T-Span, T-Edge Span Critical Graphs

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Abstract—Given a graph $G = (V, E)$ and a finite set $T$ of positive integers containing 0, a $T$-coloring of $G$ is a function $f : V(G) \rightarrow Z^* \cup \{0\}$ for all $u \neq w$ in $V(G)$ such that if $uw \in E(G)$ then $f(u) - f(w) \notin T$. For a $T$-coloring $f$ of $G$, the $f$-span $sp_f(G)$ is the maximum value of $|f(u) - f(w)|$ over all pairs $u, w$ of vertices of $G$. The $T$-span $sp_T(G)$ is the minimum $f$-span over all $T$-colorings $f$ of $G$. The $f$-edge span $esp_f(G)$ of a $T$-coloring is the maximum value of $|f(u) - f(w)|$ over all edges $uw$ of $G$. The $T$-edge span $esp_T(G)$ is the minimum $f$-edge span over all $T$-colorings $f$ of $G$. It is known that $sp_T(H) \leq sp_T(G)$ and $esp_T(H) \leq esp_T(G)$ for every graph $G$. In this paper we classify graphs containing a sub graph $H$ such that $sp_T(H) \leq sp_T(G)$ and $esp_T(H) < esp_T(G)$. Also we discuss the Mycieleiskian of $T$-coloring.

Key words: $T$-coloring, $T$-span, $T$-edge spanAMS subject classification 05C15

I. INTRODUCTION AND DEFINITIONS

All graphs considered in this paper are finite, simple and undirected. All the definitions which are not discussed in this paper one may refer [1]. Let $G = (V, E)$ be a graph and let $T$ be any finite set of positive integers containing 0. A $T$-coloring of $G$ is a function $f : V(G) \rightarrow Z^* \cup \{0\}$ for all $u \neq w$ in $V(G)$ such that if $uw \in E(G)$ then $f(u) - f(w) \notin T$. For a $T$-coloring $f$ of $G$, the $f$-span $sp_f(G)$ is the maximum value of $|f(u) - f(w)|$ over all pairs $u, w$ of vertices of $G$. The $T$-span $sp_T(G)$ is the minimum $f$-span over all $T$-colorings $f$ of $G$. The $f$-edge span $esp_f(G)$ of a $T$-coloring is the maximum value of $|f(u) - f(w)|$ over all edges $uw$ of $G$. The $T$-edge span $esp_T(G)$ is the minimum $f$-edge span over all $T$-colorings $f$ of $G$. In this paper we consider these sets.

II. MAIN RESULTS

Definition 2.1. A subset $V'$ of the vertex set $V(G)$ of a connected graph $G$ is called span altering set if $sp_T(G-V') < sp_T(G)$.

Definition 2.2. A subset $V'$ of the vertex set $V(G)$ of a connected graph $G$ is called edge span altering set if $esp_T(G-V') < esp_T(G)$.

In their papers of Hale and Robert, defined a parameter $sp_T(G)$ which is trying to save the spectrum when the frequencies allotted to the transmitter. According to the $T$-coloring problems vertices are considered as transmitters. Thus removal of one transmitter may reduce more span which is related to spectrum and more edge span which is related to bandwidth in such a way that we define the removal of some vertices implies $sp_T(G-V') < sp_T(G)$ and $esp_T(G-V') < esp_T(G)$ where $V'$ is a proper subset of $V(G)$. For more results on $T$-coloring one may refer [3,5,6,7,8] and survey on $T$-coloring as discussed in [9].

Previous Results:

Theorem 1.1. [6] Let $W_n$ be a Wheel graph with $n$ even. For a $k$-multiple of $s$ set $T$,

$$sp_T(W_n) = sp_T(W_n) = 3k + 3s = 1$$

$$sp_T(W_n) = sp_T(W_n) = 2k + 3s = 2$$

$$sp_T(W_n) = sp_T(W_n) = 3k + 3s = 3$$

$$sp_T(W_n) = sp_T(W_n) = 3s \geq 4$$

Theorem 1.2. [6] Let $W_n$ be a Wheel graph with $n$ odd. For $T$ is a $k$-multiple of $s$ set,

$$sp_T(W_n) = sp_T(W_n) = 2k + 2s = 1,2$$

$$sp_T(W_n) = sp_T(W_n) = 2s \geq 3$$

Theorem 1.3. [8] For any odd cycle $C_n$ and $T = \{1,2, \ldots , k-1\}$, $esp_T(G) = \left(\frac{n}{k}\right)$.

Theorem 1.4. [5] For any odd cycle $C_n$ and a $k$-multiple of $s$ set $T$,

$$sp_T(C_n) = sp_T(C_n) = 2k + 2s = 2$$

$$sp_T(C_n) = sp_T(C_n) = 2s \geq 3$$

Theorem 1.5. [2] If $G$ is weakly $\gamma$ perfect, then for all sets $T$, $esp_T(G) = sp_T(G) = sp_T(K_{\gamma}(G))$.

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Theorem 2.2. Let $W_n$ be a wheel graph with $n$ even. For a $k$-multiple of $s$ set $T$, $\zeta(W_n) = \zeta(W_n) = 1$.

**Proof.** Let $T$ be a $k$-multiple of $n$ set. Using Theorem 1.1, $sp_T(W_n) = esp_T(W_n) = 3k + 3$ as $s = 1$ and $sp_T(W_n) = esp_T(W_n) = 3k + 3$ as $s = 3$ and $sp_T(W_n) = esp_T(W_n) = 3k + 3$ as $s = 4$. Now remove the maximum degree vertex of $W_n$ which causes more interferences, it reduces to an odd cycle by Theorem 1.3 and Theorem 1.4. $sp_T(C_n) < sp_T(W_n)$ and $esp_T(C_n) < esp_T(W_n)$ Hence, $\zeta(W_n) = \zeta(W_n) = 1$, $n$ is even.

Theorem 2.3. Let $W_n$ be a wheel graph with $n$ odd. For a $k$-multiple of $s$ set $T$, $\zeta(W_n) = \zeta(W_n) = 1$.

**Proof.** Let $T$ be a $k$-multiple of $n$ set. Invoking Theorem 1.2, $sp_T(W_n) = esp_T(W_n) = 2k + 2$ as $s = 1,2$ and $sp_T(W_n) = esp_T(W_n) = 3k + 3$ as $s = 3$. Now remove the maximum degree vertex of $W_n$ reduces to an odd cycle by Theorem 1.3 and Theorem 1.4. $sp_T(C_n) < sp_T(W_n)$ and $esp_T(C_n) < esp_T(W_n)$ Hence, $\zeta(W_n) = \zeta(W_n) = 1$, $n$ is odd.

Theorem 2.4. Let $C_n$ be a cycle with $n$ odd. For a $k$-multiple of $s$ set $T$, $\zeta(C_n) = \zeta(C_n) = 1$.

**Proof.** Let $T$ be a $k$-multiple of $n$ set. Remove any one of the vertices of $C_n$ implies a tree $T$. Since tree is a weakly $\gamma$ perfect using Theorem 1.5, $sp_T(T) = sp_T(K_n)$. When $s = 1$, $sp_T(K_2) = k + 1 < sp_T(C_n) = 2k + 2$ and $esp_T(K_2) = k + 1 < sp_T(C_n) = 2k + 2$ and when $s = 2, sp_T(K_2) = esp_T(K_2) = 1 < sp_T(C_n) = esp_T(C_n) = 2$. Hence, $\zeta(C_n) = \zeta(C_n) = 1$.

Theorem 2.5. Let $G$ be a bipartite graph. Then $\zeta(G) = \zeta(G) = n - 1$.

**Proof.** Since $G$ is a bipartite graph which is weakly $\gamma$ perfect using Theorem 1.5, $sp_T(G) = esp_T(K_2)$, $sp_T(K_2) = esp_T(K_2)$. Therefore the only way to reduce span and edge span is remove $n-1$ vertices from $G$. Hence, $\zeta(G) = \zeta(G) = n - 1$.

The following observation is immediate:

(i) $\zeta(G) = 1$ or $\zeta(G) = 1$ if and only if there is an optimal T-coloring in which largest integer vertex $v$ is in a singleton color class.

Theorem 2.6. If $G$ is a graph with chromatic number $k \geq 3$ then $1 \leq \zeta(G) \leq n - 2$ and $1 \leq \zeta(G) \leq n - 2$.

**Proof.** In any graph $G$ with chromatic number $k \geq 3$ contains an odd cycle. Now remove $n-2$ vertices results a $K_2$. Hence, $1 \leq \zeta(G) \leq n - 2$ and $1 \leq \zeta(G) \leq n - 2$.

Theorem 2.7. Let $T$ be a $k$-multiple of $s$ set. If $G$ is a non-perfect graph then $1 \leq \zeta(G) \leq n - \omega(G)$.

**Proof.** Let $G$ be a non-perfect graph. Thus $\chi(G) > \omega(G)$. Label the vertices of $\omega(G)$ by $S = \{v_1, v_2, ..., v_\omega\}$. Let $V_0 = V(G) - S$. Hence, we have $sp_T(G - V_0) < sp_T(G)$. This bound is not attained and the converse is not true.

Theorem 2.8. Not all non perfect graphs has $\zeta(G) = 1$.

**Proof.** Petersen graph is not perfect $\omega(G) = 2$ and $\chi(G) = 3$ and $\zeta(G) = 2$. Since $sp_T(P) = esp_T(P) = 2k + 2$ as $s = 1,2$ and $sp_T(P) = esp_T(P) = 2k + 2$ as $s = 3$ which is greater than $sp_T(K_2) = esp_T(K_2) = k + 1$ as $s = 1$ and $sp_T(K_2) = esp_T(K_2) = k + 1$ as $s = 2$.

Theorem 2.9. Let $G$ be either unicyclic graph or hamiltonian-triangle graph. Let $T$ be a $k$-multiple of $n$ set. Then $\zeta(G) = 1$.

**Proof.** Let $G$ be an unicyclic graph. Then removal of a single vertex in that cycle produces a tree. Hence $\zeta(G) = 1$. Let $G$ be a graph that contains a triangle and an odd cycle on $n$ vertices. Thus $G$ has at most $\frac{n}{2}$ edges. Now removal of a maximum degree vertex $v$ results $\chi(G - v) < \chi(G)$. Which in turn implies that $sp_T(G - v) < sp_T(G)$. Hence, $\zeta(G) = 1$.

Theorem 2.10. Let $T$ be a $k$-multiple of $s$ set. The span critical number of a critical graph is one.

**Proof.** Let $G$ be a critical graph and $T$ be a $k$-multiple of $s$ set. Then $\chi(H) < \chi(G)$ for every proper subgraph $H$ of $G$. Thus removal of every single vertex implies that $sp_T(G - \nu < sp_T(G)$. Hence, $\zeta(G) = 1$.

Theorem 2.11. Every graph with $\zeta(G) = 1$ contains an odd cycle but the converse is not true.

**Proof.** Let $P$ be a Petersen graph with $\chi(P) = 3$ and $\zeta(G) = 2$.

Theorem 2.12. Let $\zeta(G_1) = 1$ and $\zeta(G_2) = 1$. Then the span critical number of cartesian product of $G_1$ and $G_2$ need not be equal to 1.

**Proof.** The cartesian product $G_1 \square G_2$ of two graphs $G_1$ and $G_2$ contains sub graphs that are isomorphic to both $G_1$ and $G_2$. Also we know that every graph with $\zeta(G) = 1$ contains an odd cycle. Thus $G_1$ and $G_2$ contains an odd cycle. Suppose $G_1 = C_m$ and $G_1 = C_n$, $m$, $n$ is odd. In $G_1 \square G_2$ we have $G_1$ copies of $G_2$ and $G_2$ copies of $G_1$. There are $m$ number of vertex disjoint cycles or $n$ number of vertex disjoint cycles will appear. Thus removal of single vertex will not reduce the span.

Theorem 2.13. Let $\zeta(G_1) = 1$ and $\zeta(G_2) = 1$. Then the edge span critical number of cartesian product of $G_1$ and $G_2$ need not equal to 1.

**Proof.** Proof is similar that of Theorem 2.12.

III. T-COLORING OF MYCELIASKIAN GRAPHS

One may refer for Myceliaskian Construction[11]. The construction is as follows:

Let $G$ be a triangle free graph with $\chi(G) = k$ and $V(G) = \{v_1,v_2,...,v_n\}$. Construct a graph $G'$ as follows. Let $V(G') = \{v_1,v_2,...,v_n\}$ and $E'(G') = E(G)$ and for every $u_i$, add edges from $u_i$ to all of $v_i$'s neighbours. In addition, take vertex $w$ and made adjacent to all the $u_i$’s. We call $w$ is the central vertex of $G'(G)[11]$. We denote Myceliaskian of graph $G$ by $\mu(G)$.

Previous results:

Theorem 3.1[2] For all graphs $G$ and all $T$-sets, (i) $\chi_T(G) = \chi(G)$.

(ii) $sp_T(K_{\omega(G)}) \leq sp_T(G) \leq esp_T(G) \leq sp_T(K_G)$.

Theorem 3.2[2] If $T$ is a $k$-initial set, then $sp_T(G) = esp_T(K_{\omega(G)})$.

Theorem 3.3 [10] If $T$ is a $k$ multiple of $s$ set, then for all graphs $G$,

$\chi_T(G) = \chi(T(K_{\omega(G)}))$

$\{st + sk - sk - 1$ if $\chi(G) = st$

$\{st + sk + m - 1$ if $\chi(G) = st + m, 1 \leq m \leq s - 1,$

$t \in \{1,2,3,...\}$

Theorem 3.4 [7] If $T$ is a $k$-multiple-of-s-set $T$, and
χ(G) ≤ s then spT(G) = espT(G) = χ(G) − 1.

Theorem 3.14 [5] Let H be a sub graph of a graph G. For each finite set of non-negative integers containing 0, (i) spT(H) ≤ spT(G), (ii) espT(H) ≤ espT(G).

Proof. We consider two cases.
Case 1. χ(G) + 1 = st.
spT(μ(G)) = spT(K_χ(G+1)) = (χ(G) + 1 − s)k + χ(G) = (χ(G) − s)k + χ(G) + k = spT(G) + k + 1
Case 2. χ(G) + 1 = st + m.
spT(μ(G)) = spT(K_χ(G+1)) = (χ(G) + 1 − m)k + χ(G) + k = (χ(G) − m)k + χ(G) + k = spT(G) + k + 1.

Corollary 3.15. If T is a k-multiple of s set and χ(G) + 1 ≤ s, then spT(μ(G)) = espT(μ(G)) = y(G).

Proof. Proof is immediate from Theorem 3.4.

Lemma 3.16. Let G be any graph with χ(G) = 2 and T = {0, 2, 4, ..., 2k} then, Then spT(μ(G)) = espT(μ(G)) = 2(k + 1).

Proof. Invoking Theorem 3.14, spT(μ(G)) = 2(k + 1).

Lemma 3.17. Let G be any graph with χ(G) = 2. If T is a k-multiple of 2 set, then spT(μ(G)) = espT(μ(G)) = 2(k + 1).

Proof. By Theorem 3.14, spT(μ(G)) = 2k + 2. By Theorem 3.1 (ii) we have espT(μ(G)) ≤ 2k + 2. By Lemma 3.16, we have if T = {0, 2, 4, ..., 2k}, there is no T-coloring function f of G satisfying the edge span < 2(k + 1). Then for any k-multiple of 2 sets T there is no T-coloring function f with edge span < 2(k + 1). Therefore espT(μ(G)) ≥ 2k + 2. Hence espT(μ(G)) = spT(μ(G)) = 2k + 2.

We end up the sections with the following open questions and conjecture.

(i) Prove or disprove: The edge span critical number of a critical graph is one.
(ii) For all graphs G, 1 ≤ ζ(G) ≤ number of disjoint odd cycles of G.
(iii) Classify the graphs, which have ζ(G) = ζ(G)?
(iv) Prove or disprove: ζ(G) = the number of vertex disjoint odd cycle of a graph G.
(v) Let G be any graph. Compute ζ(μ(G)) and ζ(μ(G)).

Conjecture: Let G be any graph and spT(G) = espT(G) if and only if ζ(G) = ζ′(G).
IV. CONCLUSION

In this paper, we classify which graphs containing a subgraph \( H \) such that \( sp_T(H) < sp_T(G) \) and \( esp_T(H) < esp_T(G) \). Also we discuss the Mycielskian of \( T \)-coloring.

REFERENCES


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