Aluthge Transformation and \(*\)- Aluthge Transformation on M class \(A_k\) Operator

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Abstract—Research works on Operators in Complex Hilbert spaces has been the interest of budding researchers in the recent years. In 1996, Furuta et al studied Aluthge transformation on p-hyponormal operators. Later, in 2001 Yamazaki et al studied Aluthge transformation and powers of operators for class \(A(k)\) operator. This work was further carried over by Panayappan et al and D. Senthil Kumar et al. In this school work, we studied Aluthge transformation and \(*\) Aluthge transformation for the new class of operator named M class \(A_k\) operator on a non-zero Complex Hilbert space.

Key Words: Class \(A_k\) operator, M-class \(A_k\) operator, Aluthge transformation

I. INTRODUCTION

The Banach algebra on a non-zero complex Hilbert space \(H\) of all bounded linear operators are denoted by \(B(H)\). An operator \(L\) is defined as an element in \(B(H)\). If \(L\) belongs to \(B(H)\), then \(L^*\) means the adjoint of \(L\) in \(B(H)\). Weyl and Weyl type theorems were studied for the following class of operators. Furuta et al introduced class \(A(k)\), \(k > 0\) as a class of operators and extended \(p\)-hyponormal and log-hyponormal operators. They studied Weyl and Weyl type theorems for the above operators [10]. Later, Panayappan et al extended this concept and introduced class \(A_k\) operators and verified Weyl’s theorem [3]. In 2013, Panayappan et al introduced a new class of operators in a different manner called class \(A_k\) operator, quasi class \(A_k\) operators and studied Weyl and Weyl type theorems and also proved tensor product of two quasi class \(A_k\) operators are closed [4].

It is well known that an operator can be decomposed into \(T= U |T|\) where \(U\) is partial isometry. In 2015, D. Senthil Kumar et al studied Aluthge transformation on \(N\) - Class \(A(k)\) operators [7]. They also studied Aluthge and \(*\)-Aluthge transformation of powers of \(N\)-class \(A(k)\) operators in 2016 [6]. The above research work kindles our interest on studying the Aluthge transformation for M-Class \(A_k\) operator.

Definition 1.1 An operator \(L\) is called class \(A_k\) operator if \(\|L_k\|^2 \geq |L_k|^2\) where \(k\) is a positive integer.

If \(k = 1\) then class \(A_k\) operator coincides with hyponormal operator [4].

Definition 1.2 An operator \(L \in B(H)\) is said to be M-Class \(A_k\) operator if there exists positive real numbers \(M, k\) such that \(|L_x|^2 \leq M \left(\|L_k\|^2 \right)\) [9].

Proposition 1.3.

If \(M = 1\), then M-Class \(A_k\) operator coincides with class \(A_k\) operator.

If \(M = 1\) and \(k = 1\), then M-Class \(A_k\) operator coincides with hyponormal operator.

Hence, Hyponormal operator \(\Rightarrow\) class \(A_k\) operator \(\Rightarrow\) M-Class \(A_k\) operator.

In the next section, we studied Aluthge Transformation for M-Class \(A_k\) operator.

II ALUTHGE TRANSFORMATION ON M CLASS \(AK\) OPERATOR

Assume that \(L\) is a bounded linear operator on a complex Hilbert space \(H\). In [1], Aluthge introduced the \(\tilde{L}\) operator for an operator \(L\) with its polar decomposition \(L = U |L| = |L| U\) and Takashi [10] defined \(\tilde{L}\) and \(\tilde{L}^*\) as below:

\[
\tilde{L} = |L| U |L|, \quad \tilde{L}^* = |L| U |L|.
\]

Theorem 2.2 An operator \(L\) is called M-Class \(A_k\) operator if and only if

\[
\|L^* x\|^2 \leq M \left(\|L_k\|^2 \right) \|x\|^{2k - 2} \text{ for all } x \in H.
\]

Proof. We know that \(|L_x|^2 \leq M \left(\|L_k\|^2 \right)\) and

\[
(LL^*) \leq M \left(\|L^* k L_k\|^2 \right) \|x\|^{2k - 2} \text{ for all } x \in H.
\]

(By Theorem 6, [7])

\[
\|L^* x\|^2 \leq M \left(\|L_k\|^2 \right) \|x\|^{2k - 2} \text{ for all } x \in H.
\]
Hence, proved.

**Theorem 2.3** If $L = U \| L \|$ and $L^* = U^* ^\dagger \| L^\dagger$ is the polar decomposition of $L$, then $L$ is M-Class $A_k^*$ operator.

Proof. By the definition of M-Class $A_k^*$ operator,

$$
\begin{align*}
\left( U \| L \| U^* \| L^* \right) & \leq M \left[ \left( U^* \| L \| U \right)^k \right]^\frac{1}{k} \\
\left( L^* \| U \| U^* \| L^* \right) & \leq M \left[ \left( L^\dagger \| U \| U \right)^k \right]^\frac{1}{k} \\
\left( L^\dagger \right)^2 & \leq M \left( \| L \| \right)^2.
\end{align*}
$$

So if $L = U \| L \|$ and $L^* = U^* ^\dagger \| L^\dagger$ is the polar decomposition of $L$ then it is M-Class $A_k^*$ operator.

**Theorem 2.4** If $L$ is M-Class $A_k^*$ operator and $S$ is an unitary operator such that $LS = SL$, then $C = LS$ is also M-Class $A_k^*$ operator.

Proof. By M-Class $A_k^*$ operator definition,

$$
\begin{align*}
(CC^*) & \leq M \left[ \left( C \| S \right)^k \right]^\frac{1}{k} \\
(LS)^* \| L^* & \leq M \left[ \left( L \| S \right)^k \right]^\frac{1}{k} \\
\left( L^\dagger \right)^2 & \leq M \left( \| L \| \right)^2.
\end{align*}
$$

Hence $C = LS$ is also M-Class $A_k^*$ operator.

**Theorem 2.5** Let $\lambda$ and $\beta$ be positive operators. Then for each $p \geq 0$ and $r \geq 0$ the following assertions hold:[2]

1. If $\left( \beta^\frac{p}{r} A^\frac{p}{r} \right)^\frac{1}{p+r} \geq \beta^r$ then $\left( \beta^\frac{p}{r} A^\frac{p}{r} \right)^\frac{p}{p+r} \leq A^p$

2. If $\left( \beta^\frac{p}{r} A^\frac{p}{r} \right)^\frac{1}{p+r} \leq A^p$ and $\text{N}(A) \subset \text{N}(\beta)$ then

$$
\left( \beta^\frac{p}{r} A^\frac{p}{r} \right)^\frac{1}{p+r} \geq \beta^r
$$

**Theorem 2.6** Let $L = U \| L \|$ be the polar decomposition of $L$ is M-Class $A_k^*$ operator for $0 < p < 1$, then $L_{s,t} = \left[ \left( L^\dagger \right)^s U \right]^t$ is $2 (p + \min(s, t))$ M-Class $A_k^*$ operator for $s, t > 0$ such that max$(s, t) \geq p$ and $U^* = U$.

Proof. By M-Class $A_k^*$ operator definition,

$$
\begin{align*}
\left( \left( L_{s,t} \right)^{p+\min(s,t)} \right)^\frac{1}{s+t} & \leq M \left[ \left( \left( L_{s,t} \right)^{p+\min(s,t)} \right)^k \right]^\frac{1}{s+t} \\
\left( \left( U \| L \| U \right)^k \right)^\frac{1}{p+\min(s,t)} & \leq M \left[ \left( \left( U \| L \| U \right)^k \right)^k \right]^\frac{1}{p+\min(s,t)} \\
\left( \left( U \| L \| U \right)^k \right) & \leq \left( \left( U \| L \| U \right)^k \right)^\frac{1}{p+\min(s,t)}
\end{align*}
$$

by (Theorem F (3.2.1),[10])

$$
\begin{align*}
\left( U \| L \| \right)^{2(p+\min(s,t))} & \leq M \left( \left( U \| L \| \right)^{k \left( 2(p+\min(s,t)) \right)} \right)^\frac{1}{k} \left( U \| L \| \right)^{2(p+\min(s,t))} \leq M \left( \left( U \| L \| \right)^{k \left( 2(p+\min(s,t)) \right)} \right)^\frac{1}{k}
\end{align*}
$$

Hence the proof.

**Theorem 2.7** If $L = U \| L \|$ is M-class $A_k^*$ operator for some positive real numbers $M$, $k$ and $U$ is isometry then $\tilde{L}$ is also M-class $A_k^*$ operator.

Proof. Given $L$ is M-class $A_k^*$ operator,
Theorem 2.8 If L and \( \tilde{L} \) is M-class \( A_k^* \) operator then \( \tilde{L} \) is also M-class \( A_k^* \) operator for some positive real numbers M, k.

Proof. Given \( L \) and \( \tilde{L} \) is M-class \( A_k^* \) operator

\[
U^* (U|U|U^*|U|) \leq M U^* \left( U|U|U|U^* \right) \frac{1}{k} U
\]

\[
U^* \left( U|U|U|U^* \right) \leq M U^* \left( U|U|U|U^* \right) \frac{1}{k} U
\]

\[
U^* \left( U|U|U|U^* \right) \leq M U^* \left( U|U|U|U^* \right) \frac{1}{k} U
\]

\[
U^* \left( U|U|U|U^* \right) \leq M U^* \left( U|U|U|U^* \right) \frac{1}{k} U
\]

\[
U^* \left( U|U|U|U^* \right) \leq M U^* \left( U|U|U|U^* \right) \frac{1}{k} U
\]

\[
U^* \left( U|U|U|U^* \right) \leq M U^* \left( U|U|U|U^* \right) \frac{1}{k} U
\]

\[
U^* \left( U|U|U|U^* \right) \leq M U^* \left( U|U|U|U^* \right) \frac{1}{k} U
\]

\[
U^* \left( U|U|U|U^* \right) \leq M U^* \left( U|U|U|U^* \right) \frac{1}{k} U
\]

\[
U^* \left( U|U|U|U^* \right) \leq M U^* \left( U|U|U|U^* \right) \frac{1}{k} U
\]

Hence, \( \tilde{L} \) is also M-class \( A_k^* \) operator.
\[ |\mathcal{L}^*|^2 \leq M |\mathcal{L}^k|^2 \]

Therefore \( \mathcal{L} \) is M-class \( A_k^* \) operator

### III. \( ^* \)-ALUTHGE TRANSFORMATION OF M-CLASS \( A_k^* \) OPERATORS & RESULTS

In this part, we discussed \( ^* \)-aluthge transformation and adjoint of \( ^* \)-aluthge transformation of M-class \( A_k^* \) operator.

**Theorem 3.1.** If \( L \) is bounded linear operator on a complex Hilbert space, then we know that

(i) \( \mathcal{L} = |L|^\frac{1}{2} U |L|^\frac{1}{2} \) is the Aluthge transformation then the adjoint of Aluthge transformation \( \mathcal{L}^* \) is given by \( \mathcal{L}^* = |L|^\frac{1}{2} U^* |L|^\frac{1}{2} \).

(ii) \( \mathcal{L}^{(*)} = (\mathcal{L}^*)^* = \mathcal{L}^* = |L|^\frac{1}{2} U^* |L|^\frac{1}{2} \) is the \( ^* \)-Aluthge transformation then adjoint of \( ^* \)-Aluthgetransformation \( \mathcal{L}^{(*)} = |L|^\frac{1}{2} U^* |L|^\frac{1}{2} \) \[5\][8].

**Theorem 3.2** An operator \( L=U|L| \) is M-class \( A_k^* \) operator and \( U \) is isometry operator if and only if \( \mathcal{L}^{(*)} \) is also M-class \( A_k^* \) operator.

**Theorem 3.3** Assume \( L \in B(H) \), \( \mathcal{L}^* \) is M-Class \( A_k^* \) operator then \( \mathcal{L}^{(*)} \) is M-class \( A_k^* \) operator.

Proof. Given that \( \mathcal{L} \) is M-Class \( A_k^* \) operator

\[ |\mathcal{L}^*|^2 \leq M |\mathcal{L}^k|^2 \]

\[ \left( |L|^\frac{1}{2} U |L|^\frac{1}{2} |L|^\frac{1}{2} U^* |L|^\frac{1}{2} \right) \leq M \left( \left( |L|^\frac{1}{2} U^* |L|^\frac{1}{2} |L|^\frac{1}{2} U \right)^k \right)^\frac{1}{k} \]

\[ U \left( |L|^\frac{1}{2} U^* |L|^\frac{1}{2} |L|^\frac{1}{2} U^* \right) \leq M U \left( \left( |L|^\frac{1}{2} U^* |L|^\frac{1}{2} |L|^\frac{1}{2} U \right)^k \right)^\frac{1}{k} U \]

\[ U (\mathcal{L}^* \mathcal{L}^*) U^* \leq M U^* \left( \mathcal{L}^k \mathcal{L}^* \mathcal{L}^k \right)^\frac{1}{k} U \]

\[ |\mathcal{L}^{(*)}|^2 \leq M |\mathcal{L}^k|^2 \]

Hence, \( \mathcal{L}^{(*)} \) is M-class \( A_k^* \) operator

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