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Northward Transport in a Stress Driven Flow Confined to a Porous Medium (Sverdrup Relation in a Porous Medium)

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***Abstract:** We obtain an expression for the total northward mass transport in a porous bed of large extant subject to a stress applied at the top free surface using β -plane approximation. It is found that β -effect is found to be responsible for western intensification of ocean currents. The total northward transport of fluid (See Sverdrup [11]) is found to be closed through the western boundary layer (See: Stommel [4]) in the presence of porous medium.*

***Keywords :** Porous medium, Porous medium, Coriolis force, Rossby waves, β -plane approximation.*

I. INTRODUCTION

Geophysical fluid dynamics deals with all naturally occurring fluid motions having a large length scales. Geophysical flows such as motions in atmosphere, oceans, earth's core and interstellar system connected with fluid motions fall under this branch of fluid mechanics. The large-scale motions are significantly influenced by the earth's rotation; the Coriolis force plays an important part.

Boundary layer flow problems are connected with the frictional effects. In the case of the two most important fluids, namely air and water, the viscosity is very small, and hence the viscosity effects may not be perceptible except near the boundaries. Sometimes the deriving force itself may be of frictional origin, as in the case of wind driven ocean circulation.

In general the vertical friction layers which occur on side boundaries differ from that of the Ekman layers. Also, vertical boundary layers become important while studying motions in the oceans, earth's core, and stellar bodies such as sun and stars, because they should close meridional circulation of all fluid motions. Proudman [6], Stewartson [7] ,Pedlosky [2], Stommel [4] and also several authors studied the vertical boundary layers. Proudman [6] considered the motion of an incompressible viscous fluid confined between two differentially rotating concentric spheres. Stewartson [7] investigated the problem of steady flow between two co-axial planes rotating almost as if rigid by integral transform techniques. He obtained one outer vertical shear layer of thickness $O(E^{1/4})$ and an inner layer of thickness $O(E^{1/3})$, which transports the mass flux from one Ekman layer to another. Greenspan [5] investigated vertical free shear layers by combining both Fourier transform and boundary layer techniques, and using Ekman compatibility condition. His results agree with those obtained by Stewartson [7]. Hydrodynamic and Hydromagnetic vertical shear layer problems have been analyzed by several authors. (For example: Howard [8], Kroll[9],Nield[10], S.Vempaty and Sundaram [1,3]).Many authors discussed about the rotating fluid due to an applied stress and tangential stress at the free surface(for examples: K. Jagadeshkumar, V. Somaraju and S. Srinivas, Satyanarayana Badeti,[12,13,14,15].

The frictional force and its distribution, gives the wind-driven oceanic circulation. Infact, the stress on the ocean surface produces an important effect on the oceanic circulation. The winds blowing on the surface of ocean produce ocean currents. The effect of variation of the Coriolis parameter with latitude is important in ocean currents. The β -plane approximation takes care of the linear variation of the Coriolis parameter f with latitude. In this paper is concerned with the dynamics of rotating saturated porous medium using β -plane approximation.

Quantitatively, the Coriolis force is expressed as the product of velocity and a factor known as the Coriolis parameter $f = 2\Omega \sin\phi$, where Ω denotes the earth's angular velocity, and ϕ is the latitude. f is called the Coriolis parameter. The Coriolis parameter f is very important in many applications. Consider a rectilinear $(\hat{x}, \hat{y})$ plane to the spherical earth at ϕ_0 , with $y(\phi_0) = 0$ and $f(\phi_0) = f_0$. We write f in a Taylor series as, $f(y) = f_0 + \left.\frac{df}{a\, d\phi}\right|_{\phi_0} y + ..$. We neglect the second or higher order terms. Because f's $y-$derivative is evaluated at ϕ_0, we get $f(y) = f_0 + \beta y$,

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where $\beta = \frac{df}{a\,d\phi}\Big|_{\phi_0} = \frac{2\Omega}{a}\cos\phi_0$ here a is the earth's radius.

Modelling variation of Coriolis parameter as $f = f_0 + \beta y$ is called β-plane approximation. β-effect is found to be responsible for western intensification of ocean currents.

II. GOVERNING EQUATIONS

The governing equations for the steady flow of an incompressible fluid in a rotating porous medium referred to a frame rotating with an angular velocity $\Omega\hat{k}$ are given below:

Momentum equation:

$$\vec{V}.\nabla\vec{V} + 2\Omega\hat{k}\times\vec{V} = -\frac{1}{\rho}\nabla(P - \frac{1}{2}\rho\Omega^2 r^2) - \frac{\mu}{k}\vec{V} - \upsilon\nabla^2\vec{V} \quad (1)$$

Conservation of Mass:

$$\nabla.\vec{V} = 0 \quad (2)$$

Here $\vec{V}$ denotes the velocity field, ρ is the density, P is the pressure, r is the radial distance and $\hat{k}$ a unit vector in the axial direction, μ is the viscosity and $\upsilon(=\mu/\rho)$ is the kinetic viscosity. The centrifugal acceleration is combined with pressure P to form the reduced pressure p. The mathematical formulation of the problem is presented here.

III. MATHEMATICAL FORMULATION

We shall discuss the total northward mass transport in a porous medium due to stress at the top free surface. We study this problem in Cartesian coordinate system. The Coriolis parameter is considered to be varying with respect to latitude. The governing equation in a rotating porous medium may be written in the Cartesian system of coordinates as,

$$-f\,\hat{k}\times\vec{V} = -\nabla p - \Lambda\vec{V} + E\nabla^2\vec{V} \quad (3)$$

Here, f is the variable. For small variations in latitude we approximate nondimensional f by,

$$f = 1 + \beta\, y$$

We propose to discuss a simple relation between the stress field and the Northward transport of fluid in a saturated porous medium including the effect of Ekman pumping. The governing equations in a porous medium may be written as *(f can vary with latitude y)*,

$$-vf = -p_x - \Lambda u + E\nabla^2 u \quad (4)$$

$$uf = -p_y - \Lambda v + E\nabla^2 v \quad (5)$$

$$0 = -p_z + \Lambda w - E\nabla^2 w \quad (6)$$

The interior solutions are

$$-vf = -p_x - \Lambda u \quad (7)$$

$$uf = -p_y - \Lambda v \quad (8)$$

$$0 = -p_z - \Lambda w \quad (9)$$

Using β-plane approximation and eliminating 'p' we get from (7) and (9). The vorticity equation,

$$\Lambda\zeta = f\frac{\partial w}{\partial z} - \beta^* v \quad (10)$$

Here, $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

β^* – is the rate of change of the Coriolis parameter.

$f = 1 + \beta^* y$

As $\Lambda \to 0$, this equation reduces to that derived for pure hydrodynamic case.

IV. EKMAN LAYER AT A FREE SURFACE

The equations (4) and (5) becomes

$$-vf = -\Lambda u + u_{\xi\xi} \quad (11)$$

$$uf = -\Lambda v + v_{\xi\xi} \quad (12)$$

$$E^{-1/2}\frac{\partial w}{\partial \xi} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Let the x and y components of the stress $\vec{\tau}$ be denoted by τ_x and τ_y. The boundary conditions on vertical velocity is $w = 0$ at $z = 1$.

In the upper Ekman layer, let us write,

$$E^{1/2}\xi = 1 - z$$

From (1) and (2)

$$-v = \frac{1}{f}[-\Lambda u + u_{\xi\xi}] \quad (13)$$

$$u = \frac{1}{f}[-\Lambda v + v_{\xi\xi}] \quad (14)$$

Using (13) and (14) we get

$$u_{\xi\xi\xi\xi} - 2\Lambda u_{\xi\xi} + (f^2 + \Lambda^2)u = 0 \quad (15)$$

Where $\beta,\gamma = \left(\frac{(\Lambda^2 + f^2)^{1/2} \pm \Lambda}{2}\right)^{1/2}$

$$\beta^2 - \gamma^2 = \Lambda,\ \beta^2 + \gamma^2 = f,$$

$$(\beta^2 + \gamma^2)^2 = f^2$$

$$u = e^{-\beta\xi}[C_1\cos\gamma\xi + C_2\sin\gamma\xi] \quad (16)$$

$$v = e^{-\beta\xi}[-C_1\sin\gamma\xi + C_2\cos\gamma\xi] \quad (17)$$

Subject to the conditions

$\frac{\partial u}{\partial \xi} = -\tau_x$, $\frac{\partial v}{\partial \xi} = -\tau_y$, $w = 0$ at $\xi = 1$

$$\tau_x = \beta C_1 - \gamma C_2 \quad (18)$$

$$\tau_y = \gamma C_1 + \beta C_2 \quad (19)$$

From (18) and (19) we get

$$C_1 = \frac{1}{(\beta^2 + \gamma^2)}(\beta\tau_x + \gamma\tau_y) \quad (20)$$

$$C_2 = -\frac{1}{(\beta^2+\gamma^2)}(\gamma\tau_x - \beta\tau_y) \qquad (21)$$

Substitute C_1 and C_2 in (16) and (17),

$$u = \frac{e^{-\beta\xi}}{(\beta^2+\gamma^2)}\left[(\beta\tau_x+\gamma\tau_y)\cos\gamma\xi - (\gamma\tau_x-\beta\tau_y)\sin\gamma\xi\right] \qquad (22)$$

$$v = \frac{e^{-\beta\xi}}{(\beta^2+\gamma^2)}\left[(\beta\tau_y-\gamma\tau_x)\cos\gamma\xi - (\beta\tau_x+\gamma\tau_y)\sin\gamma\xi\right] \qquad (23)$$

The results will naturally be the same as those presented by S.Vempaty and R. Balasubramanian [3] for the cylindrical geometry at the free surface Ekman boundary layer in the presence of magnetic field.

The northward mass transport in the Ekman layer is given

$$T_{NE} = E^{1/2}\int_0^\infty v\,d\xi$$

$$= E^{1/2}\int_0^\infty \frac{e^{-\beta\xi}}{(\beta^2+\gamma^2)}\begin{bmatrix}(\beta\tau_y-\gamma\tau_x)\cos\gamma\xi \\ -(\beta\tau_x+\gamma\tau_y)\sin\gamma\xi\end{bmatrix}d\xi$$

$$T_{NE} = \frac{E^{1/2}}{(\beta^2+\gamma^2)^2}\left[\Lambda\tau_y - f\tau_x\right]$$

Where $f = 1+\beta^* y$

Integrating the continuity equation

$$E^{-1/2}\frac{\partial w}{\partial\xi} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

We have

$$w\big|_{z=0} = \frac{E^{1/2}}{(\beta^2+\gamma^2)^2}\left[\Lambda\nabla.\underline{P} + f\hat{k}.\nabla\times\underline{P} + \beta^*\tau_x\right] \qquad (24)$$

Integration of (7) over the whole depth of the porous bed gives

$$w\big|_{z=0} = \frac{\beta^* V}{f} + \frac{\Lambda}{f}\zeta \qquad (25)$$

$$V = \frac{E^{1/2}}{(\beta^2+\gamma^2)^2}\left[\frac{f^2}{\beta^*}\hat{k}.\nabla\times\underline{P} + \frac{f\Lambda}{\beta^*}\nabla.\underline{P} + f\tau_x\right] - \frac{\Lambda\xi}{\beta^*}$$

(26)

We obtain the total northwards transports, namely

$$T_N = T_N^{Ekman} + \int_0^{h=1} v\,dz$$

$$T_N = \frac{E^{1/2}}{(\beta^2+\gamma^2)^2}\left[\frac{f^2}{\beta^*}\hat{k}.\nabla\times\underline{P} + \Lambda\left(\tau_y + \frac{f}{\beta^*}\nabla.\underline{P}\right)\right] - \frac{\Lambda\zeta}{\beta^*}$$

(27)

Using vector identities we get,

$$T_N = \frac{E^{1/2}}{\beta^*(\beta^2+\gamma^2)^2}\left[f^2\hat{k}.\nabla\times\underline{P} + \Lambda(\nabla.f\,\underline{P})\right] - \frac{\Lambda\zeta}{\beta^*} \qquad (28)$$

This is the expression for total northward transport in a large depth porous bed.

V.RESULTS AND CONCLUSION

In this paper, we discussed some effects of the stress on flow in a porous medium and obtain the expression for a Northward transport (see: equation (28)). It is found that as $\Lambda \to 0$, our expression reduces to the familiar Sverdrup relation (see: Sverdrup [11]). It is expected that the Northward transport is to be closed through the western boundary layer.

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