Aluthge Transformation and *- Aluthge Transformation on M class A_k^* Operator

P.Shanmugapriya, P. Maheswari Naik

Abstract*—Research works on Operators in Complex Hilbert spaces has been the interest of budding researchers in the recent years. In 1996, Furuta et al studied Aluthge transformation on p-hyponormal operators. Later, in 2001 yamazaki et al studied Aluthge transformation and powers of operators for class A(k)operator. This work was further carried over by Pannayappan et al and D.Senthil kumar et al. In this school work, we studied Aluthge transformation and *- Aluthge transformation for the new class of operator named M class A_k^* operator on a non-zero Complex Hilbert space.

Key Words: Class A_k^* operator, M-class A_k^* operator, Aluthge transformation

I. INTRODUCTION

The Banach algebra on a non-zero complex Hilbert space H of all bounded linear operators are denoted by B(H). An operator L is defined as an element in B(H). If L belongs to B(H), then L^* means the adjoint of L in B(H). Weyl and Weyl type theorems where studied for the following class of operators. Furuta et al introduced class A(k), k > 0 as a class of operators and extended p-hyponormal and log-hyponormal operators. They studied Weyl and Weyl type theorems for the above operators [10]. Later, Panayappan et al extended this concept and introduced class A_k operators and verified Weyl's theorem [3]. In 2013, Panayappan et al introduced a new class of operators in a different manner called class A_k^* operator, quasi class A_k^* operators and studied Weyl and Weyl type theorems and also proved tensor product of two quasi class A_k^* operators are closed [4].

It is well known that an operator can be decomposed into T= U |T| where U is partial isometry. In 2015, D.Senthil kumar et al studied Aluthge transformation on N –Class A(k) operators [7]. They also studied Aluthge and *-Aluthge transformation of powers of N-class A(k) operators in 2016 [6]. The above research work kindles our interest on studying the Aluthge transformation for M-Class A_k^* operator.

Definiton 1.1 An operator L is called class A_k^* operator if

$$\left|L^k\right|^{\frac{2}{k}} \geq \left|L^*\right|^2 \quad \text{where k is a positive integer.}$$

If k = 1 then class A_k^* operator coincides with hyponormal operator [4].

Definition 1.2 An operator $L \in B(H)$ is said to be M-Class A_k^* operator if there exists positive real numbers M, k such that $\left|L^*\right|^2 \leq M\left(\left|L^k\right|^{\frac{2}{k}}\right)$ [9].

Proposition: 1.3.

If M = 1, then M-Class A_k^* operator coincides with class A_k^* operator.

If M = 1 and k = 1, then M-Class A_k^* operator coincides with hyponormal operator.

Hence, Hyponormal operator $\Rightarrow$ class A_k^* operator $\Rightarrow$ M-Class A_k^* operator.

In the next section, we studied Aluthge Transformation for M-Class A_k^*operator.

II ALUTHGE TRANSFORMATION ON M CLASS AK* OPERATOR

Assume that L is a bounded linear operator on a complex Hilbert space H. In [1], Aluthge introduced the $\tilde{L}$ operator for an operator L with its polar decomposition $L = U|L| = |L^*|U$ and Takashi [10] defined $\tilde{L}$ and $\tilde{L}^*$ as below:

$$\tilde{L}_{s,l} = |L|^s\, U\, |L|^l$$

$$\tilde{L}^*_{s,l} = (\tilde{L}_{s,l})^* = |L|^l\, U\, |L|^s$$

.**Theorem 2.2** An operator L is called M-Class A_k^* operator if and only if

$$\left\|L^* x\right\|^2 \leq M\left\|L^k x\right\|^{\frac{2}{k}}\left\|x\right\|^{\frac{2k-2}{k}} \text{ for all } x \in H.$$

Proof. We know that $\left|L^*\right|^2 \leq M\left(\left|L^k\right|^{\frac{2}{k}}\right)$

$$(LL^*) \leq M\left\{(L^{*k} L^k)\right\}^{\frac{1}{k}}$$

$$\langle LL^* x, x\rangle \leq M\left\langle \left\{(L^{*k} L^k)\right\}^{\frac{1}{k}} x, x\right\rangle$$

$$\langle L^* x, L^* x\rangle \leq M\left\langle \left\{(L^k x, L^k x)\right\}^{\frac{1}{k}}\right\rangle \|x\|^{\frac{2k-2}{k}}$$

(By Theorem 6, [7])

$$\|L^* x\|^2 \leq M\,\|L^k x\|^{\frac{2}{k}}\,\|x\|^{\frac{2k-2}{k}} \text{ for all } x \in H$$

Revised Manuscript Received on August 05, 2019.

P.Shanmugapriya, Department of Mathematics, Sri Ramakrishna Engineering College,Coimbatore, Tamil Nadu, India.
(E-mail: prsrinithin@gmail.com)
P. Maheswari Naik, Department of Mathematics, Sri Ramakrishna Engineering College,Coimbatore, Tamil Nadu, India.
(E-mail: maheswarinaik21@gmail.com)

Hence, proved.

Theorem 2.3 If L =U |L| and $L^* = U^*|L^*|$ is the polar decomposition of L, then L is M-Class A_k^* operator.

Proof. By the definition of M-Class A_k^* operator ,

$$\left(U|L|U^*|L*|\right) \le M\left\{\left(U^{*k}|L*^k|U^k|L^k|\right)\right\}^{\frac{1}{k}}$$

$$\left(|L*|UU*|L*|\right) \le M\left\{\left(|L^k|U*^k U^k|L^k|\right)\right\}^{\frac{1}{k}}$$

$$\left(|L*|^2\right) \le M|L^k|^{\frac{2}{k}}$$

So if L =U |L| and $L^* = U^*|L^*|$ is the polar decomposition of L then it is M-Class A_k^* operator.

Theorem 2.4 If L is M-Class A_k^* operator and S is an unitary operator such that LS = SL, then C = LS is also M-Class A_k^* operator.

Proof. By M-Class A_k^* operator definition,

$$\left(CC*\right) \le M\left\{\left(C^{*k}C^k\right)\right\}^{\frac{1}{k}}$$

$$\left(L\ S\ S^*L^*\right) \le M\left\{\left(L^{*k}S^{*k}S^kL^k\right)\right\}^{\frac{1}{k}}$$

$$|L^*|^2 \le M|L^k|^{\frac{2}{k}}.$$

Hence C = LS is also M-Class A_k^* operator.

Theorem 2.5 Let A and β be positive operators. Then for each $p \ge 0$ and $r \ge 0$ the following assertions hold:[2]

1. If $\left(\beta^{\frac{r}{2}} A^p \beta^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \ge \beta^r$ then $\left(\beta^{\frac{p}{2}} A^r \beta^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \le A^p$

2. If $\left(\beta^{\frac{p}{2}} A^r \beta^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \le A^p$ and $N(A) \subset N(\beta)$ then

$$\left(\beta^{\frac{r}{2}} A^p \beta^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \ge \beta^r$$

Theorem 2.6 Let L = U |L| be the polar decomposition of L is M-Class A_k^* operator for 0 < p < 1, then $\tilde{L}_{s,l} = |L|^s U |L|^t$ is 2 (p + min(s, t)) M-Class A_k^* operator for s, t > 0 such that max(s, t) ≥ p and U*=U .

Proof. By M-Class A_k^* operator definition,

$$\left(\tilde{L}_{s,l}\tilde{L}^*_{s,l}\right)^{\frac{p+\min(s,l)}{s+l}} \le M\left[\left\{\tilde{L}^{*k}_{s,l}\tilde{L}^k_{s,l}\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}$$

$$\left(|L|^s U|L|^l|L|^l U^*|L|^s\right)^{\frac{p+\min(s,l)}{s+l}} \le$$

$$M\left[\left\{\left(|L|^l U^*|L|^s|L|^s U|L|^l\right)^k\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}$$

$$\left(U|L^*|^s|L|^{2l}|L^*|^s U^*\right)^{\frac{p+\min(s,l)}{s+l}} \le$$

$$M\left[\left\{\left(U^*|L^*|^l|L|^{2s}|L^*|^l U\right)^k\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}$$

$$U\left(|L^*|^s|L|^{2l}|L^*|^s\right)^{\frac{p+\min(s,l)}{s+l}}U^* \le$$

$$MU^*\left[\left\{\left(|L^*|^l|L|^{2s}|L^*|^l\right)^k\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}U$$

$$U\left(B^{\frac{s}{2}}A^l B^{\frac{s}{2}}\right)^{\frac{p+\min(s,l)}{s+l}}U^* \le$$

$$MU^*\left[\left\{\left(B^{\frac{1}{2}}A^s B^{\frac{1}{2}}\right)^k\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}U$$

$$U\left(B^{s+l}\right)^{\frac{p+\min(s,l)}{s+l}}U^* \le$$

$$MU^*\left[\left\{\left(B^{s+l}\right)^k\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}U$$

by (Theorem F (3.2.1),[10])

$$U\left(|L^*|^{2(p+\min(s,l))}\right)U^* \le MU^*\left[\left\{\left(|L|\right)^k\right\}^{\frac{2(p+\min(s,l))}{k}}\right]U$$

$$\left(|L^*|^{2(p+\min(s,l))}\right) \le M\left[\left\{\left(|L|\right)^k\right\}^{\frac{2(p+\min(s,l))}{k}}\right]$$

Hence the proof.

Theorem 2.7 If L = U|| is M-class A_k^* operator for some positive real numbers M, k and U is isometry then $\tilde{L}$ is also M-class A_k^* operator.

Proof. Given L is M-class A_k^* operator,

Published By:
Blue Eyes Intelligence Engineering
& Sciences Publication

$$\left(U|L|U^*|L^*|\right) \le M\left\{\left(U^{*k}|L^{*k}|U^k|L^k|\right)\right\}^{\frac{1}{k}}$$

$$\left(U|L|^{\frac12}|L|^{\frac12}U^*|L^*|^{\frac12}|L^*|^{\frac12}\right)^2 \le M\left\{\left(U^*|L^*|^{\frac12}|L^*|^{\frac12}U|L|^{\frac12}|L|^{\frac12}\right)^k\right\}^{\frac2k}$$

$$\left(|L^*|^{\frac12}U|L|^{\frac12}|L|^{\frac12}U^*|L^*|^{\frac12}\right)^2 \le M\left\{\left(|L|^{\frac12}U^*|L^*|^{\frac12}|L^*|^{\frac12}U|L|^{\frac12}\right)^k\right\}^{\frac2k}$$

$$\left(|L|^{\frac12}U|L|^{\frac12}|L|^{\frac12}U^*|L|^{\frac12}\right)^2 \le M\left\{\left(|L|^{\frac12}U^*|L|^{\frac12}|L|^{\frac12}U|L|^{\frac12}\right)^k\right\}^{\frac2k}$$

$$\left(\tilde L\tilde L^*\right) \le M\left\{\left(\tilde L^*\tilde L\right)^k\right\}^{\frac2k}$$

$$|\tilde L^*|^2 \le M|\tilde L^k|^{\frac2k}$$

Hence, $\tilde L$ is M-class A_k^* operator.

Theorem 2.8 If L and $\tilde L$ is M-class A_k^* operator then $\tilde L^*$ is also M-class A_k^* operator for some positive real numbers M, k.

Proof. Given L and $\tilde L$ is M-class A_k^* operator

$$U^*\left(U|L|U^*|L^*|\right)U \le MU^*\left(U^*|L^*|^k U|L|^k\right)^{\frac1k}U$$

$$U^*\left(U|L|^{\frac12}|L|^{\frac12}U^*|L^*|^{\frac12}|L^*|^{\frac12}\right)U \le MU^*\left(U^*|L^*|^{\frac k2}|L^*|^{\frac k2}U|L|^{\frac k2}|L|^{\frac k2}\right)^{\frac1k}U$$

$$U^*\left(|L^*|^{\frac12}U|L|^{\frac12}|L|^{\frac12}U^*|L^*|^{\frac12}\right)U \le MU^*\left(|L|^{\frac k2}U^*|L^*|^{\frac k2}|L^*|^{\frac k2}U|L|^{\frac k2}\right)^{\frac1k}U$$

$$U^*\left(\left(|L^*|^{\frac12}U|L^*|^{\frac12}\right)\left(|L^*|^{\frac12}U^*|L^*|^{\frac12}\right)\right)U \le MU^*\left(|L^*|^{\frac k2}U^*|L^*|^{\frac k2}|L^*|^{\frac k2}U|L^*|^{\frac k2}\right)^{\frac1k}U$$

$$U^*\left(\tilde L^*(\tilde L^*)^*\right)U \le MU^*\left((\tilde L^{*k})^*\tilde L^{*k}\right)^{\frac1k}U$$

$$U^*|(\tilde L^*)^*|^2U \le MU^*\left(|\tilde L^{*k}|^{\frac2k}\right)U$$

$$|(\tilde L^*)^*|^2 \le M\left(|\tilde L^{*k}|^{\frac2k}\right).$$

Hence, $\tilde L^*$ is also M-class A_k^* operator.

Theorem 2.9 If $L\in B(H)$, $\tilde L^*$ is M-Class A_k^* operator and U is isometry then $\tilde L$ is M-class A_k^* operator.

Proof.

Since, $\tilde L^*$ is M-Class A_k^* operator

$$|(\tilde L^*)^*|^2 \le M\left(|\tilde L^{*k}|^{\frac2k}\right)$$

$$\left(\tilde L^*(\tilde L^*)^*\right) \le M\left((\tilde L^{*k})^*\tilde L^{*k}\right)^{\frac1k}$$

$$\left(|L|^{\frac12}U^*|L|^{\frac12}|L^*|^{\frac12}U|L^*|^{\frac12}\right) \le M\left(|L^{*k}|^{\frac12}U|L^{*k}|^{\frac12}|L^k|^{\frac12}U^*|L^k|^{\frac12}\right)^{\frac1k}$$

$$\left(U^*|L^*|^{\frac12}|L|^{\frac12}|L^*|^{\frac12}|L|^{\frac12}U\right) \le M\left\{\left(U|L|^{\frac12}|L^*|^{\frac12}|L|^{\frac12}|L^*|^{\frac12}U^*\right)^k\right\}^{\frac1k}$$

$$U^*\left(|L^*|^{\frac12}U|L^*|^{\frac12}U^*|L^*|^{\frac12}|L|^{\frac12}\right)U \le MU\left\{\left(|L|^{\frac12}U^*|L|^{\frac12}U|L|^{\frac12}|L^*|^{\frac12}\right)^k\right\}^{\frac1k}U^*$$

$$U^*\left(|L^*|^{\frac12}U|L^*|^{\frac12}U^*|L^*|^{\frac12}|L|^{\frac12}\right)U \le MU\left\{\left(|L|^{\frac12}U^*|L|^{\frac12}U|L|^{\frac12}|L^*|^{\frac12}\right)^k\right\}^{\frac1k}U^*$$

$$U^*\left(|L^*|^{\frac12}U|L^*|^{\frac12}|L|^{\frac12}U^*|L|^{\frac12}\right)U \le MU\left\{\left(|L|^{\frac12}U^*|L|^{\frac12}|L^*|^{\frac12}U|L^*|^{\frac12}\right)^k\right\}^{\frac1k}U^*$$

$$U^*\left((\tilde L^*)^*\tilde L^*\right)U \le MU\left\{\left(\tilde L^*(\tilde L^*)^*\right)^k\right\}^{\frac1k}$$

$$|\tilde L^*|^2 \le M\left|(\tilde L^*)^{*k}\right|^{\frac2k}$$

$$|\widetilde{L}^*|^2 \le M|\widetilde{L}^k|^{\frac{2}{k}}$$

Therefore $\widetilde{L}$ is M-class A_k^* operator

III. *- ALUTHGE TRANSFORMATION OF M-CLASS AK* OPERATORS & RESULTS

In this part, we discussed *- aluthge transformation and adjoint of *-aluthge transformation of M-class A_k^* operator.

Theorem3.1. If L is bounded linear operator on a complex Hilbert space, then we know that

(i) $\widetilde{L}=|L|^{\frac{1}{2}}U|L|^{\frac{1}{2}}$ is the Aluthge transformation then the adjoint of Aluthge transformation $\widetilde{L}^*$ is given by $\widetilde{L}^*=|L|^{\frac{1}{2}}U^*|L|^{\frac{1}{2}}$.

(ii) $\widetilde{L}^{(*)}=(\widetilde{L^*})^*=|L^*|^{\frac{1}{2}}U|L^*|^{\frac{1}{2}}$ is the *- Aluthge transformation then adjoint of *-Aluthgetransformation $(\widetilde{L}^{(*)})^*=|L^*|^{\frac{1}{2}}U^*|L^*|^{\frac{1}{2}}$ [5][8].

Theorem 3.2 An operator L=U|L| is M-class A_k^* operator and U is isometry operator if and only if $(\widetilde{L}^{(*)})^*$ is also M-class A_k^* operator.

Theorem 3.3 Assume $L\in B(H)$, $\widetilde{L}^*$ is M-Class A_k^* operator then $(\widetilde{L}^{(*)})^*$ is M-class A_k^* operator.

Proof**.** Given that $\widetilde{L}^*$ is M-Class A_k^* operator

$$|(\widetilde{L}^*)^*|^2 \le M\left(|\widetilde{L}^{*k}|^{\frac{2}{k}}\right)$$

$$\left(|L^*|^{\frac{k}{2}}U|L^*|^{\frac{k}{2}}|L^*|^{\frac{k}{2}}U^*|L^*|^{\frac{k}{2}}\right) \le M\left(\left(|L^*|^{\frac{k}{2}}U^*|L^*|^{\frac{k}{2}}|L^*|^{\frac{k}{2}}U|L^*|^{\frac{k}{2}}\right)^k\right)^{\frac{1}{k}}$$

$$\left(U|L|^{\frac{k}{2}}|L^*|^k|L|^{\frac{k}{2}}U^*\right) \le M\left(\left(U^*|L|^{\frac{k}{2}}|L^*|^k|L|^{\frac{k}{2}}U\right)^k\right)^{\frac{1}{k}}$$

$$U\left(|L|^{\frac{k}{2}}U^*|L|^kU|L|^{\frac{k}{2}}\right)U^* \le MU^*\left(\left(|L|^{\frac{k}{2}}U|L^*|^kU^*|L|^{\frac{k}{2}}\right)^k\right)^{\frac{1}{k}}U$$

$$U\left(|L|^{\frac{k}{2}}U^*|L|^{\frac{k}{2}}|L|^{\frac{k}{2}}U|L|^{\frac{k}{2}}\right)U^* \le MU^*\left(\left(|L|^{\frac{k}{2}}U|L|^{\frac{k}{2}}|L|^{\frac{k}{2}}U^*|L|^{\frac{k}{2}}\right)^k\right)^{\frac{1}{k}}U$$

$$U(\widetilde{L}^*\widetilde{L})U^* \le MU^*(\widetilde{L}^k\widetilde{L}^{*k})^{\frac{1}{k}}U$$

$$|\widetilde{L}^*|^2 \le M|\widetilde{L}^k|^{\frac{2}{k}}$$

$$|(\widetilde{L}^{(*)})^*|^2 \le M|\widetilde{L}^{*k}|^{\frac{2}{k}}$$

Hence, $(\widetilde{L}^{(*)})^*$ is M-class A_k^* operator

IV. ACKNOWLEDGEMENT

The authors would like to thank the management of Sri Ramakrishna Engineering College for their continuous support and encouragement to complete this research work.

REFERENCES

1. A.Aluthge, on P-hyponormal operators for 0 < P <1, Integral equations and operator theory,Vol.13(1990),307-315.
2. H.Heuser, Functional Analysis, Marcel Dekker, New York 1982
3. S.Panayappan ,N.Jayanthi and D.Sumathi, Weyl's theorem and Tensor product for class A_k Operators,Pure Mathematical Sciences,Vol.1,2012,no.1,13-23.
4. S.Panayappan and N.Jayanthi, Weyl and Weyl type theorems for class A_k^* and Quasi Class A_k^* operators. Int. Journal of Math. Analysis,Vol.7,2013,no.14,683-698, HIKARI Ltd.
5. D.Senthilkumar,D.Kiruthika and P.Maheswari Naik, $\approx-$ Aluthge transformation and adjoint of $*-$ Aluthge transformation, Scientia Mangna,Vol.6(2010),no.2,59-66.
6. D.Senthilkumar,R.Murugan, Aluthge transformation on powers of N-class A_k operators,Mathematical sciences international research journal,Vol.4(2015),2278-2286
7. D.Senthilkumar,R.Murugan, Aluthge transformation of N-class A_k operators, International journal of Pure and Applied Mathematics,Vol.106,No.8,2016,27-31.
8. D.Senthilkumar, S.Shylaja, Aluthge and $*-$ Aluthge transformation of powers of N-class A(k) operators, International journal of Pure and Applied Mathematics,Vol.106,No.8,2016,53-58.
9. P.Shanmugapriya and P.Maheswari Naik, "Spectral properties of M- class A_k^* operator", International Journal of Mathematics and its Applications, Vol.7(2)(2019),171-176 .
10. Takayuki Furuta "Invitation to Linear Operators- From matrices to bounded linear operators on a Hilbert Space", CRC Press , Taylor and Francis Group,FL.