Strong Geodetic Domination of Graphs

D. Antony Xavier, Deepa Mathew, Santiagu Theresal

***Abstract*: The problem to find a $S \subseteq V(G)$ where all points of the graph $G(V(G), E(G))$ are covered by a unique fixed geodesic between the pair of points in S is called the strong geodetic problem. Here the domination concept is combined with strong geodetic concept resulting in Strong geodetic domination of graphs and few results are derived. Also the computational complexity part of this concept with respect to general, chordal, bipartite, chordal- bipartite graphs are explained.**

***Keywords* : strong geodetic, geodetic domination problem, NP-completeness.**

I. INTRODUCTION

Consider the graph G(V(G), E(G)), with order $o(G) = n$. Let $\{a, b\} \subseteq V(G)$, then a $a - b$ geodesic is the shortest path between the vertices a and b in G [2]. Harary et al introduced a graph theoretical parameter in [3] called the geodetic number of a graph and it was further studied in [4]. In [3] the geodetic number of a graph is explained as follows, let $I(a, b)$ be the set of all vertices lying on some $a - b$ geodesic , and for some non empty subset S of $V(G), I(S) = \cup_{a,b \in S} I(a, b)$.

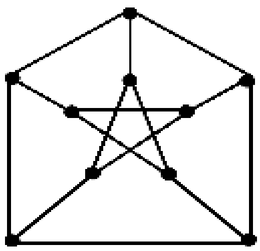

Figure 1: The strong geodetic domination, $\gamma_{sg}(GP(5,2)) = 4$

The set S of vertices of G is called a geodetic set of G, if $I(S) = V$ and a geodetic set of minimum cardinality is called minimum geodetic number, denoted as $g(G)$[7] and it is shown to be an NP-complete problem in [11]. Escuadro et. al studied the geodetic concepts in relation to domination for the first time in [5] .

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A set T of points in G(V(G), E(G)) is a dominating set if each point of $V(G)$ except the points in T is adjacent to at least one point of . A set H of points of G is a geodetic dominating set if H is both a geodetic set as well as the dominating set.[5]. All through the paper, G is assumed to be a connected graph. The strong geodetic, geodetic domination and strong geodetic domination is abbreviated as SG, GD and SGD respectively.

II. STRONG GEODETIC DOMINATION

The SG problem defined in [10] is a variety of geodetic problem that finds its application in social networks. The cardinality of the minimum SG set is the SG number, denoted as $sg(G)$. It is stated in [5] that dominating set in general is not a geodetic set and the vice-versa is not valid. Thus, arises a new class of sets called geodetic dominating sets. Similarly, the SG set is not in general a dominating set and vice-versa. A set $S \subseteq V(G)$, is called a SGD set if S is both a SG set as well as a domination set. The SGD problem is to find the set S with minimum cardinality, denoted by $\gamma_{sg}(G)$. For $K_n - e$, where $n \geq 4$ the GD number is 2, while the SGD number is equal to $(n - 1)$. Thus, the SGD number is not in the near bound of the GD number. This motivates us to study the strong geodetic domination problem separately. For a Peterson graph $\gamma_{sg}(G)\ (GP\ (5, 2)) = 4$. Refer Figure 1:

III. COMPUTATIONAL COMPLEXITY

We prove that the *SGD* problem is NP-complete for general, n-partite, chordal and chordal bipartite graphs. Finding the dominating set is NP- complete for general, bipartite and chordal graphs. The proof of NP- completeness of *SGD* problem is by polynomial reduction from dominating set problem. For the construction of $G'(V', E')$ we follow Theorem 2.1 of [7].

Theorem 1. The *SGD* set problem is NP - complete.

Proof. Given a $G(V, E)$, we construct $G'(V', E')$.For $x \in V(G)$, add two new vertices x' and x'' in the vertex set of G'. Let xx', $x'x'' \in E(G')$ and $G' = (V', E')$ where $V' = V \cup V_1 \cup V_2$ with $V_1 = \{x_1, x \in V\}$ and $V_2 = \{x_2, x \in V\}$. Then $E' = E \cup E_1 \cup E_2$ where $E_1 = \{xx_1, x \in V\}$ and $E_2 = \{x_1x_2, x \in V\}$. We prove that if $S \subseteq V$ is a domination set to G iff $S \cup V_2$ is a strong geodetic domination set of G'.Suppose S is a dominating set of G. Now consider $S \cup V_2$ in G'. Clearly $\binom{|V_2|}{2}$ geodesics between the pair of vertices of V_2 form a strong geodetic set of G'.

Also, the points of V_2 dominates the points of V_1 and S dominates the points of G. Therefore, $S \cup V_2$ is a *SGD* set of G'.

Conversely let D be a *SGD* set of G'. Since the vertices in V_2 are simplicial, $V_2 \subseteq D$. Suppose $x_1 \in D$, then replace x_1 by the corresponding x. Define $D' = \{x/x' \in D\} \cup \{D \cap V(G)\} \cup V_2$. Clearly $|D'| = |D|$. Also D' is a *SGD* set and $D' \backslash V_2$ is a domination set in G.

Theorem 2. The *SGD* problem for chordal graph is NP complete

Theorem 3. The *SGD* problem for n-partite graphs is NP complete

Theorem 4. The *SGD* problem for chordal bipartite graphs is NP complete

The proofs of Theorems 2, 3, 4 are similar to that of Theorem 1. So we omit the proof.

IV. MAIN RESULTS

Result 1. For a G of order $n \ge 2$,
$2 \le max(\gamma(G), sg(G)) \le \gamma_{sg}(G) \le n$

Result 2. For a G of order $n \ge 2$, $\gamma_g(G) \le \gamma_{sg}(G) \le n$

Theorem 5. For a G of order $n > 2$, $\gamma_{sg}(G) = 2$ iff $G \cong P_3$ *or* P_4

Proof. If $G \cong P_3$ *or* P_4, then $\gamma_{sg}(G) = 2$. Conversely, assume $\gamma_{sg}(G) = 2$, which implies $sg(G) = 2$. But when $sg(G) = 2$, $G \cong P_n$ where the set of end points forms the minimum *SG* set. In the case of domination problem for P_n, the set of end points forms a domination set only when $n = 3$ (or) 4 . Thus $G \cong P_3$ *or* P_4

Theorem 6. For a G with domination number equal to 1, then $\gamma_{sg}(G) = sg(G)$

Proof. For K_n, the domination number will be 1 and $\gamma_{sg}(G) = sg(G) = n$. Hence the condition is true in the case of complete graphs. Now the condition has to be checked where $G \neq K_n$,. The condition that $G \neq K_n$ implies that $\exists$ at least two non adjacent points, so that $diam(G) \ge 2$. But the domination number equal to 1, implies that the $diam(G) \le 2$ and the maximum degree of any vertex is equal to $(n-1)$. Thus it can be concluded that diameter is equal to 2.

Let T be a minimum *SG* set of G. For $x \in V \backslash T$, then $\exists$ a unique fixed $a - b$ path which contains x where $a, b \in T$. Then the length of the $a - b$ path is 2 with x dominated by both a and b. Thus $\gamma_{sg}(G) \le sg(G)$. By Result 1: $\gamma_{sg}(G) \ge sg(G)$. Thus $\gamma_{sg}(G) = sg(G)$.

Note 1. The converse need not be true. For a cycle on 5 vertices C_5, $\gamma_{sg}(C_5) = sg(C_5)$ but $\gamma(C_5) \neq 1$

Note 2. The above theorem is not true for $\gamma(G) = 2$. For P_6, $\gamma(P_6) = sg(P_6) = 2$, but $\gamma_{sg}(P_6) = 3$.

Theorem 7. For a G of order $n \ge 2$, $\gamma_{sg}(G) \le n - \lfloor 2diam(G)/3 \rfloor$

Proof. For integers $0 \le a \le 2$, b, and G with diameter $p = 3a + b$. Consider a diametrical path $P = x_0 x_1 \ldots x_p$. Also let $Q = \{x_0, x_3, x_6 \ldots x_{3a}, x_{3a+b}.\}$ where $|Q| = a + 1$ when $b = 0$ and $|Q| = a + 2$ otherwise. Let $D = V(G) \backslash [V(P) \backslash V(Q)]$ where D forms a SGD set. Then $|D| = |V(G) \backslash [V(P) \backslash V(Q)]| = n - \lfloor (6a + 2b)/3 \rfloor$. Thus $\gamma_{sg}(G) \le n - \lfloor 2diam(G)/3 \rfloor$

Theorem 8. For a split graph with complete set *K*, stable set T . The $sg(G) \ge \frac{n-s_2}{s_1}$, where $s_1 = |T|$ and s_2 denotes number of simplicial points in complete set *K*.

Proof. Let S be a minimum *SG* set of *G* and let *A* be set of simplicial vertices in *K* with $|A| = s_2$. Also, let *B* be set of non simplicial vertices in S, $|B| = s_3$. Clearly $T \subseteq S$, $A \subseteq S$ and $|S| = s_1 + s_2 + s_3$

The strong geodetic set contains three components including the stable set, the set of simplicial vertices and the set of non-simplicial vertices. It is clear that s_1 and s_2 are fixed, hence to get the minimum *SG* set $sg(G)$, s_3 should be minimized. The geodesics between any pair of vertices in the stable set *T* cover a maximum of two vertices. Also, the geodesics between any pair of vertices with one vertex in *T* and other vertex in *A* cover one vertex. Thus the optimization problem reduces to *minimize*(s_3).

subject to: $0 \le (s_1 + s_2 + s_3) \le n$;

$$n - |S| \le \binom{s_1}{2} 2 + s_1 s_2 + s_3(s_1 - 1);$$

From this we obtain $|S| \ge \frac{n-s_2}{s_1}$. For the split graph shown in Figure 2 this bound is sharp.

Remark 1. Suppose $A = \varphi$. Then $sg(G) \ge \frac{n}{s_1}$ and this bound is sharp.(Refer Fig 3)

Remark 2. Suppose $A = B = \varphi$. Then the inequality reduces to $sg(G) \ge \frac{n}{s_1}$. Each pair of vertices in T can cover utmost 2 vertices in its geodesic. This bound attains equality for graph in Figure 4.

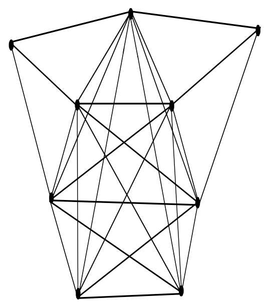

Fig 2: Split graph with |S|=4

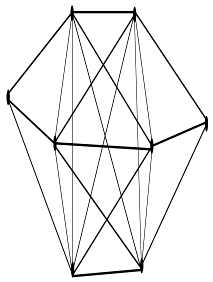

Fig 3: Split graph with |S|=4

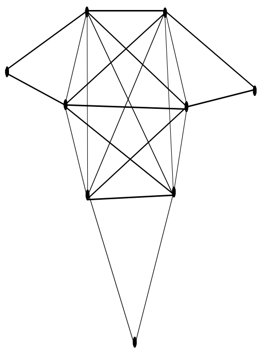

Fig 4: Split graph with |S|=3

Strong geodetic domination for Split graphs It can be observed that $\gamma_{sg}(G) = sg(G)$, where G is a split graph.

Theorem 9. For G, a split graph where complete set is K, stable set is T. Then $\gamma_{sg}(G) \geq \frac{n-s_2}{s_1}$, where $s_1 = |T|$ and s_2 denotes number of simplicial points in complete set K.

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Theorem 10. For a G, the $\gamma_{sg}(G) = n$ iff $G \cong K_n$

Proof. Let $\gamma_{sg}(G) = n$ and assume that it is not a complete graph. Then there exists two non-adjacent points x, y. Let $v \in V(G)$ belongs in $x - y$ geodesic This implies $V(G)$ except the point v forms a SG set which implies $\gamma_{sg}(G) \leq (n-1)$ a contradiction. Conversely in a complete graph, all the vertices are simplicial vertices, giving $sg(G) = n$. Therefore $\gamma_{sg}(G) = n$.

Theorem 11. For graph G of order n, $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n$ iff $G = K_n$ or $\bar{G} = K_n$

Proof. From Theorem 10; $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n$ when $G = K_n$ or $\bar{G} = K_n$. Conversely, suppose $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n$ giving $\gamma_{sg}(G)$ *and* $\gamma_{sg}(\bar{G}) = n$. But from Theorem 10, this happens only when $G = K_n$ or $\bar{G} = K_n$

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Theorem 12. Let $|V(G)| = n$,then for every edge $xy \in E(G)$, $2 \leq \gamma_{sg}(G - xy) \leq \gamma_{sg}(G)) + 2$

Proof. Let $S \subseteq V(G)$ be a *SGD* set for G. We have two cases.

Case 1: Let $xy = e$, lies in some fixed geodesic P between the pair of points u, v of S. With removal of the edge $e = xy$, P can be splitted into two unique fixed geodesics P' and P'', where $P' = u - x$ geodesic and $P'' = y - v$ geodesic. Now the vertices covered by P will be covered by P' and P''. Thus $S \cup \{x, y\}$ a strong geodetic dominating set for $(G - e)$. This implies $\gamma_{sg}(G - e) \leq \gamma_{sg}(G)) + 2$

Case 2: The edge $e = xy$ does not lie in any fixed geodesic between any pair of points in S. Then clearly S is a *SGD* set in $(G - e)$. This implies $\gamma_{sg}(G - e) \leq \gamma_{sg}(G))$. Hence $\gamma_{sg}(G - e) \leq \gamma_{sg}(G)) + 2$

The *SGD* number for theta-graphs is computed and is given below.

Result 3. For a $\theta(l, n)$ with l levels and n vertices on each level, the strong geodetic domination number is as follows:

$\gamma_{sg}\ \theta(l, 1)) = l$

$\gamma_{sg}\ (\theta(l, 2)) = l$

$\gamma_{sg}\ (\theta(l, n)) = l\lfloor n/3 \rfloor + 1$ when $n \cong 0 mod 3$

$\gamma_{sg}\ (\theta(l, n)) = l\lfloor n/3 \rfloor + 2$ when $n \cong 1,2 mod 3$

Theorem 13. For G with $\delta(G) \geq 2$,and girth at least 7, then $\gamma_{sg}(G) = \gamma(G)$

Proof. Consider U , a minimum dominating set in G. Let $A = V(G) \backslash \tilde{I}(U)$ with $p \in A$. The point p will have an adjacent vertex q in U . Since $\delta(G) \geq 2$, there would exist a vertex $r \in N(p) \backslash \{q\}$. As the girth of G is assumed to be at least 7, there exists no cycle on 3 vertices, hence $qr \notin E(G)$. Suppose $r \in U$,then p lies on the path rpq and since girth of G is atleast 7, rpq is the unique path which starts and ends at r and p respectively. Therefore, $p \in \tilde{I}\{q, r\} \subseteq \tilde{I}(U)$, a contradiction. Thus $r \in V \backslash U$ with a vertex $t \in U \backslash N(r)$. The $trpq$ is a path of length 3, which starts at t, ends at q, with r and p as the internal vertices. Since the girth of G is at least 7, $trpq$ is the unique path of length 3 which starts and ends at points t and q respectively. This implies that $p \in \tilde{I}\{q, t\} \subseteq \tilde{I}(U)$,, a contrary. Thus A is empty and U is a SGD set. Thus $\gamma_{sg}(G) \leq |U| = \gamma(G)$. Also $\gamma(G) \leq \gamma_{sg}(G)$. Therefore $\gamma_{sg}(G) = \gamma(G)$.

From [1] $\gamma(G) \leq 1 + \frac{In(\delta+1)}{\delta+1} n$ and the bound for the strong geodetic problem given in [9]. A lower and a upper bound of graphs with girth 7 can be derived for the strong geodetic domination problem.

Corollary 1. For G with $\delta(G) \geq 2$, diameter d and girth at least 7. Then $\left\lceil \frac{d-3+\sqrt{(d-3)^2+8n(d-1)}}{2(d-1)} \right\rceil \leq \gamma_{sg}(G) \leq 1 + \frac{In(\delta+1)}{\delta+1} n$

Lemma 1. For a G ,where $sg(G) = n - i$. Then $diam(G) \leq i + 1$ where $i < n$

Proof. Let the $diam(G) = i + 2$. This implies $\exists$ a diametrical path $P = v_0 v_1, \ldots, v_{i+2}$. Thus $V(G) \backslash \{v_1, \ldots, v_{i+1}\}$ forms a SG set for G, giving the SG number, $sg(G) \leq n - \{i + 1\} < n - i$, giving a contradiction.

Lemma 2. For a G with $sg(G) = n - 1$, the diameter will be 2.

Proof. Let sg(G) = n - 1. Then by Lemma 1, diam(G) = 1or 2. But if $diam = 1$, then $G \cong K_n$ and $sg(K_n) = n$. Therefore $diam(G) = 2$

Theorem 14. For a G , if $sg(G) = n - i$, where $i =$ 1 or 2 then $sg(G) = \gamma_{sg}(G)$

Proof. Let $sg(G) = n - 1$. Then by Lemma 2, $diam(G)$ is equal to 2. Let S be a minimum SG set for G. Then there exists exactly one point in x in $V \setminus S$. Also there exists $u, v \in S$ such that x lies in a unique fixed geodesic of length 2, between u and v. Here u and v dominates x. Hence $sg(G) = \gamma_{sg}(G)$.Let $sg(G) = n - 2$. Let S be a minimum SG set for G. Then there exists two points x, y in $V \setminus S$. By Lemma 1, $diam(G) = 2$ (or) 3.

Case 1: Let $diam(G) = 2$

This implies that $\exists\, u, v, u', v' \in S$ such that x and y lies in a unique fixed geodesic between $u - v$ and $u' - v'$respectively. Here u and v dominates x while u' and v' dominates y.

Hence $sg(G) = \gamma_{sg}(G)$.

Case 2: Let $diam(G) = 3$

Let $u, v \in S$ where x, y lies on a unique fixed geodesic between u and v, such that $x \in N(u)$ and $y \in N(v)$.Thus $sg(G) = \gamma_{sg}(G)$.

Considering the SGD problem instead of the GD problem, in Theorem 4.6(a)[7] the statements holds good for the strong geodetic domination problem also. Considering the strong geodetic problem in Theorem 4.6(b)[7]. But for G or $\bar{G} = K_n - e$ the $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n - 1$ Therefore Theorem 4.6 (b)[7] is not true in terms of SGD problem. In the case of (c) if $G \cong K_{n-2} \cup K_1 \cup K_1$ or $K_{1,n-2} \cup K_{1,}$ then $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n - 2$ but not for $G \cong K_2 \cup K_2$ and $\theta(l, 1)$. Considering (d) the statement holds good in SGD problem, for G or $\bar{G} \cong K_{1,n-3} \cup K_1 \cup K_1$ or $(K_1 + (K_1 \cup K_{n-3})) \cup K_1$. For $n \geq 6$ and G or $\bar{G} \cong K_{n-3} \cup K_{1,} \cup K_{1,} \cup K_{1,}$ (d) is true for strong geodetic domination problem also. The G or $\bar{G} \cong K_{2,} \cup K_3, K_{1,} \cup K_2 \cup K_2, K_{2,2} \cup K_1, K_3 \cup K_3$ doesn't hold good in (d) for the strong geodetic domination problem. Also there exists some other graphs which is not listed in the above theorem which satisfies $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n - 3$. For example, G or $\bar{G} \cong (K_n - \{e_1, e_2\}), \gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n - 3$. Hence this type of approach will not help us in proving the following theorem.

Theorem 15. For connected graphs G and $\bar{G}$ of order $n \geq 4$, the $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) \leq 2n - 4$

Proof. Suppose $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n$ This implies that $\gamma_{sg}(G) = \gamma_{sg}(\bar{G}) = n$. Thus G or $\bar{G} \cong K_n$, giving G or $\bar{G}$ a set of independent vertices contradicting the assumption in the theorem.

Suppose $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n - 1$. This implies that $\gamma_{sg}(G)$ or $\gamma_{sg}(\bar{G}) = n$. i.e either G or $\bar{G} \cong K_n$. Assume $\gamma_{sg}(G) = n$, i.e $G \cong K_n$. Thus $\bar{G}$ is not connected.

Suppose $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n - 2$. This implies that either $\gamma_{sg}(G) = n$ and $\gamma_{sg}(\bar{G}) = n - 2$ or $\gamma_{sg}(G) = n - 1$ and $\gamma_{sg}(\bar{G}) = n - 1$. Assume $\gamma_{sg}(G) = n$, i.e $G \cong K_n$. Thus $\bar{G}$ is not connected. Considering when $\gamma_{sg}(G) = n - 1$ and $\gamma_{sg}(\bar{G}) = n - 1$.By Lemma 2, $diam(G)$ and $diam(\bar{G})$ is equal to 2. Let S be a minimum SGD set for G. Since $\gamma_{sg}(G) = n - 1$, there exists exactly one point say, $x \in V \setminus S$ and this lies on a unique fixed $u - v$ path where $u, v \in S$. Also, length of $u - v$ path is 2, as the $diam(G) = 2$.

Consider uxv path in G as shown in Figure 5. Since $uv \notin E(G), uv \in E(\bar{G})$. As $ux, vx \notin E(\bar{G})$ and $diam(\bar{G}) = 2$, there exists diametrical paths urx and vsx in $(\bar{G})$ as shown in Figure 6. If r and s are distinct vertices, then $V(\bar{G})$ except the points$\{r, s\}$ is a SGD set giving $\gamma_{sg}(\bar{G}) \leq n - 2$, a contradiction.

Suppose r and s are not distinct. Consider $\bar{G}$ as shown in Figure 7 and x is connected to u and v through the paths utx and vtx respectively.

Suppose there exists diametrical paths tau, tbx, tcv where a, b, c are distinct vertices in $V(G)$, as shown in Figure 8. Then $diam(G) = 3$, which is a contradiction.

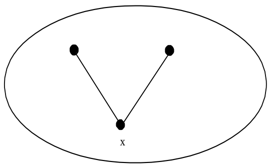

Fig 5: G

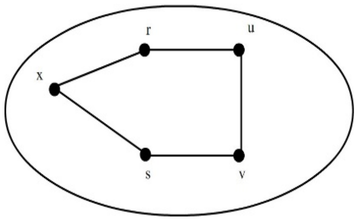

Fig 6 : $(\bar{G})$

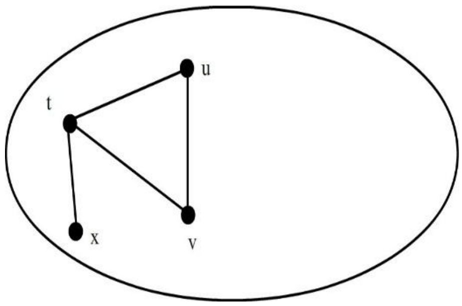

Fig 7 : $(\bar{G})$

Suppose two points of a, b, c are not same and $b = c$. Then $V(G) \setminus \{a, b\}$ is a SGD set. Thus $\gamma_{sg}(G) \leq n - 2$ which is a contradiction.(Refer Figure 9).

Suppose $a = b = c$. In this case all the diametrical paths from t to u, x and y share a same internal vertex a (Refer Figure 10). Clearly $V(G) \setminus \{a, x\}$ is a strong geodetic domination set giving $\gamma_{sg}(G) \leq n - 2$, a contradiction.

Suppose $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) = 2n - 3$. If $\gamma_{sg} = n$ and $\gamma_{sg}(\bar{G}) = n - 3$,then $G \cong K_n$, $\bar{G}$ a set of independent vertices contradicting the assumption in the theorem.

The second possibility is either $\gamma_{sg}(G) = n - 1$ and $\gamma_{sg}(\bar{G}) = n - 2$ or $\gamma_{sg}(G) = n - 2$ and $\gamma_{sg}(\bar{G}) = n - 1$. Assume that $\gamma_{sg}(G) = n - 1$and $\gamma_{sg}(\bar{G}) = n - 2$. This implies by lemma 2, $diam(G) = 2$ and by lemma 1, $diam(\bar{G}) \leq 3$.

Case 1: Suppose $diam(G) = 2$ and $diam(\bar{G}) = 3$. Considering a $u - v$ path of length 3 containing x, y as internal vertices in $(\bar{G})$ as shown in Figure 11. Since $\gamma_{sg}(\bar{G}) = n - 2$, $S = V(\bar{G})$ except the vertices $\{x, y\}$ is a minimum SGD set in $\bar{G}$.

Case 1(a): Consider the diametrical paths $x - u, x - y, y - v$ in G with the internal points a, b, c respectively. Let a, b, c are distinct.

Then $V(G)$except the points $\{a, b, c\}$ is a SGD set giving, $\gamma_{sg}(G) \leq n - 3$, which is a contradiction.(Refer Figure 12.)

Case 1(b): Consider the diametrical paths $x - u, x - y, y - v$ of length 2 in G. Sup- pose these paths have two distinct internal vertices say a, b. Here $V(G)$except the points $\{a, b\}$ is a SGD set giving, $\gamma_{sg}(G) \leq n - 2$, which is a contradiction. (Refer Figure 13.)

Case 1(c) Suppose the diametrical paths $x - u, x - y$ and $y - v$ in G share a same internal vertex t. The vertex t lies in the unique fixed path ytx and the vertex u lies in the unique fixed path yuv. This implies $V(G)$ except $\{t, v\}$ is a SGD set giving, $\gamma_{sg}(G) \leq n - 2$, a contradiction.(Refer Figure 14)

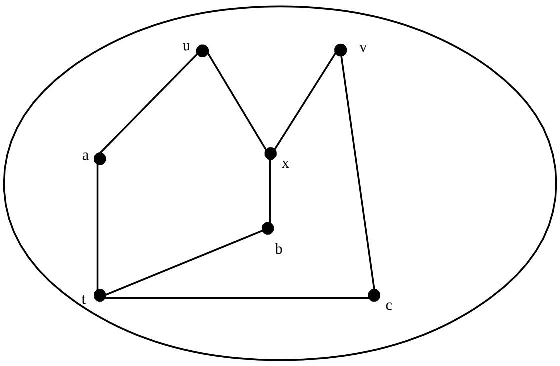

Fig 8: G

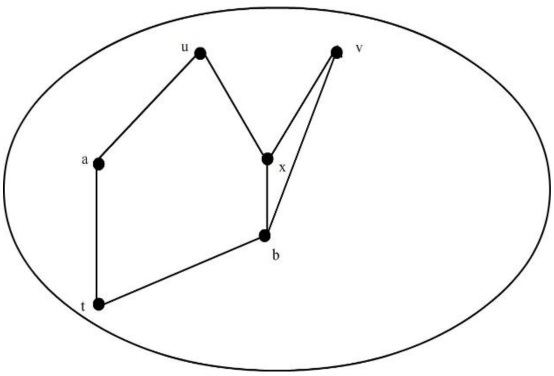

Fig 9: G

u lies in the unique fixed path yuv. This implies $V(G)$ except $\{t, v\}$ is a SGD set giving, $\gamma_{sg}(G) \leq n - 2$, a contradiction.(Refer Figure 14)

Case 2: Suppose $diam(G) = 2$ and $diam(\bar{G})$ is equal to 2. Let S be a minimum SGD set in $\bar{G}$. As $\gamma_{sg}(\bar{G}) = n - 2$, $|S| = n - 2$. Therefore $\exists x, y \in V(G) \backslash S$.

Case 2(a): Consider $\bar{G}$ with the points x and y as the internal vertices in fixed geodesics, of axb and ayc and $bc \in E(\bar{G})$ as shown in the Figure 15.

When $o(G) = 5$, then either $diam(G) > 2$ or $\gamma sg(G) \leq n - 2$, a contrary (Refer Figure 16)

Let $o(G) > 5$. Since $diam(G) = 2$ and G is connected, the non-adjacent pair of points in G are connected by diametrical paths of length 2.

Suppose there exists at least 2 internally disjoint vertices $\{s, t\}$ in the diametrical paths. Then $V(G) \backslash \{s, t\}$ is a SG set. Thus $\gamma_{sg}(G) \leq n - 2$, a contradiction.

Suppose there exists exactly one internal vertex t, common to all the diametrical paths. Then $V(G) \backslash \{x, t\}$ is a SG set. Thus $\gamma_{sg}(G) \leq n - 2$, a contradiction.

Case 2(b): Suppose in $\bar{G}$ x and y are the internal points of different fixed geodesics, say axb and ayc with $xy, bc \in E(\bar{G})$.

Case 2(c): Suppose in $\bar{G}$ x and y are the internal points of different fixed geodesics, say axb and ayc with a $xc \in E(\bar{G})$

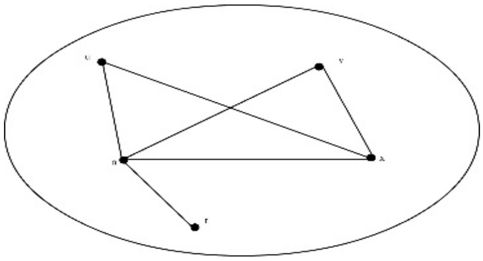

Fig 10: G

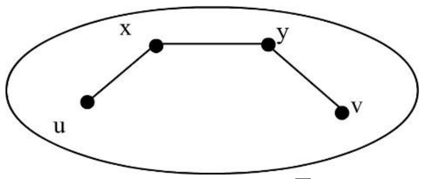

Fig 11 : *Case* 1 $(\bar{G})$

Case 2(d): Suppose in $(\bar{G})$ x and y are the internal vertices of different fixed geodesics, say axb and ayc with $\{xc, xy\} \in E(\bar{G})$.

Case 2(e): Suppose in $(\bar{G})$ x and y are the internal vertices of different fixed geodesics, say axb *and* ayc with $xc, bc \in E(\bar{G})$.

Case 2(f): Suppose in $(\bar{G})$ x and y are the internal vertices of different fixed geodesics, say axb and ayc with edges $xc, xy, bc \in E(\bar{G})$.

Case 2(g): Suppose in $(\bar{G})$ x and y are the internal vertices of different fixed geodesics, say axb and ayc with edges $xc, xy, by \in E(\bar{G})$.

Case 2(h): Suppose in $(\bar{G})$ x and y are the internal vertices of different fixed geodesics, say axb and ayc with edges $xc, by, bc \in E(\bar{G})$.

Case 2(i): Suppose in $(\bar{G})$ x and y are the internal vertices of different fixed geodesics, say axb and ayc with edges $xc, by, bc, xy \in E(\bar{G})$.

In all the cases from 2(b) to 2(i), the minimum SGD number, $\gamma_{sg}(G) \leq n-2$, a contradiction. (Proof is similar to Case 2(a))

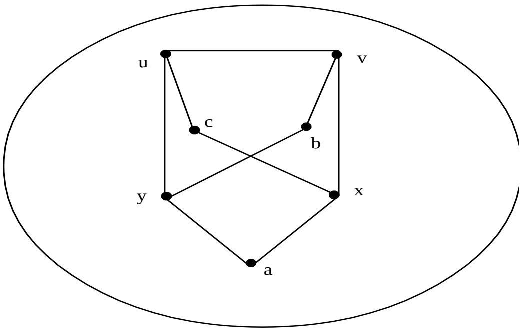

Fig 12 : Case 1(a) G

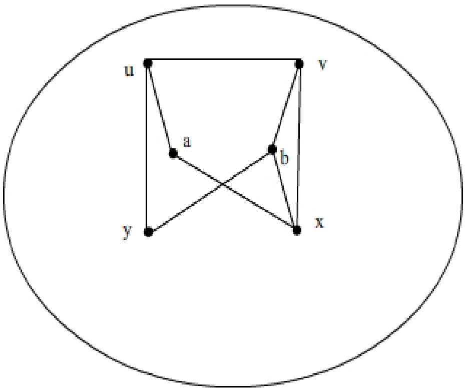

Fig 13 : Case 1(b) G

Case 2(j): Suppose in $(\bar{G})$, x and y are the internal points of different fixed geodesics, say axb and ayc with edges $xc, by \in E(\bar{G})$.
When $o(G) = 5$, then G is disconnected which is a contradiction. Let $o(G) > 5$. Since $diam(G) = 2$ and G is connected, the non-adjacent pair of vertices in G are connected by diametrical paths of length 2.
Suppose there exists at least 2 internally disjoint vertices $\{s,t\}$ in the diametrical paths. Then $V(G)\backslash\{s,t\}$ is a strong geodetic domination set. This implies $\gamma_{sg}(G) \leq n-2$, a contradiction. Suppose there exists exactly one internal vertex t, common to all the diametrical
paths in G. Retracting $(\bar{G})$ from $(\bar{G})$, there would exists diametrical paths from t
to $\{a,b,c\}$. Let tra be one such path. Here $V(G)\backslash\{r,x,y\}$ forms a SGD set for $(\bar{G})$, a contradiction.
Case 2(k): Let x and y be the internal points of different fixed geodesics, axc and dye where a,c,d,e are disjoint. Since $diam(\bar{G}) = 2$,it is clear that $ad, dc, ce, ae, xy \in E((\bar{G}))$. If $o(G) = 6$, then G is disconnected, a contradiction. Let $o(G) > 6$. Since $diam(G) = 2$ and G is connected ,the non-adjacent pair of vertices among a,d,c,e,x and y in G are connected by diametrical paths of length 2. Suppose there exists at least two disjoint internal vertices s,t in the diametrical paths. Then $V(G)$ except points $\{t,s\}$ is a SGD set giving, $\gamma_{sg}(G) \leq n-2$, a contra-diction.

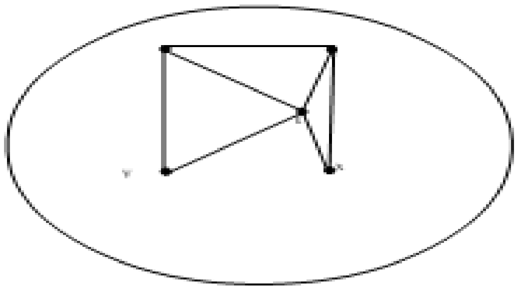

Fig 14 : Case 1(c) G

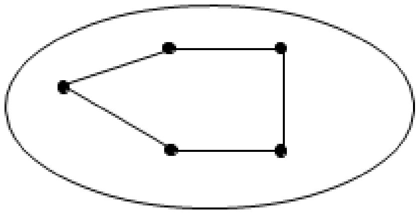

Fig 15 : Case 2(a) $(\bar{G})$

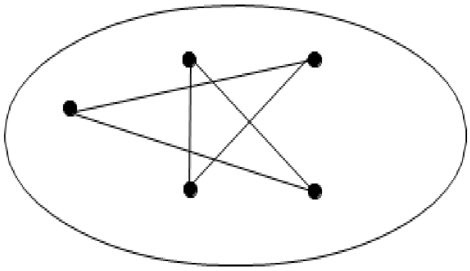

Fig 16 : Case 2(a) G

Suppose there exists only one internal vertex t in all diametrical paths of G. Retracting $(\bar{G})$ from G, there would exist diametrical paths $tse, tsa, tsx, dye, axc\ etc.$
Here $V\backslash\{x,y,s\}$ forms a SGD set for $(\bar{G})$, contradiction as $\gamma_{sg}(\bar{G}) \leq n-3$.
Case 2(l): Suppose in $(\bar{G})$, x and y are the internal vertices of different fixed geodesics, say axc and dye where a,c,d,e are disjoint with the edges $ad, dc, ce, dx, ae \in E(\bar{G})$.
Case 2(m): Suppose in$(\bar{G})$, x and y are the internal vertices of different fixedngeodesics, say axc and dye where a,c,d,e are disjoint with the edges $ad, dc, ce, dx, ae, ex \in E(\bar{G})$.
Case 2(n): Suppose in $(\bar{G})$, x and y are the internal vertices of different fixed geodesics, say axc and dye where a,c,d,e are disjoint and $ad, dc, ce, ae, ay, xd, xe \in E(\bar{G})$.
Case 2(o): Let x and y be the internal points of different geodesics, axc and dye where a,c,d,e are disjoint with $ad, dc, ce, ae, ay, cd, cy \in E(\bar{G})$.
Case 2(p): Suppose in $(\bar{G})$, x and y are the internal vertices of different fixed geodesics, say axc and dye where a,c,d,e are disjoint with the edges $ad, dc, ce, dx, ae, xy \in E(\bar{G})$.
Case 2(q): Suppose in $(\bar{G})$, x and y are the internal vertices of different fixed geodesics, say axc and dye where a,c,d,e are disjoint with the edges $ad, dc, ce, dx, ae, ex, xy \in E(\bar{G})$.

Case 2(r): Suppose in $(\bar{G})$, x and y are the internal vertices of different fixed geodesics, say axc and dye where a, c, d, e are disjoint and $ad, dc, ce, ae, ay, xd, xe, xy \in E(\bar{G})$.

Case 2(s): Let x and y be the internal points of different geodesics, axc and dye where a, c, d, e are disjoint with $ad, dc, ce, ae, ay, cd, cy, xy \in E(\bar{G})$.

The cases, 2(l) -2(s) also gives a contradiction and the proof is similar to Case 2(k).

Hence $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) \neq 2n - 3$. Thus for connected graphs G and $\bar{G}$ of order $n \geq 4$, the $\gamma_{sg}(G) + \gamma_{sg}(\bar{G}) \leq 2n - 4$

Remark 3. For this theorem there exists connected graphs with connected complements, where the bound is sharp. The equality is attained for $G \cong P_4$ or C_5

Proposition 4. For G with a vertex x such that the graph $(G - x)$ is the union of at least two complete graphs and the vertex x is adjacent to all other vertices of G. Then $\gamma_{sg}(G) = (n - 1)$.

The converse need not be true. For example, $\gamma_{sg}(K_n - e) = n - 1$, where $e \in E(K_n)$

Proposition 5. Let $diam(G) = i$, where $i = 1, 2, 3$. Then $sg(G) = \gamma_{sg}(G)$

V. REALIZATION RESULTS

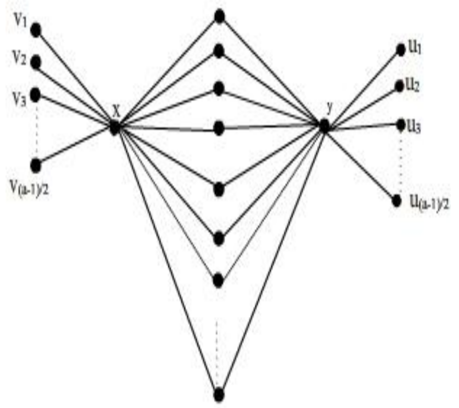

Fig 17: Graph G with $\gamma_{sg}(G) = a$ where n is odd.

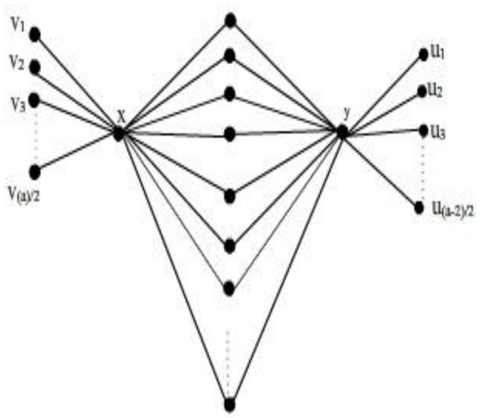

Fig 18: Graph G with $\gamma_{sg}(G) = a$ where n is even

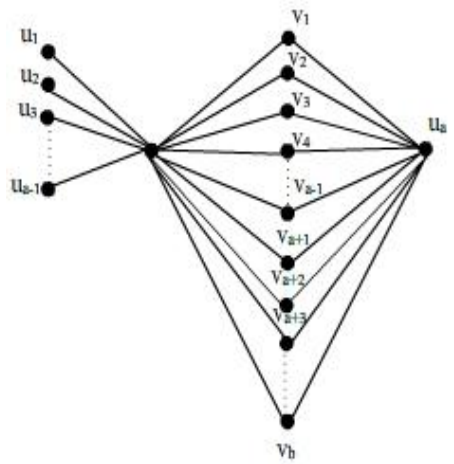

Fig 19: Graph G with $\gamma_g(G) = a, \gamma_{sg}(G) = b$

Theorem 16. For a and n with $a \geq 3$ with $n \leq \left(\frac{(a-1)^2}{4} + a + 2\right)$, when a is odd and $n \leq \left(\frac{(a-2)}{2}\frac{a}{2} + a + 2\right)$ when a is even, $\exists$ a G with $\gamma_{sg}(G) = a$ and $o(G) = n$.

Proof. Let $a = n$, then $G = K_n$ and when $a = n - 1$, assume G to be $K_{1,n-1}$.Now the case left to prove is when $a \leq n - 2$. The a can be either odd or even. When a is odd, construct G from a $\theta-$ graph with $n - a - 1 \leq \frac{(a-1)^2}{4}$ levels and adding $\frac{(a-1)}{2}$ pendant vertices to the vertices x and y as shown in the Figure 17. There exists $\frac{(a-1)^2}{4}$ distinct fixed geodesics between each of these pendent vertices and they cover $(n - a)$ vertices. These $(a - 1)$ pendant vertices with the vertex x is a minimum SGD set for G, giving $\gamma_{sg}(G) = a$. When a is even, construct a graph G from a $\theta-$ graph with $n - a - 1 \leq \frac{(a-2)}{2}\frac{a}{2}$ levels and adding $\frac{(a-2)}{2}$ pendant vertices to the vertex x and $\frac{(a)}{2}$ pendant vertices to vertex y as shown in the Figure 18. There exists $\frac{(a-2)}{2}\frac{a}{2}$ distinct fixed geodesic between each of these pendent vertices and they cover (n − a) vertices. These $(a - 1)$ pendant vertices with the vertex x is a minimum SGD set for G, giving $\gamma_{sg}(G) = a$.

Theorem 17. For two positive integers a, b where $a < b$, then $\exists$ G with $\gamma_g(G) = a$ and $\gamma_{sg}(G) = b$

Proof. Form a graph G as shown in Figure 19, The vertices $\{u_1, u_2, \ldots, u_{a-1}, u_a\}$ forms a minimum GD set for G, thus giving $\gamma_g(G) = a$. Here utmost $(a - 1)$ points from the set $\{v_1, v_2, \ldots, v_{a-1}, v_{a+1}, v_{a+2}, \ldots, v_b\}$ exists in a unique fixed path between the geodesics in the vertices of $\{u_1, u_2, \ldots, u_{a-1}, u_a\}$. Hence to cover the remaining vertices in a unique fixed path at least $(b - a)$ vertices from $\{v_1, v_2, \ldots, v_{a-1}, v_{a+1}, v_{a+2}, \ldots, v_b\}$ has to be choosen, thus giving the strong geodetic domination number $\gamma_{sg}(G) = b$.

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