

# On Fibonacci Numbers and its Applications

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**Abstract:** In this article, we explore the representation of the product of  $k$  consecutive Fibonacci numbers as the sum of  $k$ th power of Fibonacci numbers. We also present a formula for finding the coefficients of the Fibonacci numbers appearing in this representation. Finally, we extend the idea to the case of generalized Fibonacci sequence and also, we produce another formula for finding the coefficients of Fibonacci numbers appearing in the representation of three consecutive Fibonacci numbers as a particular case. Also, we point out some amazing applications of Fibonacci numbers.

**Keywords:** Fibonacci sequence, Generalized Fibonacci sequence, Golden ratio, Linear combination of numbers.

## I. INTRODUCTION

The Fibonacci sequence named after Leonardo Fibonacci enchants mathematicians over worldwide with their unpredictable nature. The Fibonacci numbers, denoted by  $F_n$  are the numbers occurring in the system 1,1,2,3,5,8,13, ... in which each term is the sum of the previous two terms. This is defined recursively as

$$F_1 = 1, F_2 = 1 \text{ and} \\ F_n = F_{n-1} + F_{n-2}; n \geq 2 \quad (1)$$

If the first two elements of this system be taken as arbitrary integers  $a$  and  $b$  then the resultant sequence  $a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, \dots$  is called the generalized Fibonacci sequence. It is worthy to note that Lucas sequence 2,1,3,4,7, ... falls under the generalized Fibonacci sequence. The Fibonacci sequence has wide range applications in different engineering fields like sorting algorithm, architectural engineering and financial engineering. The Fibonacci numbers which are the basis for the golden ratio are widely used in architecture and design.

In a human body, the bone length in each hand is actually Fibonacci numbers. Also, the cochlea of our ear follows the golden ratio. Another application is in the field of music. Fibonacci compositions give more sweet voices. Also, in a piano, the interval between keys are Fibonacci numbers. The idea of golden ratio is widely used in geometry like in regular pentagon, star fish and in the body of dolphin. Nowadays, the Fibonacci sequence is used in cryptographic coding. Raphel and Sundaram [1] showed that the security of data communication can be improved while using the properties of Fibonacci numbers. Recently, the students from the

polytechnic institute of New York University constructed a robot that moves based on the Fibonacci sequence of numbers and they explored how fast the numbers in the sequence grow with the help of their robotics kit which works using computer programs. We can visualize many applications of Fibonacci numbers in other branches of science and engineering in [2]-[4].

A lot of studies were carried out worldwide by different mathematicians for a long time. The earlier literature on Fibonacci numbers were found in [5]. One of a remarkable discovery about Fibonacci numbers was by Kepler [6]. He explored that the proportion of successive Fibonacci numbers converges and this ratio is known as the golden ratio. Recently Adegoke [7] exhibits some generalizations of Fibonacci-Lucas sums of Brousseau.

In the present investigation, we are trying to prove that the product of  $k$  consecutive Fibonacci numbers can be denoted by the sum of  $k$ th power of Fibonacci numbers and also develop the formula for finding the coefficients of Fibonacci numbers involved in this representation. Table 1 exhibits a list of Fibonacci numbers as per the convention followed in this article.

## II. RESULTS AND DISCUSSIONS

It is quite interesting to note that the product  $F_2 \times F_3 = 1 \times 2 = 2$  can be represented as  $a_1 F_1^2 + a_2 F_2^2$  where  $a_1 = a_2 = 1$ . This idea motivated the authors to generalize it for the product of  $k$  consecutive Fibonacci numbers and thus we arrived at the following result.

**Theorem 1.** The product of  $k$  consecutive Fibonacci numbers can be expressed as the sum of  $k$ th power of Fibonacci numbers.

*proof.* Consider the Fibonacci sequence 1,1,2,3,5, ... This can be defined as in (1). Consider  $k$  successive Fibonacci numbers say  $F_n, F_{n-1}, F_{n-2}, \dots, F_{n-k+1}$ . Then,  $F_n \times F_{n-1} \times F_{n-2} \times \dots \times F_{n-k+1} = (F_{n-1} + F_{n-2}) \times F_{n-1} \times F_{n-2} \times \dots \times F_{n-k+1} = F_{n-1}^2 F_{n-2} \dots F_{n-k+1} + F_{n-1} F_{n-2}^2 \dots F_{n-k+1} + (F_{n-2} + F_{n-3})^2 F_{n-2} F_{n-3} \dots F_{n-k+1} + (F_{n-2} + F_{n-3} + F_{n-4})^2 F_{n-2} F_{n-3} \dots F_{n-k+1} + \dots + F_{n-k+1}^k$  and continuing to expand  $F_{n-2} = F_{n-3} + F_{n-4}$  and proceed as above, the RHS reduces to  $u_1 F_1^k + \dots + u_{n-k} F_{n-k}^k$  where  $u_i$ 's are constants.

### A. Illustration of Theorem 1

Consider three consecutive Fibonacci numbers say  $F_5, F_4, F_3$ . Then,

$$F_5 \times F_4 \times F_3 = (F_4 + F_3) \times F_4 F_3 = F_4^2 F_3 + F_4 F_3^2 \\ = (F_3 + F_2)^2 \times F_3 + (F_3 + F_2) F_3^2$$

If we continue like this the RHS reduces to  $2F_3^3 + 4F_2^3 + 10F_1^3$ .

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**B. Method for finding  $u_i$ .**

Consider the pyramid

$$\begin{array}{cccc}
 & & 1 & & \\
 & & 1 & & 1 \\
 & 2 & 3 & & 1 \\
 6 & 13 & 9 & & 2
 \end{array}$$

Let  $a_{x,y}$  represents the entries in the  $x^{th}$  row and  $y^{th}$  column of the pyramid. These  $a_{x,y}$ 's can be defined as follows:

$$a_{1,1} = 1 \text{ and } a_{x,y} = \sum_{i=1}^{x-1} \binom{x-i}{y-1} a_{x-1,i}$$

At the  $k$ th step of the equation (2) we have the relation  $a_{k,1}F_{n-k+1}^k + a_{k,2}F_{n-k+1}^{k-1}F_{n-k} + a_{k,3}F_{n-k+1}^{k-2}F_{n-k}^2 + \dots + a_{k,k}F_{n-k+1}^{k-1}F_{n-k}^k$  (3)

Now, consider the coefficients  $a_{k,2}, \dots, a_{k,k}$  of (2) and apply the recursive formula

$$b_{x,y} = \sum_{i=1}^{k-1} \binom{k-i}{y-1} b_{x-1,i+1} \quad (3)$$

on  $a_{k,2}, \dots, a_{k,k}$ . Then at the  $(n-k)^{th}$  step, we get

$$F_n \times F_{n-1} \times F_{n-2} \times \dots \times F_{n-k+1} = b_{1,1}F_{n-k+1}^k + b_{2,1}F_{n-k}^k + b_{3,1}F_{n-k-1}^k + b_{n-k,1}F_2^k + cF_1^k \quad (4)$$

where  $c = \sum_{i=1}^{k-1} 2^{k-i} b_{n-k,i}$ .

**C. Example**

Consider the four consecutive Fibonacci numbers  $F_3 = 3, F_4 = 5, F_5 = 8$  and  $F_6 = 13$ . Let  $S = F_6 \times F_5 \times F_4 \times F_3 = 1560$  be their product. Now construct the pyramid as explained in section 2.1. So, we have

$$\begin{array}{cccc}
 & & 1 & & \\
 & & 1 & & 1 \\
 & 2 & 3 & & 1 \\
 6 & 13 & 9 & & 2
 \end{array}$$

So, at the 4<sup>th</sup> step, we have

$S = 6 \times F_3^4 + 13 \times F_3^3 \times F_2 + 9 \times F_3^2 F_2^2 + 2 \times F_3 F_2^3$   
 Now apply the formula (3) on the coefficients 13, 9 and 2. Then, without modifying the fourth step, we get

$$\begin{array}{cccc}
 6 & 13 & 9 & 2 \\
 24 & 59 & 48 & 13
 \end{array}$$

So

$$S = 6 \times F_3^4 + 24 \times F_2^4 + (2^3 \times 59 + 2^2 \times 48 + 2^1 + 13)F_1^4.$$

Hence  $S = 6 \times F_3^4 + 24 \times F_2^4 + 690 \times F_1^4$ .

**Remark 1.** If  $F_1 = a, F_2 = b$  and  $F_n = F_{n-1} + F_{n-2}; n \geq 2$  where  $a$  and  $b$  are any two positive integers, then the resultant sequence is a generalized Fibonacci sequence and Theorem 1 can apply to this sequence.

**Remark 2.** As per Theorem 1, the product of three consecutive Fibonacci numbers can be denoted as the sum of third power of Fibonacci numbers. Now, the coefficients (except the coefficient of  $F_1$ ) for the product of three consecutive Fibonacci numbers are given by

$$u_n = \frac{1}{2} \left\{ (1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right\}; n = 2,3.$$

From Theorem 1 we can write the product of three consecutive Fibonacci numbers as  $\sum_{i=2}^k u_{k-i+1} F_i^3 + uk+1F13$ . Consider the pyramid

$$\begin{array}{ccc}
 & 1 & \\
 & 1 & 1 \\
 2 & 3 & 1
 \end{array}$$

for this product. Take  $u_i = a_i + b_i$  where  $a_1 = 3, b_1 = 1$  and  $a_i = 2a_{i-1} + b_{i-1}; b_i = a_{i-1}$ . Consider the polynomial

$$x^2 = 2x + 1 \quad (5)$$

By multiplying  $x$  on both sides, we get  $x^3 = 2x^2 + x = 2(2x + 1) + x = 5x + 2$ . Similarly, we get  $x^4 = 12x + 5$ . Repeat this process. In general, we can write

$$x^n = c_n x + c_{n-1}; c_n = 2c_{n-1} + c_{n-2}$$

whereas observed in (5). We use induction to prove that each term in the series  $u_n$  is twice of each term in the series  $c_n$ . For, it is enough to prove that

$$\left. \begin{aligned}
 a_n &= c_{n+1} + c_n \\
 b_n &= c_{n+1} - c_n
 \end{aligned} \right\} \quad (6)$$

This is because,  $u_n = a_n + b_n = c_n + c_{n+1} + c_{n+1} - c_n = 2c_{n+1}$ .

To prove (6), let  $n = 1$ . Then  $c_1 = 1, c_2 = 2$  and  $a_1 = 3, b_1 = 1$ . Also note that  $a_1 = c_2 + c_1$  and  $b_1 = c_2 - c_1$ . Now assume that the result is true for some  $n$ . That is, we have  $a_n = c_{n+1} + c_n$  and  $b_n = c_{n+1} - c_n$ . Now consider  $a_{n+1}$  and  $b_{n+1}$ . We know that  $a_{n+1} = 2a_n + b_n$  and  $b_{n+1} = b_n$ . From the induction hypothesis,

$$\begin{aligned}
 a_{n+1} &= 2(c_{n+1} + c_n) + c_{n+1} - c_n \text{ and} \\
 b_{n+1} &= c_{n+1} - c_n
 \end{aligned}$$

So,  $a_{n+1} = 3c_{n+1} + c_n = (2c_{n+1} + c_n) + c_{n+1} = c_{n+2} + c_{n+1}$ . Also, note that  $c_{n+2} = 2c_{n+1} + c_n$ . So,  $c_{n+2} - c_{n+1} = c_{n+1} + c_n$ . Thus, we get  $b_{n+1} = c_{n+1} + c_n = c_{n+2} - c_{n+1}$ . Thus, by induction, we proved (6).

We know that the solutions of (5) are  $1 + \sqrt{2}$  and  $1 - \sqrt{2}$  and let it be  $p$  and  $q$  which satisfy the equation  $x^n = c_n x + c_{n-1}$ . Therefore,  $p^n = c_{n+1}p + c_n$  and  $q^n = c_{n+1}q + c_n$ . This implies that  $c_{n+1} = \frac{p^n - q^n}{p - q}$ .

Thus,

$$c_{n+1} = \frac{1}{2\sqrt{2}} \left[ (1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right].$$

Hence

$$u_n = \frac{1}{2} \left\{ (1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right\}$$

**III. CONCLUSION**

We proved that the product of  $k$  consecutive Fibonacci numbers can be depicted as the sum of  $k$ th power of Fibonacci numbers.



We extended this result to the case of generalized Fibonacci sequence. Also, we developed a formula for finding the coefficients of the Fibonacci numbers appearing in the obtained linear combination. In particular, we developed one more formula for the coefficients of the product of three successive Fibonacci numbers. Also, we pointed out some applications of Fibonacci numbers. We left open the problem of finding the method for the coefficients in the case of generalized Fibonacci sequence.

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