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# Aluthge Transformation and *- Aluthge Transformation on M class $A_k^*$ Operator

**P.Shanmugapriya, P. Maheswari Naik**

***Abstract—Research works on Operators in Complex Hilbert spaces has been the interest of budding researchers in the recent years. In 1996, Furuta et al studied Aluthge transformation on p-hyponormal operators. Later, in 2001 yamazaki et al studied Aluthge transformation and powers of operators for class A(k)operator. This work was further carried over by Pannayappan et al and D.Senthil kumar et al. In this school work, we studied Aluthge transformation and *- Aluthge transformation for the new class of operator named M class $A_k^*$ operator on a non-zero Complex Hilbert space.***

***Key Words: Class $A_k^*$ operator, M-class $A_k^*$ operator, Aluthge transformation***

## I. INTRODUCTION

The Banach algebra on a non-zero complex Hilbert space H of all bounded linear operators are denoted by B(H). An operator L is defined as an element in B(H). If L belongs to B(H), then $L^*$ means the adjoint of L in B(H). Weyl and Weyl type theorems where studied for the following class of operators. Furuta et al introduced class A(k), k > 0 as a class of operators and extended p-hyponormal and log-hyponormal operators. They studied Weyl and Weyl type theorems for the above operators [10]. Later, Panayappan et al extended this concept and introduced class $A_k$ operators and verified Weyl's theorem [3]. In 2013, Panayappan et al introduced a new class of operators in a different manner called class $A_k^*$ operator, quasi class $A_k^*$ operators and studied Weyl and Weyl type theorems and also proved tensor product of two quasi class $A_k^*$ operators are closed [4].

It is well known that an operator can be decomposed into T= U |T| where U is partial isometry. In 2015, D.Senthil kumar et al studied Aluthge transformation on N –Class A(k) operators [7]. They also studied Aluthge and *-Aluthge transformation of powers of N-class A(k) operators in 2016 [6]. The above research work kindles our interest on studying the Aluthge transformation for M-Class $A_k^*$ operator.

**Definiton 1.1** An operator L is called class $A_k^*$ operator if $\left|L^k\right|^{\frac{2}{k}} \geq \left|L^*\right|^2$ where k is a positive integer.

If k = 1 then class $A_k^*$ operator coincides with hyponormal operator [4].

**Definition 1.2** An operator $L \in B(H)$ is said to be M-Class $A_k^*$ operator if there exists positive real numbers M, k such that $\left|L^*\right|^2 \leq M\left(\left|L^k\right|^{\frac{2}{k}}\right)$ [9].

**Proposition: 1.3.**

If M = 1, then M-Class $A_k^*$ operator coincides with class $A_k^*$ operator.

If M = 1 and k = 1, then M-Class $A_k^*$ operator coincides with hyponormal operator.

Hence, Hyponormal operator $\Rightarrow$ class $A_k^*$ operator $\Rightarrow$ M-Class $A_k^*$ operator.

In the next section, we studied Aluthge Transformation for M-Class $A_k^*$ operator.

## II ALUTHGE TRANSFORMATION ON M CLASS AK* OPERATOR

Assume that L is a bounded linear operator on a complex Hilbert space H. In [1], Aluthge introduced the $\tilde{L}$ operator for an operator L with its polar decomposition $L = U|L| = |L^*|U$ and Takashi [10] defined $\tilde{L}$ and $\tilde{L}^*$ as below:

$$\tilde{L}_{s,l} = |L|^s\, U\, |L|^l$$

$$\tilde{L}^*_{s,l} = (\tilde{L}_{s,l})^* = |L|^l\, U\, |L|^s$$

.**Theorem 2.2** An operator L is called M-Class $A_k^*$ operator if and only if

$$\left\|L^* x\right\|^2 \leq M \left\|L^k x\right\|^{\frac{2}{k}} \left\|x\right\|^{\frac{2k-2}{k}} \text{ for all } x \in H.$$

Proof. We know that $\left|L^*\right|^2 \leq M\left(\left|L^k\right|^{\frac{2}{k}}\right)$

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* Correspondence Author
**P.Shanmugapriya*,** Department of Mathematics, Sri Ramakrishna Engineering College,Coimbatore, Tamil Nadu, India.
(E-mail: prsrinithin@gmail.com)
**P. Maheswari Naik,** Department of Mathematics, Sri Ramakrishna Engineering College,Coimbatore, Tamil Nadu, India.
(E-mail: maheswarinaik21@gmail.com)

$$(LL^*) \le M\{(L^{*k}L^k)\}^{\frac{1}{k}}$$

$$\langle LL^*x, x\rangle \le M\left\langle \{(L^{*k}L^k)\}^{\frac{1}{k}}x, x\right\rangle$$

$$\langle L^*x, L^*x\rangle \le M\left\langle \{(L^k x, L^k x)\}^{\frac{1}{k}}\right\rangle \|x\|^{\frac{2k-2}{k}}$$

(By Theorem 6, [7])

$$\|L^*x\|^2 \le M\,\|L^k x\|^{\frac{2}{k}}\|x\|^{\frac{2k-2}{k}} \text{ for all } x \in H$$

Hence, proved.

**Theorem 2.3** If L =U |L| and $L^* = U^*|L^*|$ is the polar decomposition of L, then L is M-Class $A_k^*$ operator.

Proof. By the definition of M-Class $A_k^*$ operator ,

$$(U|L|U^*|L^*|) \le M\{(U^{*k}|L^{*k}|U^k|L^k|)\}^{\frac{1}{k}}$$

$$(|L^*|UU^*|L^*|) \le M\{(|L^k|U^{*k}U^k|L^k|)\}^{\frac{1}{k}}$$

$$(|L^*|^2) \le M|L^k|^{\frac{2}{k}}$$

So if L =U |L| and $L^* = U^*|L^*|$ is the polar decomposition of L then it is M-Class $A_k^*$ operator.

**Theorem 2.4** If L is M-Class $A_k^*$ operator and S is an unitary operator such that LS = SL, then

C = LS is also M-Class $A_k^*$ operator.

Proof. By M-Class $A_k^*$ operator definition,

$$(CC^*) \le M\{(C^{*k}C^k)\}^{\frac{1}{k}}$$

$$(L\ S\ S^*L^*) \le M\{(L^{*k}S^{*k}S^kL^k)\}^{\frac{1}{k}}$$

$$|L^*|^2 \le M|L^k|^{\frac{2}{k}}.$$

Hence C = LS is also M-Class $A_k^*$ operator.

**Theorem 2.5** Let A and $\beta$ be positive operators. Then for each $p \ge 0$ and $r \ge 0$ the following assertions hold:[2]

1. If $\left(\beta^{\frac{r}{2}}A^p\beta^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \ge \beta^r$ then $\left(\beta^{\frac{p}{2}}A^r\beta^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \le A^p$

2. If $\left(\beta^{\frac{p}{2}}A^r\beta^{\frac{p}{2}}\right)^{\frac{p}{p+r}} \le A^p$ and $N(A) \subset N(\beta)$ then

$$\left(\beta^{\frac{r}{2}}A^p\beta^{\frac{r}{2}}\right)^{\frac{r}{p+r}} \ge \beta^r$$

**Theorem 2.6** Let L = U |L| be the polar decomposition of L is M-Class $A_k^*$ operator for 0 < p < 1, then $\tilde{L}_{s,l} = |L|^s U|L|^t$ is 2 (p + min(s, t)) M-Class $A_k^*$ operator for s, t > 0 such that max( s, t) ≥ p and U*=U .

Proof. By M-Class $A_k^*$ operator definition,

$$\left(\tilde{L}_{s,l}\tilde{L}^*_{s,l}\right)^{\frac{p+\min(s,l)}{s+l}} \le M\left[\{\tilde{L}^{*k}_{s,l}\tilde{L}^k_{s,l}\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}$$

$$(|L|^s U|L|^l|L|^l U^*|L|^s)^{\frac{p+\min(s,l)}{s+l}} \le$$

$$M\left[\left\{\left(|L|^l U^*|L|^s|L|^s U|L|^l\right)^k\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}$$

$$\left(U|L^*|^s|L|^{2l}|L^*|^s U^*\right)^{\frac{p+\min(s,l)}{s+l}} \le$$

$$M\left[\left\{\left(U^*|L^*|^l|L|^{2s}|L^*|^l U\right)^k\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}$$

$$U\left(|L^*|^s|L|^{2l}|L^*|^s\right)^{\frac{p+\min(s,l)}{s+l}}U^* \le$$

$$MU^*\left[\left\{\left(|L^*|^l|L|^{2s}|L^*|^l\right)^k\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}U$$

$$U\left(B^{\frac{s}{2}}A^l B^{\frac{s}{2}}\right)^{\frac{p+\min(s,l)}{s+l}}U^* \le$$

$$MU^*\left[\left\{\left(B^{\frac{1}{2}}A^s B^{\frac{1}{2}}\right)^k\right\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}U$$

$$U(B^{s+l})^{\frac{p+\min(s,l)}{s+l}}U^* \le$$

$$MU^*\left[\{(B^{s+l})^k\}^{\frac{1}{k}}\right]^{\frac{p+\min(s,l)}{s+l}}U$$

by (Theorem F (3.2.1), [10])

$$U\left(|L^*|^{2(p+\min(s,l))}\right)U^* \le MU^*\left[\{(|L|)^k\}^{\frac{2(p+\min(s,l))}{k}}\right]U$$

$$\left(|L^*|^{2(p+\min(s,l))}\right) \le M\left[\{(|L|)^k\}^{\frac{2(p+\min(s,l))}{k}}\right]$$

Hence the proof.

**Theorem 2.7** If L = U|| is M-class $A_k^*$ operator for some positive real numbers M, k and U is isometry then $\tilde{L}$ is also M-class $A_k^*$ operator.

Proof. Given L is M-class $A_k^*$ operator,

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$$\left(U|L|U^*|L^*|\right) \le M\left\{\left(U^{*k}|L^{*k}|U^k|L^k|\right)\right\}^{\frac{1}{k}}$$

$$\left(U|L|^{\frac12}|L|^{\frac12}U^*|L^*|^{\frac12}|L^*|^{\frac12}\right)^2 \le M\left\{\left(U^*|L^*|^{\frac12}|L^*|^{\frac12}U|L|^{\frac12}|L|^{\frac12}\right)^k\right\}^{\frac2k}$$

$$\left(|L^*|^{\frac12}U|L|^{\frac12}|L|^{\frac12}U^*|L^*|^{\frac12}\right)^2 \le M\left\{\left(|L|^{\frac12}U^*|L^*|^{\frac12}|L^*|^{\frac12}U|L|^{\frac12}\right)^k\right\}^{\frac2k}$$

$$\left(|L|^{\frac12}U|L|^{\frac12}|L|^{\frac12}U^*|L|^{\frac12}\right)^2 \le M\left\{\left(|L|^{\frac12}U^*|L|^{\frac12}|L|^{\frac12}U|L|^{\frac12}\right)^k\right\}^{\frac2k}$$

$$\left(\tilde{L}\tilde{L}^*\right) \le M\left\{\left(\tilde{L}^*\tilde{L}\right)^k\right\}^{\frac2k}$$

$$|\tilde{L}^*|^2 \le M|\tilde{L}^k|^{\frac2k}$$

**Hence,** $\tilde{L}$ is M-class $A_k^*$ operator.

**Theorem 2.8** If L and $\tilde{L}$ is M-class $A_k^*$ operator then $\tilde{L}^*$ is also M-class $A_k^*$ operator for some positive real numbers M, k.

Proof. Given L and $\tilde{L}$ is M-class $A_k^*$ operator

$$U^*\left(U|L|U^*|L^*|\right)U \le MU^*\left(U^*|L^*|^k U|L|^k\right)^{\frac1k}U$$

$$U^*\left(U|L|^{\frac12}|L|^{\frac12}U^*|L^*|^{\frac12}|L^*|^{\frac12}\right)U \le$$

$$MU^*\left(U^*|L^*|^{\frac k2}|L^*|^{\frac k2}U|L|^{\frac k2}|L|^{\frac k2}\right)^{\frac1k}U$$

$$U^*\left(|L^*|^{\frac12}U|L|^{\frac12}|L|^{\frac12}U^*|L^*|^{\frac12}\right)U \le$$

$$MU^*\left(|L|^{\frac k2}U^*|L^*|^{\frac k2}|L^*|^{\frac k2}U|L|^{\frac k2}\right)^{\frac1k}U$$

$$U^*\left(\left(|L^*|^{\frac12}U|L^*|^{\frac12}\right)\left(|L^*|^{\frac12}U^*|L^*|^{\frac12}\right)\right)U \le$$

$$MU^*\left(|L^*|^{\frac k2}U^*|L^*|^{\frac k2}|L^*|^{\frac k2}U|L^*|^{\frac k2}\right)^{\frac1k}U$$

$$U^*\left(\tilde{L}^*(\tilde{L}^*)^*\right)U \le MU^*\left((\tilde{L}^{*k})^*\tilde{L}^{*k}\right)^{\frac1k}U$$

$$U^*|(\tilde{L}^*)^*|^2U \le MU^*\left(|\tilde{L}^{*k}|^{\frac2k}\right)U$$

$$|(\tilde{L}^*)^*|^2 \le M\left(|\tilde{L}^{*k}|^{\frac2k}\right).$$

Hence, $\tilde{L}^*$ is also M-class $A_k^*$ operator.

**Theorem 2.9** If $L\in B(H)$, $\tilde{L}^*$ is M-Class $A_k^*$ operator and U is isometry then $\tilde{L}$ is M-class $A_k^*$ operator.

Proof.

Since, $\tilde{L}^*$ is M-Class $A_k^*$ operator

$$|(\tilde{L}^*)^*|^2 \le M\left(|\tilde{L}^{*k}|^{\frac2k}\right)$$

$$\left(\tilde{L}^*(\tilde{L}^*)^*\right) \le M\left((\tilde{L}^{*k})^*\tilde{L}^{*k}\right)^{\frac1k}$$

$$\left(|L|^{\frac12}U^*|L|^{\frac12}|L^*|^{\frac12}U|L^*|^{\frac12}\right) \le$$

$$M\left(|L^{*k}|^{\frac12}U|L^{*k}|^{\frac12}|L^k|^{\frac12}U^*|L^k|^{\frac12}\right)^{\frac1k}$$

$$\left(U^*|L^*|^{\frac12}|L|^{\frac12}|L^*|^{\frac12}|L|^{\frac12}U\right) \le$$

$$M\left\{\left(U|L|^{\frac12}|L^*|^{\frac12}|L|^{\frac12}|L^*|^{\frac12}U^*\right)^k\right\}^{\frac1k}$$

$$U^*\left(|L^*|^{\frac12}U|L^*|^{\frac12}U^*|L^*|^{\frac12}|L|^{\frac12}\right)U \le$$

$$MU\left\{\left(|L|^{\frac12}U^*|L|^{\frac12}U|L|^{\frac12}|L^*|^{\frac12}\right)^k\right\}^{\frac1k}U^*$$

$$U^*\left(|L^*|^{\frac12}U|L^*|^{\frac12}U^*|L^*|^{\frac12}|L|^{\frac12}\right)U \le$$

$$MU\left\{\left(|L|^{\frac12}U^*|L|^{\frac12}U|L|^{\frac12}|L^*|^{\frac12}\right)^k\right\}^{\frac1k}U^*$$

$$U^*\left(|L^*|^{\frac12}U|L^*|^{\frac12}|L|^{\frac12}U^*|L|^{\frac12}\right)U \le$$

$$MU\left\{\left(|L|^{\frac12}U^*|L|^{\frac12}|L^*|^{\frac12}U|L^*|^{\frac12}\right)^k\right\}^{\frac1k}U^*$$

$$U^*\left((\tilde{L}^*)^*\tilde{L}^*\right)U \le MU\left\{\left(\tilde{L}^*(\tilde{L}^*)^*\right)^k\right\}^{\frac1k}$$

$$|\tilde{L}^*|^2 \le M|(\tilde{L}^*)^{*k}|^{\frac2k}$$

$$\left|\tilde{L}^*\right|^2 \le M\left|\tilde{L}^k\right|^{\frac{2}{k}}$$

Therefore $\tilde{L}$ is M-class $A_k^*$ operator

### III. *- ALUTHGE TRANSFORMATION OF M-CLASS AK* OPERATORS & RESULTS

In this part, we discussed *- aluthge transformation and adjoint of *-aluthge transformation of M-class $A_k^*$ operator.

**Theorem3.1.** If L is bounded linear operator on a complex Hilbert space, then we know that

**(i)** $\tilde{L} = |L|^{\frac{1}{2}} U |L|^{\frac{1}{2}}$ is the Aluthge transformation then the adjoint of Aluthge transformation $\tilde{L}^*$ is given by $\tilde{L}^* = |L|^{\frac{1}{2}} U^* |L|^{\frac{1}{2}}$.

**(ii)** $\tilde{L}^{(*)} = \left(\tilde{L}^*\right)^* = |L^*|^{\frac{1}{2}} U |L^*|^{\frac{1}{2}}$ is the *- Aluthge transformation then adjoint of *-Aluthgetransformation $\left(\tilde{L}^{(*)}\right)^* = |L^*|^{\frac{1}{2}} U^* |L^*|^{\frac{1}{2}}$ [5][8].

**Theorem 3.2** An operator L=U|L| is M-class $A_k^*$ operator and U is isometry operator if and only if $\left(\tilde{L}^{(*)}\right)^*$ is also M-class $A_k^*$ operator.

**Theorem 3.3** Assume $L \in B(H)$, $\tilde{L}^*$ is M-Class $A_k^*$ operator then $(\tilde{L}^{(*)})^*$ is M-class $A_k^*$ operator.

Proof**.** Given that $\tilde{L}^*$ is M-Class $A_k^*$ operator

$$\left|(\tilde{L}^*)^*\right|^2 \le M\left(\left|\tilde{L}^{*k}\right|^{\frac{2}{k}}\right)$$

$$\left(|L^*|^{\frac{k}{2}} U |L^*|^{\frac{k}{2}} |L^*|^{\frac{k}{2}} U^* |L^*|^{\frac{k}{2}}\right) \le$$

$$M\left(\left(|L^*|^{\frac{k}{2}} U^* |L^*|^{\frac{k}{2}} |L^*|^{\frac{k}{2}} U |L^*|^{\frac{k}{2}}\right)^k\right)^{\frac{1}{k}}$$

$$\left(U|L|^{\frac{k}{2}} |L^*|^k |L|^{\frac{k}{2}} U^*\right) \le M\left(\left(U^* |L|^{\frac{k}{2}} |L^*|^k |L|^{\frac{k}{2}} U\right)^k\right)^{\frac{1}{k}}$$

$$U\left(|L|^{\frac{k}{2}} U^* |L|^k U |L|^{\frac{k}{2}}\right) U^* \le MU^*\left(\left(|L|^{\frac{k}{2}} U |L^*|^k U^* |L|^{\frac{k}{2}}\right)^k\right)^{\frac{1}{k}} U$$

$$U\left(|L|^{\frac{k}{2}} U^* |L|^{\frac{k}{2}} |L|^{\frac{k}{2}} U |L|^{\frac{k}{2}}\right) U^* \le$$

$$MU^*\left(\left(|L|^{\frac{k}{2}} U |L|^{\frac{k}{2}} |L|^{\frac{k}{2}} U^* |L|^{\frac{k}{2}}\right)^k\right)^{\frac{1}{k}} U$$

$$U(\tilde{L}^*\tilde{L})U^* \le MU^*(\tilde{L}^k \tilde{L}^{*k})^{\frac{1}{k}} U$$

$$\left|\tilde{L}^*\right|^2 \le M\left|\tilde{L}^k\right|^{\frac{2}{k}}$$

$$\left|(\tilde{L}^{(*)})^*\right|^2 \le M\left|\tilde{L}^{*k}\right|^{\frac{2}{k}}$$

Hence, $(\tilde{L}^{(*)})^*$ is M-class $A_k^*$ operator

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### REFERENCES


1. A.Aluthge, on P-hyponormal operators for 0 < P <1, Integral equations and operator theory,Vol.13(1990),307-315.
2. H.Heuser, Functional Analysis, Marcel Dekker, New York 1982
3. S.Panayappan ,N.Jayanthi and D.Sumathi, Weyl's theorem and Tensor product for class $A_k$ Operators,Pure Mathematical Sciences,Vol.1,2012,no.1,13-23.
4. S.Panayappan and N.Jayanthi, Weyl and Weyl type theorems for class $A_k^*$ and Quasi Class $A_k^*$ operators. Int. Journal of Math. Analysis,Vol.7,2013,no.14,683-698, HIKARI Ltd.
5. D.Senthilkumar,D.Kiruthika and P.Maheswari Naik, $\approx -$ Aluthge transformation and adjoint of $*-$ Aluthge transformation, Scientia Mangna,Vol.6(2010),no.2,59-66.
6. D.Senthilkumar,R.Murugan, Aluthge transformation on powers of N-class $A_k$ operators,Mathematical sciences international research journal,Vol.4(2015),2278-2286
7. D.Senthilkumar,R.Murugan, Aluthge transformation of N-class $A_k$ operators, International journal of Pure and Applied Mathematics,Vol.106,No.8,2016,27-31.
8. D.Senthilkumar, S.Shylaja, Aluthge and $*-$ Aluthge transformation of powers of N-class A(k) operators, International journal of Pure and Applied Mathematics,Vol.106,No.8,2016,53-58.
9. P.Shanmugapriya and P.Maheswari Naik, "Spectral properties of M- class $A_k^*$ operator", International Journal of Mathematics and its Applications, Vol.7(2)(2019),171-176 .
10. Takayuki Furuta "Invitation to Linear Operators- From matrices to bounded linear operators on a Hilbert Space", CRC Press , Taylor and Francis Group,FL.