International Journal of Innovative Technology and Exploring Engineering (IJITEE)
ISSN: 2278-3075 (Online), Volume-8 Issue-12, October 2019

# Dynamic Stress and Displacement Fields on Concrete Gravity Dams due to Harmonic Ground Motion

**Charles Chinwuba IKE, Hyginus Nwankwo ONAH**

***Abstract*: The partial differential equations (PDE) of equilibrium governing the natural vibrations of concrete gravity dams were derived in this work such that the fluid structure interactions were accounted for. The displacement formulation is a system of two coupled PDE in two unknown displacement components. For seismic ground motion assumed to be horizontal harmonic motion whose amplitude and period are known, the system of two coupled PDEs were solved subject to the boundary conditions using the method of undetermined parameters. In applying the method of undetermined parameters to the PDE, displacement shape functions constructed to satisfy the displacement boundary conditions were used in assuming the trial dynamic displacement fields in terms of two unknown parameters that were determined by substitution into the governing equations. Conditions for the trial dynamic displacement fields to be solutions to the governing PDE were sought by solving the resulting system of equations. The problem reduced to an algebraic eigenvalue eigenvector problem which was solved for nontrivial cases to obtain the characteristic frequency equation and the eigenvalues. Modal superposition technique was employed to obtain the general solution for the displacement fields. The use of the displacement boundary conditions at the upstream face and Fourier series theory yielded the dynamic displacement field components, and the dynamic stress fields. The maximum value of the hydrodynamic pressure distribution on the upstream face is found to occur at the bottom of the dam and is found mathematically to be a convergent series of infinite terms. The maximum hydrodynamic force was calculated by integration of the hydrodynamic pressure distribution over the upstream face of the dam, and found to be a convergent series. Values of the maximum hydrodynamic force computed in this work agree with solutions from the technical literature.***

***Index Terms*: Seismic ground motion, hydrodynamic pressure distribution, concrete gravity dam, modal superposition method, algebraic eigenvalue eigenvector problem.***

## I. INTRODUCTION

Dams are designed to withstand all applied loads namely, gravity load, hydrostatic, hydrodynamic pressures, etc.

**Revised Manuscript Received on October 30, 2019.**
***** Correspondence Author
**Charles Chinwuba IKE***, Department of Civil Engineering, Enugu State University of Science and Technology, Enugu, Enugu State, Nigeria. Email: ikecc2007@yahoo.com, charles.ike@esut.edu.ng.
**Hyginus Nwankwo ONAH**, Department of Civil Engineering, University of Nigeria, Nsukka, Enugu State, Nigeria. Email: hyginus.onah@unn.edu.ng.



The proper analysis and design of dams require a good understanding of their structural response under both static and dynamic loads [1].

In general, vibration is a major challenge in the analysis and design of civil engineering structures. Vibrations due to earthquakes are known to be major causes of destruction leading to loss of lives and properties. Since the sources of vibrations may not be completely eliminated, it is important to reduce the effects of vibration in order to minimize the loss of lives and properties.

The analysis of seismic behaviour of concrete dams is vital in the design of such structures [2, 3]. Eigen frequency analysis of dams is one of the vital steps in the analysis of their behaviour due to seismic excitation. Several methods exist for performing the eigenfrequency analysis of structures under seismic motion.

Parish *et al* [4] presented a Fourier series method for the determination of natural frequencies of foundation dam systems under free vibration of the dam. Watanabe *et al* [5] used a modal decomposition method and the Fourier spectra of microtremors to determine the first five natural frequencies and their corresponding modal shape functions. Zhen *et al* [6] determined the dynamic characteristics, namely, the eigen frequencies and modal shape functions for nonhomogeneous earth dams in triangular canyons for the case of transverse vibration.

The dynamic modal spectral method which is commonly used in the finite element analysis of structures under earthquake excitation expresses the equations of motion as differential equations [7, 8].

For structures modelled as one degree of freedom systems, the differential equation of motion for ground motion excitation is [9, 10]:

$$m\ddot{u} + c\dot{u} + ku = p(t) = -m\ddot{u}_g \qquad (1)$$

where $m$ is the mass, $u$ is the displacement, $c$ is the viscous damping constant and $k$ is the elastic stiffness constant.

The equation is the differential equation of motion for a single degree of freedom vibration excited by a forcing function $p(t)$ whose intensity is the mass multiplied by the ground acceleration $\ddot{u}_g$.

Dividing the equation by the mass yields:

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = -\ddot{u}_g \qquad (2)$$

The right hand side of the differential equation – Equation (2) – is the accelerogram or time history of a given earthquake. Methods used to obtain solutions to the differential equations are: response spectrum method, time history method and equivalent static method [11, 12].

The dynamic stresses on concrete dams can significantly be affected by the effects of ground motion caused by seismic activity [13], and this underscores the need for effective methods of analysis of dynamic response of such dams under ground motion. Research studies on the analysis of hydrodynamic stresses on concrete dams were pioneered by Westergaard [14]. Liu [15] have also studied the analysis of hydrodynamic stress distribution on rigid dams by assuming the impounded water in the dam reservoir to be incompressible. Zienkiewicz and Nath [16], based on experimental procedure, have also determined the distribution of hydrodynamic stresses on concrete dams. Chopra [17] investigated the effect of compressibility of water impounded in the dam reservoir on the distribution of hydrodynamic stresses and found it to significantly affect the hydrodynamic stress distribution.

Saimi *et al* [18], Patil and Katti [19], Hall and Chopra [20], Fenves and Chopra [21], Lofti *et al* [22] have used the finite element method to analyse the problem of hydrodynamic stresses on concrete gravity dams. Formulations using boundary element methods of the problem of hydrodynamic stresses on concrete dams have been derived by Humar and Jablonski [23], Medina and Dominguez [24], Cho and Liu [25], and Antes and Von Estorff [26]. Ike and Onah [27] presented analytical solution for the hydrodynamic pressure on concrete gravity dams based on the classical Westergaard's theory.

In this study, the problem of concrete dams impounding reservoir of water where the dam reservoir system is under seismic ground motion is formulated mathematically using the mathematical theory of elasticity. The resulting system of partial differential equations which govern the seismic motion of the dam-foundation-reservoir system is solved in closed form assuming harmonic ground motion to obtain the eigen frequencies, eigenvalues and dynamic displacement and stresses on the face of the dam.

### A. Research Aim and Objectives

The main goal of this research is to determine the closed form expressions for the stress and displacement fields in concrete gravity dams due to seismic ground motion idealized as a horizontal sinusoidal vibration. The specific objectives are as follows:

(i) to formulate the governing partial differential equations of equilibrium for the free vibration analysis of concrete gravity dams and embankment dams considering fluid-structure interaction.

(ii) to solve the governing equation of free vibration for the case of horizontal sinusoidal (harmonic) vibration of known amplitude and period using the method of undetermined parameters (trial functions).

(iii) to find the eigen frequencies and eigen modal shapes of the vibrating system.

(iv) to determine the hydrodynamic pressure distribution on the upstream surface of the concrete gravity dam, and the maximum values of the hydrodynamic pressure.

(v) to determine the expression for the resultant hydrodynamic force on the upstream face of concrete gravity dams due to seismic ground horizontal sinusoidal motion, and the maximum resultant force on the wall.

## II. METHODOLOGY

### A. Fundamental Assumptions of Problem Formulation

The problem is idealized as a two dimensional elasto-dynamic problem defined using the two dimensional Cartesian coordinate system shown in Fig. 1.

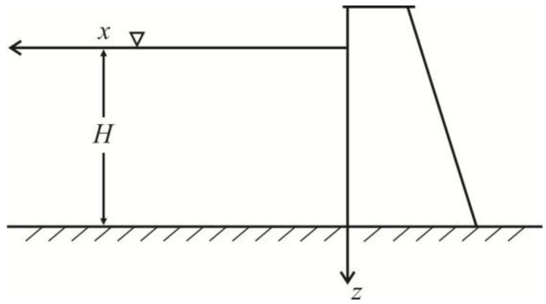


Fig. 1: Concrete gravity dam under horizontal ground motion-description of Cartesian coordinate.

The basic assumptions of the formulation are as follows:

(i) The seismic activity occurs at the bottom of the reservoir and is modelled as a ground motion described using the sinusoidal functions, with known amplitude and period. The ground motion occurs in the horizontal direction only and in the vertical direction, there is no ground motion.

(ii) The differential equations of dynamic equilibrium are satisfied for any two dimensional element of water in the reservoir.

(iii) The stress-strain law is satisfied.

(iv) The strain-displacement relations for infinitesimal deformations are used.

(v) The upstream surface of the dam is vertical as shown in Figure 1 and is given by the equation $x = 0$.

(vi) The surface of water in the reservoir is horizontal.

### B. Differential Equations of Dynamic Equilibrium

The displacement field components in the $x$ and $z$ coordinate directions at any time $t$, are denoted respectively by $u(x, z, t)$ and $w(x, z, t)$. The stress field is given by $\sigma(x,z,t)$. By considering the dynamic equilibrium of an infinitesimal two-dimensional element ($dx \times dz$) of water in the reservoir, the dynamic equilibrium of hydrodynamic pressures acting on the surface are obtained as:

$$\frac{\partial\sigma(x,z,t)}{\partial x} = \rho\frac{\partial^2 u(x,z,t)}{\partial t^2} \quad (3)$$

$$\frac{\partial\sigma(x,z,t)}{\partial z} = \rho\frac{\partial^2 w(x,z,t)}{\partial t^2} \quad (4)$$

where $\rho$ is the mass density of water.

### C. Stress-strain Equation

Water in the reservoir is considered incompressible; hence the stress field is directly proportional to the volumetric strain $\varepsilon_v$, where the proportionality constant is the bulk modulus of elasticity of water, denoted by $K_b$.

The stress-strain equation is thus given by

$$\sigma(x,z,t) = K_b \varepsilon_v \quad (5)$$

$$\sigma(x,z,t) = K_b(\varepsilon_{xx} + \varepsilon_{zz}) \quad (6)$$

where $\varepsilon_{xx}$, $\varepsilon_{zz}$ are the normal strains in the $xx$ and $zz$ coordinate directions.

### D. Strain-Displacement Relations

For infinitesimal deformations, the strain-displacement relations are given by

$$\varepsilon_{xx} = \frac{\partial u(x,z,t)}{\partial x} \quad (7)$$

$$\varepsilon_{zz} = \frac{\partial w(x,z,t)}{\partial z} \quad (8)$$

$$\gamma_{xz} = \frac{\partial u(x,z,t)}{\partial z} + \frac{\partial w(x,z,t)}{\partial x} \quad (9)$$

where $\gamma_{xz}$ is the shear strain.

### E. Displacement Formulation of the Elastodynamic Problem

From the strain-displacement relations, the stress-displacement equations are given by:

$$\sigma(x,z,t) = K_b\left(\frac{\partial u}{\partial x}(x,z,t) + \frac{\partial w}{\partial z}(x,z,t)\right) \quad (10)$$

By substitution of Equation (10) in Equations (3) and (4) the differential equations of equilibrium are expressed in terms of the displacement field components as follows:

$$\frac{\partial}{\partial x} K_b\left(\frac{\partial u}{\partial x}(x,z,t) + \frac{\partial w}{\partial z}(x,z,t)\right) = \rho\frac{\partial^2 u}{\partial t^2}(x,z,t) \quad (11)$$

$$\frac{\partial}{\partial z} K_b\left(\frac{\partial u}{\partial x}(x,z,t) + \frac{\partial w}{\partial z}(x,z,t)\right) = \rho\frac{\partial^2 w}{\partial t^2}(x,z,t) \quad (12)$$

Equations (9) and (12) are simplified to obtain:

$$\frac{\partial^2 u}{\partial x^2}(x,z,t) + \frac{\partial^2 w}{\partial x \partial z}(x,z,t) - \frac{\rho}{K_b}\frac{\partial^2 u}{\partial t^2}(x,z,t) = 0 \quad (13)$$

$$\frac{\partial^2 w}{\partial z^2}(x,z,t) + \frac{\partial^2 u}{\partial x \partial z}(x,z,t) - \frac{\rho}{K_b}\frac{\partial^2 w}{\partial t^2}(x,z,t) = 0 \quad (14)$$

The system of two coupled partial differential equations in $u(x, z, t)$ and $w(x, z, t)$ represent the governing partial differential equations in a displacement formulation for the dynamic/vibration analysis of concrete gravity dams and embankment dams.

### F. Idealization of Seismic Ground Motion

The vibration of the concrete gravity dam is due to seismic ground motion formulated/expressed in mathematical terms as a horizontal sinusoidal vibration with a known/given amplitude $u_a$ and period $T$. Let the seismic ground motion $u_g$ be given by:

$$u_g = u_a \sin(\omega_g t + \phi) \quad (15)$$

where $$\omega_g = \frac{2\pi}{T} = 2\omega f \quad (16)$$

$$f = \frac{1}{T} \quad (17)$$

$\omega_g$ is the circular frequency, $f$ is the frequency, $\phi$ is the phase.

The displacement boundary conditions are:

$$u(x,H,t) = u_g = u_a \sin(\omega_g t + \phi) \quad (18)$$

$$w(x,H,t) = 0 \quad (19)$$

$$\frac{\partial^2 u}{\partial t^2}(x,z,t) = a = -\omega_g^2 u_a \sin(\omega_g t + \phi) \quad (20)$$

If the seismic acceleration of the ground which is the base of the reservoir is $\alpha g$, then from the boundary condition Equation (18),

$$\alpha g = -\omega_g^2 u_a = -\omega_g^2 u_a \quad (21)$$

$$u_a = \frac{\alpha g}{-\omega_g^2} = -\frac{\alpha g}{\omega_g^2} = -\frac{\alpha g T^2}{4\pi^2} \quad (22)$$

$$u(x,H,t) = -\frac{\alpha g T^2}{4\pi^2}\sin(\omega_g t + \phi) \quad (23)$$

### G. Stress and Displacement Compatibility Conditions

The other boundary conditions are:

(i) the hydrodynamic pressure distribution vanishes at the surface of the reservoir where $z = 0$, i.e.

$$\sigma(x, z=0, t) = 0 \quad (24)$$

(ii) the dynamic stress fields vanish at points that are very distant from the upstream surface of the concrete dam, i.e.

$$\sigma(x \to \infty, z, t) \to 0$$

(iii) the concrete gravity dam is built using very rigid materials, hence the relative displacement of the dam with respect to the ground due to the seismic motion is infinitesimally small as to be neglected. The horizontal displacement of the upstream surface of the dam is thus equal to the horizontal displacement due to the ground motion ($u_g$).

Hence,

$$u(x=0,z,t) = u_g \sin(\omega_g t + \phi)$$

$$= -\frac{\alpha g T^2}{4\pi^2}\sin(\omega_g t + \phi) \quad (25)$$

## III. RESULTS

### A. Trial Displacement Fields

Trial displacement fields that satisfy the boundary conditions are assumed in terms of unknown displacement parameters, $c_1$, $c_2$, $\lambda$ and $\beta$ as follows:

$$u(x,z,t) = c_1 \exp(-\lambda x)\sin\beta z \sin(\omega_g t + \phi) \quad (26)$$

$$w(x,z,t) = c_2 \exp(-\lambda x)\cos\beta z \sin(\omega_g t + \phi) \quad (27)$$

where $c_1$, $c_2$, $\lambda$ and $\beta$ are unknown parameters which are to be determined.

$$u(x,z,t) = c_1 \sin(\omega_g t + \phi)\varphi_1(x,z) \quad (28)$$

$$w(x,z,t) = c_2 \sin(\omega_g t + \phi)\varphi_2(x,z) \quad (29)$$

where $$\varphi_1(x,z) = \exp(-\lambda x)\sin\beta z \quad (30)$$

$$\varphi_2(x,z) = \exp(-\lambda x)\cos\beta z \quad (31)$$

$\varphi(x,z)$ is the displacement shape function.

By substitution of Equations (26) and (27) into Equations (13) and (14) we obtain:

$$\lambda^2 c_1 \exp(-\lambda x)\sin\beta z\sin(\omega_g t+\phi) + \lambda\beta c_2 \exp(-\lambda x)\sin\beta z\sin(\omega_g t+\phi) + \frac{\rho}{K_b}\omega_g^2 c_1 \exp(-\lambda x)\sin\beta z\sin(\omega_g t+\phi) = 0 \quad (32)$$

$$-\beta^2 c_2 \exp(-\lambda x)\cos\beta z\sin(\omega_g t+\phi) - \lambda\beta c_1 \exp(-\lambda x)\cos\beta z\sin(\omega_g t+\phi) + \frac{\rho}{K_b}\omega_g^2 c_2 \exp(-\lambda x)\cos\beta z\sin(\omega_g t+\phi) = 0 \quad (33)$$

Simplifying Equations (30) and (31) we obtain:

$$\left(\left(\lambda^2 + \frac{\rho\omega_g^2}{K_b}\right)c_1 + \lambda\beta c_2\right) \times\exp(-\lambda x)\sin\beta z\sin(\omega_g t+\phi) = 0 \quad (34)$$

$$\left(-\lambda\beta c_1 + \left(\frac{\rho\omega_g^2}{K_b} - \beta^2\right)c_2\right) \times\exp(-\lambda x)\cos\beta z\sin(\omega_g t+\phi) = 0 \quad (35)$$

For nontrivial solutions,

$$\exp(-\lambda x)\sin\beta z\sin(\omega_g t+\phi) \neq 0$$

$$\exp(-\lambda x)\cos\beta z\sin(\omega_g t+\phi) \neq 0$$

The characteristic frequency equations become the system of two equations.

$$\left(\lambda^2 + \frac{\rho\omega_g^2}{K_b}\right)c_1 + \lambda\beta c_2 = 0 \quad (36)$$

$$-\lambda\beta c_1 + \left(\frac{\rho\omega_g^2}{K_b} - \beta^2\right)c_2 = 0 \quad (37)$$

In matrix form, we obtain the homogeneous equation:

$$\begin{pmatrix} \left(\lambda^2 + \frac{\rho\omega_g^2}{K_b}\right) & \lambda\beta \\ -\lambda\beta & \left(\frac{\rho\omega_g^2}{K_b} - \beta^2\right) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (38)$$

The characteristic equation is:

$$\begin{vmatrix} \left(\lambda^2 + \frac{\rho\omega_g^2}{K_b}\right) & \lambda\beta \\ -\lambda\beta & \left(\frac{\rho\omega_g^2}{K_b} - \beta^2\right) \end{vmatrix} = 0 \quad (39)$$

By expansion, the characteristic equation is:

$$\left(\lambda^2 + \frac{\rho\omega_g^2}{K_b}\right)\left(\frac{\rho\omega_g^2}{K_b} - \beta^2\right) + (\lambda\beta)^2 = 0 \quad (40)$$

Further expansion yields:

$$\lambda^2 \frac{\rho\omega_g^2}{K_b} - \lambda^2\beta^2 + \left(\frac{\rho\omega_g^2}{K_b}\right)^2 - \beta^2 \frac{\rho\omega_g^2}{K_b} + \lambda^2\beta^2 = 0 \quad (41)$$

Simplifying,

$$(\lambda^2 - \beta^2)\frac{\rho\omega_g^2}{K_b} + \left(\frac{\rho\omega_g^2}{K_b}\right)^2 = 0 \quad (42)$$

Division by $\frac{\rho\omega_g^2}{K_b}$ yields:

$$\lambda^2 - \beta^2 + \frac{\rho\omega_g^2}{K_b} = 0 \quad (43)$$

Alternatively, we have:

$$\lambda^2 - \beta^2 + \frac{\rho}{K_b}\left(\frac{2\pi}{T}\right)^2 = 0 \quad (44)$$

### B. Enforcement of Boundary Conditions

The boundary conditions for the seismic horizontal ground harmonic vibratory motion of the base of the dam are given by Equations (18) and (19).

Application of the boundary conditions in Equations (26) and (27) yields:

$$u(x, z = H, t) = c_1 \exp(-\lambda x)\sin\beta H\sin(\omega_g t+\phi) = u_a \sin(\omega_g t+\phi) \quad (45)$$

$$w(x, z = H, t) = c_2 \exp(-\lambda x)\cos\beta H\sin(\omega_g t+\phi) = 0 \quad (46)$$

For nontrivial solutions of Equation (46), $c_2 \exp(-\lambda x)\sin(\omega_g t+\phi) \neq 0$

Hence, we obtain the frequency equation as

$$\cos\beta H = 0 \quad (47)$$

Solving,

$$\beta H = \cos^{-1} 0 = \pm\frac{m\pi}{2}, \quad m = 1, 3, 5, 7, 9, 11, \ldots \quad (48)$$

Hence,

$$\beta = \pm\frac{m\pi}{2H} \quad (49)$$

Then, from Equation (44), we have:

$$\lambda^2 = \beta^2 - \frac{\rho}{K_b}\left(\frac{2\pi}{T}\right)^2 \quad (50)$$

Substitution for $\beta$ from Equation (49) yields:

$$\lambda^2 = \left(\pm\frac{m\pi}{2H}\right)^2 - \frac{\rho}{K_b}\left(\frac{2\pi}{T}\right)^2 \quad (51)$$

Simplifying,

$$\lambda^2 = \left(\frac{m\pi}{2H}\right)^2 - \frac{\rho}{K_b}\frac{4\pi^2}{T^2} \quad (52)$$

Simplifying further,

$$\lambda^2 = \left(\frac{m\pi}{2H}\right)^2 \left(1 - \frac{\rho}{K_b}\frac{4\pi^2}{T^2}\left(\frac{2H}{m\pi}\right)^2\right) \quad (53)$$

Further simplification yields:

$$\lambda^2 = \left(\frac{m\pi}{2H}\right)^2 \left(1 - \frac{\rho}{K_b}\frac{4\pi^2}{T^2}\frac{4H^2}{m^2\pi^2}\right) \quad (54)$$

Hence,

$$\lambda^2 = \left(\frac{m\pi}{2H}\right)^2 \left(1 - \frac{16\rho H^2}{K_b m^2 T^2}\right) \quad (55)$$

Alternatively,

$$\lambda^2 = \beta^2 F_m^2 \quad (56)$$

where $F_m^2 = 1 - \dfrac{16\rho H^2}{m^2 K_b T^2}$ (57)

So,

$$\lambda = \beta F_m = \frac{m\pi}{2H} F_m \quad (58)$$

where $F_m = \sqrt{\left(1 - \dfrac{16\rho H^2}{m^2 K_b T^2}\right)}$ (59)

if $1 > \dfrac{16\rho H^2}{m^2 K_b T^2}$

Also, $F_m = \sqrt{\left(\dfrac{16\rho H^2}{m^2 K_b T^2} - 1\right)}$ (60)

if $\dfrac{16\rho H^2}{m^2 K_b T^2} > 1$

Then, the system of algebraic eigenvalue eigenvector equations become:

$$(\beta^2 - \lambda^2)c_1 + \lambda^2 c_1 + \lambda\beta c_2 = 0 \quad (61)$$

$$(\beta^2 - \lambda^2)c_2 - \beta^2 c_2 + \lambda\beta c_1 = 0 \quad (62)$$

Simplifying Equation (61), we have:

$$\beta^2 c_1 + \lambda\beta c_2 = 0 \quad (63)$$

So,

$$c_2 = -\frac{\beta}{\lambda} c_1 = -\frac{c_1}{F_m} \quad (64)$$

Or,

$$c_1 = -F_m c_2 \quad (65)$$

The dynamic displacement field components then become:

$$u(x,z,t) = c_1 \exp\left(-\frac{m\pi}{2H} F_m x\right) \sin\frac{m\pi z}{2H} \sin(\omega_g t + \phi) \quad (66)$$

$$w(x,z,t) = c_2 \exp\left(-\frac{m\pi}{2H} F_m x\right) \cos\frac{m\pi z}{2H} \sin(\omega_g t + \phi) \quad (67)$$

The solutions expressed by Equations (66) and (67) are for the *m*th vibratory mode. The general solutions by the modal superposition technique for the dynamic displacement field components become:

$$u(x,z,t) = \sum_{m=1}^{\infty} c_{1m} \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\frac{m\pi z}{2H} \sin(\omega_g t + \phi) \quad (68)$$

$$w(x,z,t) = \sum_{m=1}^{\infty} c_{2m} \exp\left(-\frac{m\pi F_m x}{2H}\right) \cos\frac{m\pi z}{2H} \sin(\omega_g t + \phi) \quad (69)$$

Using Equation (65), we have:

$$w(x,z,H) = \sum_{m=1}^{\infty} -\frac{c_{1m}}{F_m} \exp\left(-\frac{m\pi F_m x}{2H}\right) \cos\left(\frac{m\pi z}{2H}\right) \sin(\omega_g t + \phi) \quad (70)$$

Enforcing the displacement boundary conditions at the upstream face of the concrete gravity dam, yields:

$$w(x=0,z,t) = \sum_{m=1}^{\infty} c_{1m} \exp 0 \sin\frac{m\pi z}{2H} \sin(\omega_g t + \phi) = \frac{\alpha g T^2}{4\pi^2} \sin(\omega_g t + \phi) \quad (71)$$

Thus,

$$\left(\sum_{m=1}^{\infty} c_{1m} \sin\frac{m\pi z}{2H}\right) \sin(\omega_g t + \phi) = \frac{\alpha g T^2}{4\pi^2} \sin(\omega_g t + \phi) \quad (72)$$

Hence,

$$\sum_{m=1}^{\infty} c_{1m} \sin\frac{m\pi z}{2H} = \frac{\alpha g T^2}{4\pi^2} = \frac{\alpha g}{\omega_g} \quad (73)$$

From Fourier series theory,

$$\sum_{m=1}^{\infty} c_{1m} \sin\frac{m\pi z}{2H} \sin\frac{m\pi z}{2H} = \frac{\alpha g T^2}{4\pi^2} \sin\frac{m\pi z}{2H} \quad (74)$$

Integration of both sides of Equation (74) yields:

$$\int_0^H \sum_{m=1}^{\infty} c_{1m} \sin^2\frac{m\pi z}{2H} dz = \int_0^H \frac{\alpha g T^2}{4\pi^2} \sin\frac{m\pi z}{2H} dz \quad (75)$$

Alternatively,

$$\sum_{m=1}^{\infty} c_{1m} \int_0^H \sin^2\frac{m\pi z}{2H} dz = \frac{\alpha g T^2}{4\pi^2} \int_0^H \sin\frac{m\pi z}{2H} dz \quad (76)$$

Evaluating the integral yields:

$$\sum_{m=1}^{\infty} c_{1m} \frac{H}{2} = \frac{\alpha g T^2}{4\pi^2} \left[\frac{-2H}{m\pi} \cos\frac{m\pi z}{2H}\right]_0^H \quad (77)$$

Hence,

$$c_{1m} \frac{H}{2} = -\frac{\alpha g T^2 \cdot 2H}{4\pi^2 m\pi} \left(\cos\frac{m\pi}{2} - 1\right) \quad (78)$$

Simplifying,

$$c_{1m} \frac{H}{2} = \frac{\alpha g T^2 H}{2m\pi^3} \quad (79)$$

Further simplification yields:

$$c_{1m} = \frac{2}{H} \frac{\alpha g H T^2}{2m\pi^3} =$$

$$c_{1m} = \frac{\alpha g T^2}{m\pi^3} \quad (80)$$

From Equation (64), we have:

$$c_{2m} = -\frac{c_{1m}}{F_m} = -\frac{\alpha g T^2}{m\pi^3 F_m} \quad (81)$$

Then, the dynamic displacement field components become:

$u(x,z,t)$

$$= \sum_{m=1}^{\infty} \frac{\alpha g T^2}{m\pi^3} \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\left(\frac{m\pi z}{2H}\right) \sin(\omega_g t + \phi) \quad (82)$$

$w(x, z, t)$

$$= \sum_{m=1}^{\infty} \frac{-\alpha g T^2}{m\pi^3 F_m} \exp\left(-\frac{m\pi F_m x}{2H}\right) \cos\left(\frac{m\pi z}{2H}\right) \sin(\omega_g t + \phi) \quad (83)$$

Simplifying, we have:

$$u(x, z, t) = \frac{\alpha g T^2}{\pi^3} \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \left(\frac{1}{m}\right) \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\left(\frac{m\pi z}{2H}\right) \quad (84)$$

$$w(x, z, t) = -\frac{\alpha g T^2}{\pi^3} \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \left\{\frac{1}{m F_m} \exp\left(-\frac{m\pi F_m x}{2H}\right) \cos\left(\frac{m\pi z}{2H}\right)\right\} \quad (85)$$

The hydrodynamic stress field then becomes from Equation (10)

$$\sigma(x, z, t) = K_b \frac{\alpha g T^2}{\pi^3} \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \left(\frac{1}{m}\left(-\frac{m\pi F_m}{2H}\right)\right) \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\left(\frac{m\pi z}{2H}\right) + K_b \left(\frac{-\alpha g T^2}{\pi^3}\right) \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \frac{1}{m F_m}\left(-\frac{m\pi}{2H}\right) \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\left(\frac{m\pi z}{2H}\right) \quad (86)$$

Simplifying,

$$\sigma(x, z, t) = \frac{\alpha g T^2 K_b}{\pi^3} \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \left(\frac{\pi}{2H F_m} - \frac{\pi F_m}{2H}\right) \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\left(\frac{m\pi z}{2H}\right) \quad (87)$$

Simplifying further,

$$\sigma(x, z, t) = \frac{\alpha g T^2 K_b}{\pi^3} \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \frac{\pi}{2H}\left(\frac{1}{F_m} - F_m\right) \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\left(\frac{m\pi z}{2H}\right) \quad (88)$$

Further simplification yields:

$$\sigma(x, z, t) = \frac{\alpha g T^2 K_b}{2\pi^2 H} \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \left(\frac{1 - F_m^2}{F_m}\right) \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\left(\frac{m\pi z}{2H}\right) \quad (89)$$

From Equation (57),

$$\frac{16\rho H^2}{m^2 K_b T^2} = 1 - F_m^2 \quad (90)$$

So,

$$\frac{1 - F_m^2}{F_m} = \frac{16\rho H^2}{m^2 K_b T^2 F_m} \quad (91)$$

Hence,

$$\sigma(x, z, t) = \frac{\alpha g T^2 K_b}{2\pi^2 H} \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \frac{16\rho H^2}{m^2 K_b T^2 F_m} \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\frac{m\pi z}{2H} \quad (92)$$

Simplifying,

$$\sigma(x, z, t) = \frac{\alpha g T^2 K_b}{2\pi^2 H} \frac{16\rho H^2}{m^2 K_b T^2} \times \sum_{m=1}^{\infty} \frac{1}{m^2 F_m} \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\frac{m\pi z}{2H} \quad (93)$$

Hence,

$$\sigma(x, z, t) = \frac{8\alpha g \rho H}{\pi^2} \sin(\omega_g t + \phi) \sum_{m=1}^{\infty} \frac{1}{m^2 F_m} \exp\left(-\frac{m\pi F_m x}{2H}\right) \sin\left(\frac{m\pi z}{2H}\right) \quad (94)$$

The dynamic stress distribution $p(x = 0, z, t)$ on the upstream surface of the concrete dam is given by:

$$p(x = 0, z, t) = \sigma(x = 0, z, t) = \frac{8\alpha g \rho H}{\pi^2} \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \frac{1}{m^2 F_m} \exp\left(-\frac{m\pi F_m 0}{2H}\right) \sin\left(\frac{m\pi z}{2H}\right) \quad (95)$$

So,

$$p(x = 0, z, t) = \frac{8\alpha g \rho H}{\pi^2} \sin(\omega_g t + \phi) \times \sum_{m=1}^{\infty} \frac{1}{m^2 F_m} \sin\left(\frac{m\pi z}{2H}\right) \quad (96)$$

Hence,

$$p(0, z, t) = M_d \alpha g \sin(\omega_g t + \phi) \quad (97)$$

$$\text{where } M_d = \frac{8\rho H}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2 F_m} \sin\left(\frac{m\pi z}{2H}\right) \quad (98)$$

The maximum hydrodynamic pressure distribution on the upstream surface of the concrete dam occurs when $z = H$ and is given by:

$$p(0, z = H, t) = \frac{8\alpha g \rho H}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2 F_m} \sin\frac{m\pi}{2} \quad (99)$$

$m = 1, 3, 5, 7, 9, 11, \ldots$

Then,

$$p_{\max} = \frac{8\alpha g \rho H}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{\frac{m-1}{2}}}{m^2 F_m} \quad (100)$$

For incompressible fluids, the bulk volumetric modulus $K_b$ tend to infinity. Hence, the assumption of incompressibility of water in the reservoir leads to the consideration of the results as $K_b \to \infty$. As $K_b \to \infty$, $F_m \to 1$.

The maximum hydrodynamic pressure can thus be obtained as:

$$p_{\max} = \frac{8\alpha g\rho H}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^2} = \frac{8\alpha g\rho H}{\pi^2}$$

$$\times\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\frac{1}{13^2}-\ldots\right) \quad (101)$$

Evaluation gives:

$$p_{\max} = \frac{8\alpha g\rho H}{\pi^2}(0.9184791016)$$

$$= 0.744491\alpha g\rho H \quad (102)$$

The exact solution for $p_{\max}$ at convergence is:

$$p_{\max} = 0.743\alpha g\rho H \quad (103)$$

### C. Resultant Hydrodynamic Force on the Upstream Surface of the Dam

The resultant hydrodynamic force $F_d$ on the upstream surface of the dam is found by integration of the hydrodynamic stresses (pressures) over the vertical distance of the upstream face, as follows:

$$F_d = \int_0^H \sigma(x=0,z,t)dz \quad (104)$$

Hence,

$$F_d = \frac{8\alpha g\rho H}{\pi^2}\sin(\omega_g t+\phi)$$

$$\times\sum_{m=1}^{\infty}\frac{1}{m^2F_m}\exp(0)\int_0^H \sin\frac{m\pi z}{2H}dz \quad (105)$$

Integrating,

$$F_d = \frac{8\alpha g\rho H}{\pi^2}\sin(\omega_g t+\phi)$$

$$\times\sum_{m=1}^{\infty}\frac{1}{m^2F_m}\frac{2H}{m\pi}\left[-\cos\frac{m\pi z}{2H}\right]_0^H \quad (106)$$

Substitution of the integration limits gives:

$$F_d = \frac{16\alpha g\rho H^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{m^3F_m}\left(-\cos\frac{m\pi}{2}+\cos 0\right) \quad (107)$$

Hence,

$$F_d = \frac{16\alpha g\rho H^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{m^3F_m} = \frac{16\alpha g\rho H^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{m^3} \quad (108)$$

Evaluation for $m = 1, 3, 5, 7, 9, 11, \ldots$ gives:

$$F_d = \frac{16\alpha g\rho H^2}{\pi^3}\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}+\ldots\right) \quad (109)$$

Hence,

$$F_d = 0.541864762\alpha g\rho H^2 \quad (110)$$

for $m = 1, 3, 5, 7, 9, 11, \ldots$

Hence, approximately,

$$F_d \approx 0.5419\alpha g\rho H^2 \quad (111)$$

## IV. DISCUSSION

In this research work, analytical expressions have been derived for the stress and displacement fields in concrete gravity dams due to seismic ground motion idealized mathematically as a horizontal sinusoidal vibration. The partial differential equations of equilibrium that govern the natural vibrations of concrete gravity dams and embankment dams that account for fluid-structure interaction were derived as Equations (13) and (14) in a displacement formulation. The seismic ground motion, considered as a horizontal sinusoidal (harmonic) vibration with a known amplitude and period was given as Equation (15). The displacement boundary conditions used were given as Equations (18) and (19). Other boundary conditions considered in the analysis were given as Equations (24) and (25).

The system of equations formulated in displacement terms – Equations (13) and (14) – were solved using the method of trial functions. Trial displacement functions of the space coordinate variables and time that are constructed to automatically satisfy the displacement boundary conditions are given as Equations (26) and (27) and are presented in terms of four unknown parameters, namely, $c_1$, $c_2$, λ and β. Substitution of the trial displacement fields into the governing system of partial differential equations resulted in the system of algebraic equations – Equations (34) and (35). For nontrivial solutions, the system reduced further to a system of two algebraic homogeneous equations given in matrix form as Equation (38). The condition for the nontrivial solution of the algebraic homogeneous Equation (38) is the determinantal Equation (39). Expansion of Equation (39) yields after simplification Equation (44). The boundary conditions for the seismic horizontal ground harmonic vibratory motion of the base of the dam, given by Equations (18) and (19) are applied to Equations (26) and (27) to obtain Equations (45) and (46), from which the conditions for nontrivial solution of Equation (46) gave the frequency equation as Equation (47). Solution of the frequency equation gave Equation (49). Substitution of the solution for the frequency equation in Equation (44) gave Equation (55) which is expressed in general as Equation (56) or (58). The system of eigenvector eigenvalue equations becomes Equations (61) and (62), from which we have Equation (64) or (65). The dynamic displacement field components for the $m$th vibratory mode then become Equations (66) and (67). The general solutions by the modal superposition method for the dynamic displacement field components are obtained as Equations (68) and (69). Enforcement of displacement boundary conditions at the upstream face and the use of Fourier series theory gave the dynamic displacement field components as Equations (82) and (83). The hydrodynamic stress field is then found from Equation (10) as Equation (94) after same simplifications. The dynamic stress field at the upstream face of the dam is found as Equation (95). The maximum value of the hydrodynamic pressure distribution on the upstream face of the dam is found to occur at $z = H$ and is found as Equation (100). For incompressible fluids, the maximum hydrodynamic pressure is obtained as Equation (101) which is evaluated numerically as Equation (102) for convergence of the series at $m = 13$. The exact solution for maximum hydrodynamic pressure for very large values of $m$ is found as Equation (103). The maximum hydrodynamic force is found using Equation (104) as Equation (108).

For $m = 1, 3, \ldots, 11$, the maximum hydrodynamic force is found as Equation (110) and for very large number of terms the numerical solution converges as Equation (111).

## V. CONCLUSION

The conclusions are as follows:

(i) The governing equation for the dynamic vibration analysis of concrete gravity dams due to seismic horizontal harmonic motion is a system of two coupled partial differential equations in a displacement formulation where the dynamic displacement field components are the unknowns.

(ii) The solution of the system of two coupled PDEs reduce the boundary value problem to a system of algebraic homogeneous equations – algebraic eigenvalue problem.

(iii) The analytical expressions for the dynamic displacement field components $u(x, z, t)$ and $w(x, z, t)$ are single series of infinite terms with rapid convergence.

(iv) The analytical expression for the dynamic stress field $\sigma(x, z, t)$ is a single series of infinite terms that converges reasonably to the exact solution with seven terms of the series.

(v) The maximum hydrodynamic pressure distribution on the upstream face of the dam occurs at the base of the dam where $z = H$ and is given as a single series with infinite terms.

(vi) The resultant hydrodynamic force is similarly given by a single rapidly convergent series with infinite terms.

## REFERENCES


1. H. Ahmad, "Safety Evaluation of Bakun Concrete Faced Rockfill Dam," MSc Thesis in Structural Engineering and Construction, Department of Civil Engineering, University Putra, Malaysia, Serdang Selangor Malaysia, 2006, pp. 245.
2. M. Mahdizadeh and A. Ghanbari (2013) "Calculation of natural frequency of earth dams by means of analytical solution," *Seventh International Conference on Case Histories in Geotechnical Engineering April 29 – May 4, 2013*. Chicago Paper No 3.13a 6pp. Mossuri University of Science and Technology. Available: http://scholarsmine.mst.edu/icchge/7icchge/session03/6.
3. N. Nandi, "Free vibration characteristics of earthen dam," *International Journal of Mechanics and Solids*, Vol 3, No 1, 2008, pp. 43-59.
4. Y. Parish, M. Sadek and I. Shahvour, "Seismic response of end-bearing piles," *Soil Dynamics and Earthquake Engineering*, No 3, 1984, pp 82-93.
5. H. Watanabe, "Vibration modes of rockfill dam based on the observation of micro tremors and an earthquake," VI, No 1-4, 1996.
6. Z. Zhen, "Shear vibration of 3D inhomogeneous earth dam in triangular canyons," *Eleventh world conference of earthquake engineering*, paper No 584, 1996.
7. A. Lokke, "Earthquake analysis of concrete gravity dams," Master Thesis, Norwegian Institute of Science and Technology, 2013.
8. T. Berrabah, "Modal behavior of dam reservoir foundation system," *Electronic Journal of Geotechnical Engineering*, 2011, pp 1593-1601.
9. J.A. Varughese and S. Nikithan, "Seismic behavior of concrete gravity dams," *Advances in Computational Design*, Vol 1 No 2, 2016, pp 195-206. DoI: http:dx.doi/10.12989/acd.2016.1.2.195.
10. S. Brijesh and P. Agarwal, "Seismic response of high concrete dam including dam-reservoir foundation effect," *Journal of South Asia Disaster Studies*, Vol 2, 2009, pp 41-57.
11. A.K. Chopra and S. Gupta, "Hydrodynamic and foundation interaction effects in frequency response functions for concrete gravity dams," *Earthquake Engineering and Structural Dynamics*, Vol 10, No 1, 1982, pp 89-106.
12. A. Burman, B.V. Reddy and D. Maity, "Seismic analysis of concrete gravity dams considering foundation flexibility and nonlinearity," *The 12th International Conference of International Association for Computer Methods and Advances in Geomechanics (IACMAG)*, 1-6 October 2008, Goa India, pp 2882-2888.
13. A.R. Yucel, "Seismic analysis of concrete gravity dams including dam-foundation-reservoir interaction," Master of Science in Civil Engineering Thesis, Graduate School of Natural and Applied Sciences, Middle East Technical University, September 2013, pp 98.
14. H.M Westergaard, "Water pressures on dams during earthquake," *ASCE Transactions*, 98, 1933, pp 418-433.
15. P.L.F Liu, "Hydrodynamic pressures on rigid dams during earthquakes," *Journal of Fluid Mechanics*, Vol 165, 1986, pp 131-145.
16. O.C. Zienkiewicz and B. Nath, "Earthquake hydrodynamic pressures on arch dams; an elastic analog solution," *Proceedings Institution of Civil Engineers*, Vol 25, 1963, pp 165-176.
17. A.K. Chopra, "Earthquake behaviour of dam-reservoir systems," *Journal of Engineering Mechanics ASCE*, Vol 94, 1968, pp 1475-1499.
18. S. Saini, P. Bettess and O.C. Zienkiewicz, "Coupled hydrodynamic response of concrete dams using finite and infinite elements," *Earthquake Engineering and Structural Dynamics*, Vol 6, 1978, pp 363-374.
19. P.A. Patil and G.B. Katti, "Finite element analysis of gravity dam with drainage gallery in ANSYS workbench 14.5," *International Journal of Modern Trends in Engineering and Research*, eISSN 2349-9745 Vol 3, Issue 4, April 2016 Special Issue of ICRTET 2016, pp 544-549.
20. J.F. Hall and A.K. Chopra, "Hydrodynamic effects in the response of concrete gravity dams," *Earthquake Engineering and Structural Dynamics*, Vol 10, 1982, pp 333-345.
21. G. Fenves and A.K. Chopra, "Earthquake analysis of concrete gravity dams including bottom absorption dam water-foundation rock interaction," *Earthquake Engineering and Structural Dynamics*, Vol 12, Issue 5, 1984, pp 663-683. https://doi.org/10.1002/eqe.4290120507.
22. V. Lotfi, "Seismic analysis of concrete gravity dams by decoupled modal approach in time domain," *Electronic Journal of Structural Engineering*, Vol 3, 2003, pp 102-116.
23. J.L. Humar and A.M. Jablonski, "Boundary element reservoir model for seismic analysis of gravity dams," *Earthquake Engineering and Structural Dynamics*, Vol 16, 1988, pp 1129-1156.
24. F. Medina and J. Dominguez, "Boundary elements for the analysis of the seismic response of dams including dam-water foundation interaction effects," *Engineering Analysis*, Vol 6, 1989, pp 152-157.
25. Y.S. Cho and P.L.F. Liu, "Hydrodynamic pressures acting on rigid gravity dams during earthquakes," *Journal of Hydraulic Research*, Vol 40, No 2, 2002, pp 175-181.
26. H. Antes and O. Von Estorff, "Analysis of absorption effects on the dynamic response of dam reservoir systems by boundary element methods," *Earthquake Engineering and Structural Dynamics*, Vol 15, 1987, pp 1023-1036.
27. C.C. Ike and H. Onah, "Analytical solution for hydrodynamic pressure on concrete gravity dams," *Journal of Research in Engineering*, Vol 6, No 2, 2009, pp 54-60.


## AUTHORS PROFILE

**Engr Dr. Charles C. Ike** is a lecturer at the Enugu State University of Science and Technology, Enugu, Nigeria. He holds a PhD in structural engineering, and is a member of the Nigerian Society of Engineers. He is registered with the Council for the Regulation of Engineering in Nigeria (COREN). He has numerous publications in reputable international journals. He also holds a Master of Engineering degree in Structural Engineering and another Master of Engineering degree in Soil Mechanics and Foundation Engineering and a Bachelor of Engineering degree in Civil Engineering. He is a corporate member of the Nigerian Society of Engineers. His research interests include: theory of elasticity, theory of plates and shells, eigenvalue problems, higher order shear deformation theories of beams and plates, integral equations, integral transformation techniques for solving boundary value problems of civil engineering, soil-structure interaction and analysis.

**Hyginus Nwankwo Onah (B.Eng, M.Eng, PhD)** Born in Nigeria on May 4th 1959, he obtained B.Eng. (Civil Engineering) in 1983. After brief work in Lagos Nigeria, he travelled to France in 1987 where he obtained M.Eng (Civil Engineering) at INSA de Lyon and Ph.D (Structural Mechanics) at CNAM Paris in 1991. Then he worked as a researcher at Research and Development (R&D) unit of RENAULT Motors Reuil Malmaison France, Department of Engineering University of Manchester England and Becchis Osiride of FIAT Motors Torino Italy for one year each. From January 1996 to date he is a lecturer in Civil Engineering Department of University of Nigeria, Nsukka, where he teaches and supervises both undergraduate and postgraduate students and carries out research activities in mechanics and behavior of structures.