

A New Numerical Method to Solve Non Linear Fractional Differential Equations



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Abstract: In the present paper a new approximate analytical method, the homotopy perturbation and natural transform method namely HPNT is introduced that is blend of the homotopy perturbation method and natural transform method. The proposed method is applied for solution of the non-linear Fokker Planck equation of time fractional order. The correctness and efficacy of the proposed method is verified through graphical method and error analysis.

I. INTRODUCTION

For past two to three centuries the study of fractional calculus confined as a pure theoretical area of mathematics. It has been shown by many authors that the arbitrary order derivative and integrals are very useful for explanation of properties of various materials in mathematical modelling which are more adequate than integer order models. Arbitrary order derivatives gives an excellent descriptions of memory and hereditary properties of various process. Many important concepts in diverse areas of applied sciences and engineering [1] are well described by fractional order differential equations [2, 3, 4, 5]. Exact solutions of many fractional differential equations cannot be found thus analytical and approximated solutions have become essential tools for their solution [6-11]. In the present paper, we introduce an approximate analytical method HPNT, which is mixing of the homotopy perturbation method [12, 13, 14, 15] and natural transform method [16]. This numerical method is suitable in effective way to solve fractional order nonlinear differential equations without using Adomian and He's polynomials for computations of nonlinear terms.

II. FRACTIONAL CALCULUS

In the present portion, we impart some important definitions and results of Riemann-Liouville (RL) fractional integral operator and fractional derivative that is given in Caputo sense. The RL fractional order is most commonly used in mathematics but its requirement of fractional order initial conditions which restrict, its uses in physical modelling. Thus Caputo sense definition used in applied area as its benefit of allowing integer order initial conditions.

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Definition 2.1 The well known Riemann-Liouville fractional integral operator [17] of order $\alpha > 0$ of function $f: R^+ \rightarrow R$ is defined as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad x > 0 \quad (1)$$

Definition 2.2 The Caputo fractional derivative [17] of order $\alpha > 0$, $n-1 < \alpha \leq n$, $x > 0$, $n \in \mathbb{N}$ is defined as

$$D^\alpha f(x) = I^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt, \quad (2)$$

It is noted that $f(t)$ possess absolute continuously derivatives up to order $n-1$.

III. NATURAL TRANSFORM

This section includes the basic definitions and properties of natural transform [16, 18] as follows

If $f(t)$ is section wise continuous and of exponential order such that $f(t) > 0$ for $t > 0$ and $f(t) = 0$ for $t < 0$, and belongs to following set A as given below

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_1}}, \text{ if } t \in (-\tau_1)^j \times [0, \infty) \right\} \quad (3)$$

The Natural transform for function $f(t) > 0$ is defined by [19] as follows

$$N^+[f(t)] = R(s, u) = \int_0^\infty e^{-st} f(ut) dt; \quad s > 0, u > 0 \quad (4)$$

Where s and u are the transform variables. When $u \equiv 1$ in equation (4) converges to Laplace transform [20, 21, 22] and $s \equiv 1$ in (4) converges to Sumudu transform [23, 24] respectively defined by

$$\mathcal{A}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt$$

$$\mathcal{S}^+ F(s) = G(u) = \int_0^\infty e^{-t} f(ut) dt; \quad u \in (-\tau_1, \tau_2)$$

3.1 Natural-Laplace Duality (NLD)

If $R(s, u)$ and $F(s)$ denote Natural and Laplace transform respectively of function $f(t) \in A$ then

$$N[f(t)] = R(s, u) = \frac{1}{u} \int_0^\infty f(t) e^{-\frac{st}{u}} dt = \frac{1}{u} F\left(\frac{s}{u}\right) \quad (5)$$



3.2 Natural-Sumudu Duality (NSD)

If $R(s,u)$ and $G(u)$ are Natural and Sumudu transforms respectively of function $f(t) \in A$ then

$$N[f(t)] = R(s, u) = \frac{1}{s} \int_0^\infty f\left(\frac{ut}{s}\right) e^{-t} dt = \frac{1}{s} G\left(\frac{u}{s}\right) \quad (6)$$

3.3 Natural Transform of n^{th} derivative of function $f(t)$

If $f^n(t)$ is the n -th derivative of function $f(t)$ then due to [19]

$$N[f^n(t)] = R_n(s, u) = \frac{s^n}{u^n} R(s, u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0), \quad n \geq 1. \quad (7)$$

3.4 Convolution theorem

If $F(s, u)$, $G(s, u)$ are the Natural transform of respective functions $f(t)$, $g(t)$ both belongs to set A then Convolution theorem of Natural Transform [19] is given as

$$N[f * g] = uF(s, u)G(s, u) \quad (8)$$

where $f * g$ denote convolution of f & g which is given as

$$\int_0^t f(a)g(t-a)da = \int_0^t f(t-a)g(a)da.$$

3.5 Natural transform of Fractional Derivative

The Natural transform of fractional derivative of order (α) , due to [19] is given as

$$N[f^{(\alpha)}(t)] = \frac{s^\alpha}{u^\alpha} R(s, u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0) \quad (9)$$

$$N[f(t)] = \frac{1}{u} \int_0^\infty e^{-st/u} f(t) dt = \frac{1}{u} F\left(\frac{s}{u}\right) = \frac{1}{u} \int_0^\infty e^{-st} f(ut) dt = R(s, u) \quad (10)$$

N. Sumudu Duality

$$N[f(t)] = \frac{1}{s} \int_0^\infty e^{-t} f\left(\frac{ut}{s}\right) dt = \frac{1}{s} F\left(\frac{u}{s}\right) \quad (11)$$

$$L[f'(t)] = \frac{1}{u} \int_0^\infty e^{-st/u} f'(t) dt \quad (12)$$

$$\begin{aligned} &= \frac{1}{u} \left[\left(e^{-st/u} f(t) \right)_0^\infty - \int_0^\infty \left(-\frac{s}{u} \right) e^{-st/u} f(t) dt \right] \\ &= \frac{1}{u} \left[-f(0) + \frac{s}{u} \bar{F}\left(\frac{s}{u}\right) \right] \\ &= \frac{s}{u^2} \bar{F}\left(\frac{s}{u}\right) - \frac{1}{u} f(0) \\ &= \frac{s}{u} R(s, u) - \frac{f(0)}{u} \end{aligned}$$

Similarly we get

$$L[f''(t)] = \frac{s^2}{u^2} R(s, u) - \frac{s}{u^2} f(0) - \frac{f'(0)}{u} \quad (14)$$

$$N[f^n(t)] = \frac{s^n}{u^n} R(s, u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} f^{(k)}(0) \quad (15)$$

For $n=2$

$$L[f''(t)] = \frac{s^2}{u^2} \bar{F}\left(\frac{s}{u}\right) - \sum_{k=0}^1 \frac{s^{1-k}}{u^{2-k}} f^{(k)}(0) - \frac{s}{u^2} f(0) - \frac{f'(0)}{u} \quad (16)$$

$$\begin{aligned} L[f''(t)] &= \frac{1}{u} \left[\left(e^{-st/u} f'(t) \right)_0^\infty - \int_0^\infty e^{-st/u} \left(-\frac{s}{u} \right) f'(t) dt \right] \\ &= \frac{1}{u} \left[-f'(0) + \frac{s}{u} \left\{ \frac{s}{u^2} \bar{F}\left(\frac{s}{u}\right) - \frac{1}{u} f(0) \right\} \right] \\ &= -\frac{1}{u} f'(0) + \frac{s^2}{u^4} \bar{F}\left(\frac{s}{u}\right) - \frac{s}{u^3} f(0) \end{aligned} \quad (17)$$

Hence $N[f^n(t)] =$

$$\frac{s^n}{u^n} N[U(x, t)] - \frac{s^{n-1}}{u^n} u(x, 0) - \frac{s^{n-1}}{u^{n-1}} u'(x, 0) - \dots - \frac{s}{u} u^{n-1}(x, 0) \quad (18)$$

Sumudu Transform

$$F(u) = s[f(t)] = \int_0^\infty \frac{1}{u} e^{-t/u} f(t) dt \quad (19)$$

IV. FRACTIONAL HOMOTOPY PERTURBATION AND NATURAL TRANSFORM METHOD (HPNT)

Here we take general nonlinear homogeneous partial order differential equation with the following conditions to illustrate the basic idea of the method

$$D_t^\alpha (U(x, t)) = RU(x, t) + N_1 U(x, t) + g(x, t) \quad (20)$$

with $n-1 < \alpha \leq n$ and subjected to the following condition

$$\begin{aligned} \frac{\partial^r U(x, 0)}{\partial t^r} &= U^{(r)}(x, 0) \\ &= f_r(x) \end{aligned} \quad (21)$$

$r = 0, 1, \dots, n-1.$

Where $D_t^\alpha (U(x, t))$ denote Caputo orders Fractional Derivative, $g(x, t)$ is the source term, L is the linear operator and N_1 is the non-linear operator.

Taking the Natural transform on both side

$$N[D_t^\alpha (U(x, t))] = N[RU(x, t) + N_1 U(x, t) + g(x, t)] \quad (22)$$

(13) On using the equation (9), we get



$$s^\alpha u^{-\alpha} - N[(U(x, t))] - \sum_{k=0}^{n-1} s^{\alpha-k-1} u^{-(\alpha-k)} U^{(k)}(x, 0) = N[RU(x, t) + N_1 U(x, t) + g(x, t)] \quad (23)$$

$$N[(U(x, t))] = \frac{u^\alpha}{s^\alpha} \left[\sum_{k=0}^{n-1} s^{\alpha-k-1} u^{-(\alpha-k)} U^{(k)}(x, 0) + N[RU(x, t) + N_1 U(x, t) + g(x, t)] \right]$$

Now applying inverse Laplace transform on both side of equation, we get

$$U(x, t) = N^{-1} \left[\frac{u^\alpha}{s^\alpha} \left[\sum_{k=0}^{n-1} s^{\alpha-k-1} u^{-(\alpha-k)} U^{(k)}(x, 0) + N[RU(x, t) + N_1 U(x, t) + g(x, t)] \right] \right] \quad (24)$$

Now using perturbation technique and assuming solution of above equation is in the form

$$U(x, t) = \sum_{n=0}^{\infty} p^n U_n(x, t) \quad (25)$$

where $p \in [0, 1]$ is the homotopy parameter

The nonlinear term can be decomposed

$$N_1 U(x, t) = \sum_{n=0}^{\infty} p^n H_n(U) \quad (26)$$

where H_i are He's polynomials, can be given by the following formula

$$H_n(U_0, U_1, U_2, \dots, U_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} [N_1(\sum_{i=0}^{\infty} p^i U_i)]_{p=0}; n = 0, 1, 2, \dots \quad (27)$$

Substituting equation (25) and (26) in equation (24), we obtain

$$\sum_{n=0}^{\infty} p^n U_n(x, t) = N^{-1} \left[\frac{u^\alpha}{s^\alpha} \left(\sum_{k=0}^{n-1} s^{\alpha-k-1} u^{-(\alpha-k)} U^{(k)}(x, 0) \right) + p N^{-1} \left[\frac{u^\alpha}{s^\alpha} N \left[R \left(\sum_{n=0}^{\infty} p^n U_n(x, t) \right) + N_1 \left(\sum_{n=0}^{\infty} p^n U_n(x, t) \right) + g(x, t) \right] \right] \right]$$

Equating the terms with like powers of 'p', we get

$$p^0: U_0(x, t) = N^{-1} \left[\frac{u^\alpha}{s^\alpha} \left(\sum_{k=0}^{n-1} s^{\alpha-k-1} u^{-(\alpha-k)} U^{(k)}(x, 0) \right) \right]$$

$$p^1: U_1(x, t) = N^{-1} \left[\frac{u^\alpha}{s^\alpha} N [R(U_0(x, t)) + N_1(U_0(x, t)) + g(x, t)] \right]$$

$$\vdots$$

$$p^n: U_n(x, t) = N^{-1} \left[\frac{u^\alpha}{s^\alpha} N \left[R \left(\sum_{n=0}^{\infty} p^n U_n(x, t) \right) + N_1 \left(\sum_{n=0}^{\infty} p^n U_n(x, t) \right) + g(x, t) \right] \right]$$

Finally the suitable approximation for the solution is

$$U(x, t) = \lim_{p \rightarrow 1} \sum p^n U_n(x, t) = U_0 + U_1 + U_2 \dots \quad (28)$$

which converges rapidly.

V. APPLICATION

In this section, we present the solution of a nonlinear Fokker Planck equation of fractional order by proposed HPNT method.

Consider following fractional order Fokker Planck equation [25]-

$$\frac{\partial^\alpha U}{\partial t^\alpha} = \left[-\frac{\partial}{\partial x} \left(\frac{4}{x} U - \frac{x}{3} \right) + \frac{\partial^2}{\partial x^2} U \right] U. \quad (29)$$

Where $t > 0, x \in R, 0 < \alpha \leq 1$, subjected to the initial condition $U(x, 0) = x^2$.

Taking the Natural transform of both side of the equation (29), thus

$$N \left[\frac{\partial^\alpha U}{\partial t^\alpha} \right] = N \left[\left[-\frac{\partial}{\partial x} \left(\frac{4}{x} U(x, t) - \frac{x}{3} \right) + \frac{\partial^2}{\partial x^2} U(x, t) \right] U(x, t) \right]$$

$$N[U(x, t)] = \frac{x^2}{s} + \frac{u^\alpha}{s^\alpha} N \left[\left[-\frac{\partial}{\partial x} \left(\frac{4}{x} U(x, t) - \frac{x}{3} \right) + \frac{\partial^2}{\partial x^2} U(x, t) \right] U(x, t) \right]$$

Apply Inverse Natural transform

$$U(x, t) = x^2 + N^{-1} \left[\frac{u^\alpha}{s^\alpha} N \left[\left[-\frac{\partial}{\partial x} \left(\frac{4}{x} U(x, t) - \frac{x}{3} \right) + \frac{\partial^2}{\partial x^2} U(x, t) \right] U(x, t) \right] \right] \quad (30)$$

Now apply Homotopy perturbation method in equation (30), we get

$$\sum_{n=0}^{\infty} p^n U(x, t) = x^2 + p N^{-1} \left[\frac{u^\alpha}{s^\alpha} N \left[-4 \sum_{n=0}^{\infty} p^n H_n(x, t) + \frac{1}{3} \sum_{n=0}^{\infty} p^n U_n(x, t) + n=0 \infty p n B n x, t \right] \right] \quad (31)$$

Where $H_n = \frac{\partial}{\partial x} \left(\frac{1}{x} U_n(x, t) \right) U_n(x, t)$

$$B_n = \frac{\partial^2}{\partial x^2} U_n^2(x, t)$$

Comparing the like powers of p both sides of (31) we have

$$p^0: U_0(x, t) = x^2$$

$$p^1: U_1(x, t) = N^{-1} \left[\frac{u^\alpha}{s^\alpha} N \left[-4H_0 + \frac{1}{3} U_0 + B_0 \right] \right]$$

$$= x^2 \frac{t^\alpha}{\Gamma(\alpha + 1)}$$

$$p^2: U_2(x, t) = N^{-1} \left[\frac{u^\alpha}{s^\alpha} N \left[-4H_1 + \frac{1}{3} U_1 + B_1 \right] \right]$$

$$= x^2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$p^n: U_n(x, t) = N^{-1} \left[\frac{u^\alpha}{s^\alpha} N \left[-4H_{n-1} + \frac{1}{3} U_{n-1} + B_{n-1} \right] \right]$$

$$= x^2 \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}$$

Hence the solution of equation (29) is given by

$$U(x, t) = \lim_{p \rightarrow 1} \sum_{n=0}^{\infty} p^n U_n(x, t)$$

$$= x^2 + x^2 \frac{t^\alpha}{\Gamma(\alpha+1)} + x^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + x^2 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots$$

(32)

$$U(x, t) = x^2 e^{t^\alpha}$$

For $\alpha = 1$ the above solution reduces to exact solution [25]

$$U(x, t) = x^2 e^t$$

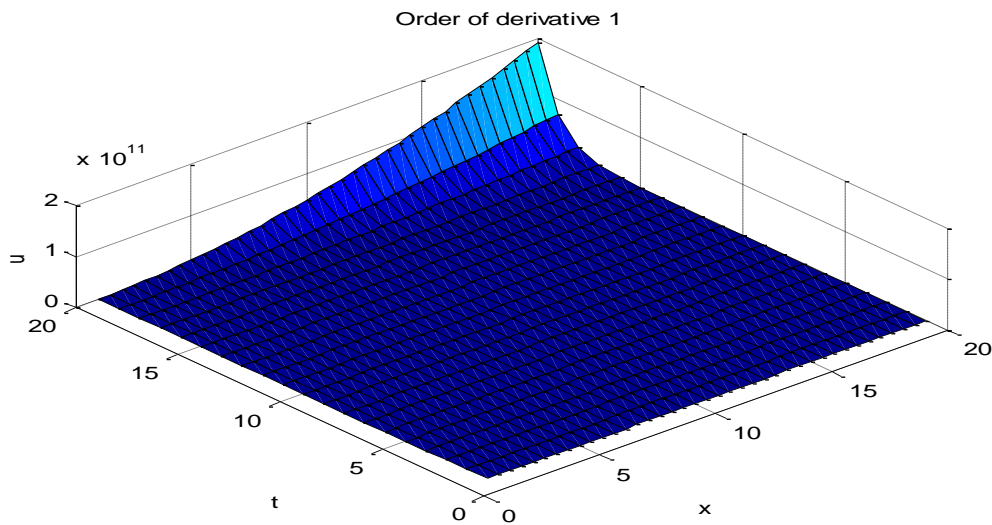


Fig. 1 Plots for u vs t and x at derivative of order 1.

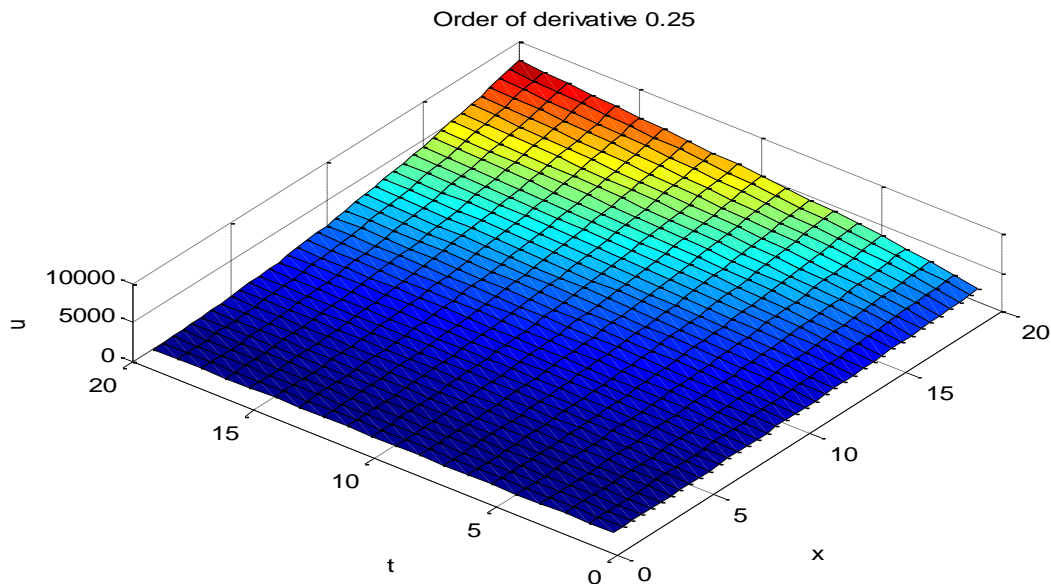


Fig. 2 Plots for u vs t and x at derivative of order 0.25.

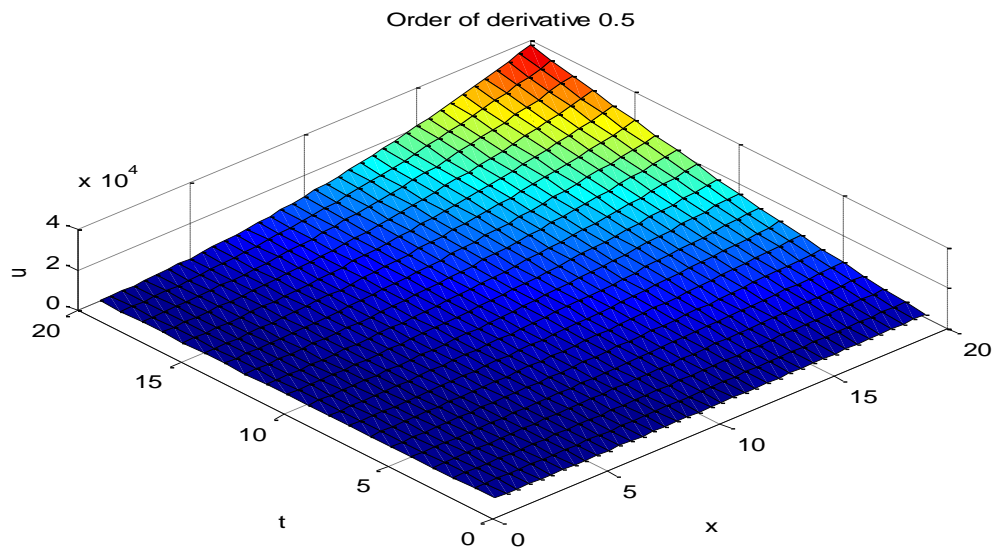


Fig. 3 Plots for u vs t and x at derivative of order 0.5.

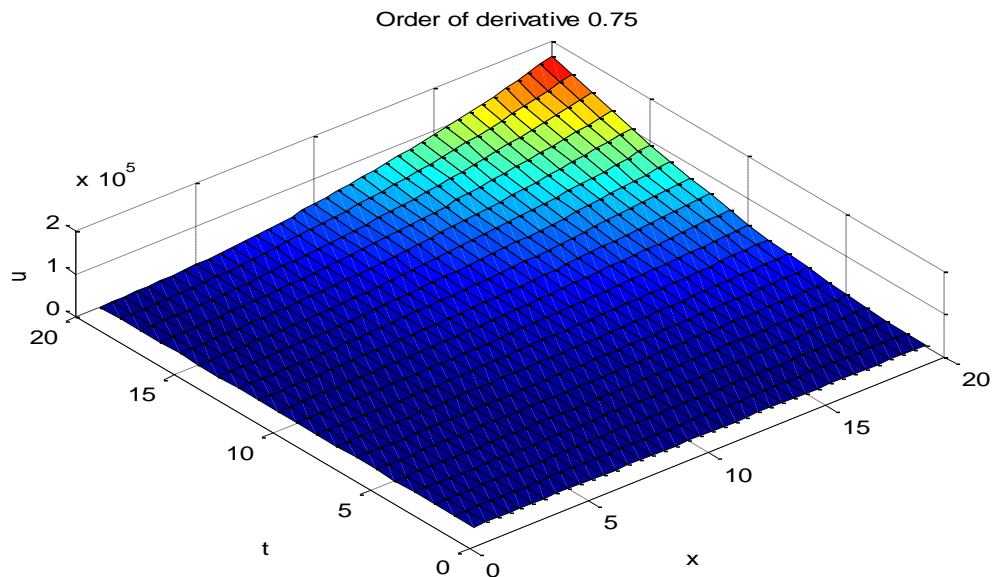


Fig. 4 Plots for u vs t and x at derivative of order 0.75.

Error diagram of Numerical Solution up to III approximation with Exact Solution

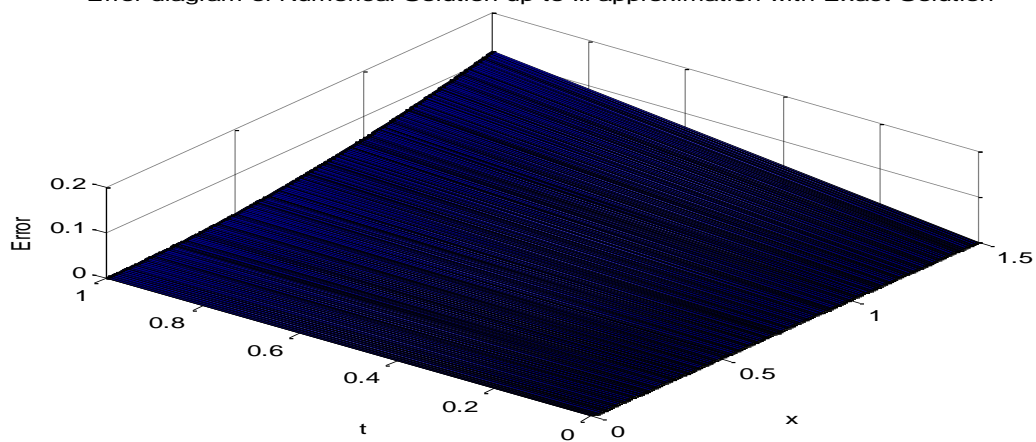


Fig. 5 Plots for Error of approximate solution.

5.1 Approximate third order solution of Fokker Planck for differential values of α and absolute error

Table 5.1

Approximate third order solution of Fokker Planck for differential values of α and absolute error at $\alpha = 1$.

Third order Approximate solution by HPNT						
		Exact		Error		
T	x	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	$\alpha = 1$	$ U_{exact} - U_{HPNT} $
0.2	0.3	0.1375	0.1215	0.109	0.1099	2×10^{-4}
	0.6	0.5499	0.4861	0.4394	0.4397	3×10^{-4}
	0.8	0.9777	0.8642	0.7813	0.7817	4×10^{-4}
0.4	0.3	0.1636	0.1491	0.1342	0.1343	1×10^{-4}
	0.6	0.6544	0.5963	0.5366	0.5371	5×10^{-4}
	0.8	1.1633	1.0601	0.9540	0.9548	8×10^{-4}

VI. CONCLUSION

In the present work, we introduce a blend of Natural transform and homotopy perturbation method. We explored the methodology for the construction of the new twisting scheme and employed to compute the analytic approximate solution of a fractional-order non-linear Fokker Planck equation. By comparing these approximate solutions with known exact solutions, it was shown that the proposed solution is rapidly convergent with high accuracy which is shown by depiction of graphs and error analysis.

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