49, 0.59)	(32, 35.3, 38.6, 42)
9	(151, 153, 155, 157)	(58, 59.3, 60.6, 62)	(0.35, 0.41, 0.47, 0.65)	(25, 26.3, 27.6, 29)
10	(265, 275.7, 286.4, 297)	(49, 51, 53, 55)	(0.68, 0.76, 0.86, 0.94)	(37, 39.7, 42.4, 45)
11	(345, 356.3, 367.6, 379)	(21, 23, 25, 27)	(0.15, 0.17, 0.19, 0.21)	(54, 56.7, 59.4, 62)
12	(215, 219.7, 224.4, 229)	(4, 4.7, 5.4, 8)	(0.19, 0.196, 0.203, 0.21)	(56, 58, 60, 62)
13	(385, 389, 394, 397)	(6, 7.3, 8.6, 10)	(0.33, 0.337, 0.343, 0.35)	(34, 35.3, 36.6, 38)
14	(454, 467.3, 480.6, 494)	(7, 8.3, 9.6, 11)	(0.44, 0.46, 0.48, 0.50)	(11, 12.3, 13.6, 15)
15	(384, 388, 392, 396)	(7, 7.7, 8.4, 9)	(0.20, 0.21, 0.23, 0.24)	(48, 50, 52, 54)
16	(384, 388.7, 393.4, 398)	(9, 10.3, 11.6, 13)	(0.16, 0.18, 0.20, 0.22)	(52, 53.3, 54.6, 56)

Table 2 : Reliability values after changes has been made.

Z= (A, B)	Membership parameters	functions
B	Sure	(0.8, 1, 1, 1)
	Likely	(0.65, 0.75, 0.85, 0.95)
	Usually	(0.5, 0.6, 0.7, 0.8)

Using the method outlined in Section 2, the efficiency values for each project is obtained as in

Table 3. According to the ranking, project number 6 is the most efficient project.

IV. CONCLUSION

Due to the real world environment, many cases involve linguistic assessment by experts. It is important to take into account expert's opinion on how certain they are to the data that has been collected since each expert has different opinions on linguistic assessment. A method called the Z-number is proposed by Zadeh that takes into account the expert's reliability on the data itself. Currently, [2] proposed an integrated Z-number CCR model using the triangular membership function. In the present study, the Z-number CCR model using the trapezoidal membership function is proposed as it is better suited for linguistic assessment which can handle vague and imprecise data. The proposed method is applied for project portfolio selection problem. The most efficient project goes to project number 6 with the efficiency score equals to 1

Table 3 : Efficiency score

Project number	Efficiency score	Ranking
1	0.6034643	4
2	0.5593223	6
3	0.5873105	5
4	0.5309568	8
5	0.6623612	3
6	1.0000000	1
7	0.5528586	7
8	0.3685344	11
9	0.9887899	2
10	0.5078825	10
11	0.3506480	12
12	0.5079757	9
13	0.2122091	13
14	0.1304303	16
15	0.1627877	15
16	0.1805543	14

Table 4 : Features of proposed model compared to previous studies.

	DEA	FDEA	Z-DEA	Deterministic	Reliability (TFN)	Z-numbers (TFN)	Reliability (TrFN)	Z-numbers (TrFN)
Proposed model	√	√	√	√			√	√
CCR (1978)	√			√				
BCC (1984)	√							
Sueyoshi (2000)	√			√				
Letworasirikul et. al (2003)	√	√						
Saati et. al (2002)	√	√		√				
Danila (1989)				√				
Cooper et. al (1997)				√				
Schmidt (1993)				√				
Bardhan et. al (2004)				√				
Eilat et. al (2006)	√			√				
Huang et. al (2008)				√				
Chen and Cheng (2009)				√				
Ghapanchi et. al (2012)	√	√		√				
Azadeh and Kokabi (2016)	√	√	√	√	√	√		

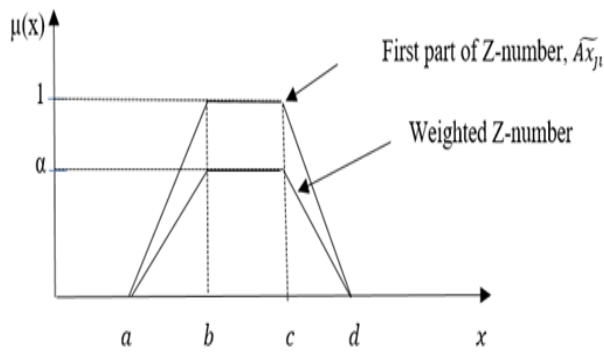


Figure 1: Z-number after multiplying with reliability value

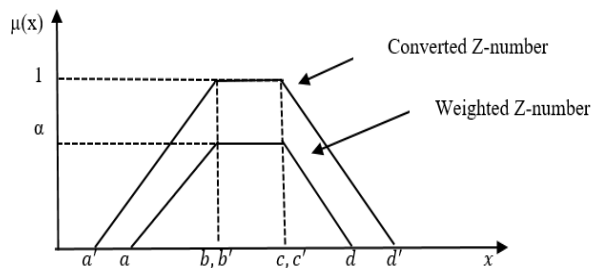


Figure 2 : Convert weighted Z-numbers to normal fuzzy numbers

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