Some constraints for the Struve function to belong to certain subclasses of Analytic functions

G. Saravanan, Muthunagai. K

Abstract: ***The objective of the present article is to obtain some constraints that are sufficient for the generalized Struve functions of first kind to belong to the subclasses*** $S^*(\alpha,\beta,\gamma)$, $R^{\lambda}(A,B,\alpha)$ ***and to study the inclusion properties.***
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I. INTRODUCTION

The class of all normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in U \ (1)$$

be denoted by A .
Let S be the subclass of A, which consists of all univalent functions in the open unit disk U.

Definition 1.1 [24]
Let $f \in A$. Then $f \in S^*(\alpha,\beta,\gamma)$ if for $0 \le \alpha < 1,\ 0 < \beta \le 1$ and $0 < \gamma \le 1$.

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\gamma\left(\frac{zf'(z)}{f(z)} - \alpha\right) - \left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < \beta \ (2)$$

For Suitable values of α , β and γ, this class reduces to the following subclasses.

1) If $\beta = 1,\ \gamma = \frac{1}{2}$, then $S^*(\alpha,\beta,\gamma) = S^*(\alpha,1,1/2)$.
2) If $\alpha = 0, \beta = 1 \, and \ \gamma = \frac{2\gamma - 1}{2\gamma}, (\gamma > 1/2), then \, S^*(\alpha,\beta,\gamma) = S^*(0,1,\frac{2\gamma-1}{2\gamma})$
3) $If \ \alpha = \frac{1-\gamma}{1+\gamma},\ \beta = 1, \gamma = \frac{1+\gamma}{2}, then \, S^*(\alpha,\beta,\gamma) = S^*\left(\frac{1-\gamma}{1+\gamma},\ 1,\ \frac{1+\gamma}{2}\right)$
4) $If \ \alpha = 1-\alpha, \beta = 1 \, and \ \gamma = 1/2, then \, S^*(\alpha,\beta,\gamma) = S^*(1-\alpha,1,1/2)$.

These classes were studied by C.P. McCarty [15], R. Singh [26], K. S. Padmanabhan[21] and P.J. Eenigenburg[12].

Definition 1.2[[1], with p=1]

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G. Saravanan, School of Advanced Sciences, VIT Chennai - 600 127 (Tamil Nadu), India.
Present Address: Department of Mathematics, Patrician College of Arts and Science, Adyar, Chennai-600020, Tamil Nadu, India
Muthunagai. K, School of Advanced Sciences, VIT Chennai - 600 127, Tamil Nadu, India.

For $-1 \le A < B \le 1$, $|\lambda| < \frac{\pi}{2}$ and $0 \le \alpha < 1$, we say that a function $f(z) \in A$ is in the class $R^{\lambda}(A,B,\alpha)$ if it satisfies

$$e^{i\lambda} f'(z) \prec \cos\lambda\left[(1-\alpha)\frac{1+Az}{1+Bz}+\alpha\right]+i\sin\lambda.$$

According to Principle of sub-ordination, $f(z) \in R^{\lambda}(A,B,\alpha)$ iff there exists a function $w(z)$ satisfying $w(0)=0$ *and* $|w(z)|<1 (z \in U)$, such that

$$e^{i\lambda} f'(z) = \cos\lambda\left[(1-\alpha)\frac{1+Aw(z)}{1+Bw(z)}+\alpha\right]+i\sin\lambda.$$

(or) equivalently,

$$\left|\frac{e^{i\lambda}(f'(z)-1)}{Be^{i\lambda}f'(z)-\left[Be^{i\lambda}+(A-B)(1-\alpha)\cos\lambda\right]}\right|<1 \quad (z \in U). (3)$$

Choosing A, B and α suitably, we obtain the following subclasses

1) If $A=-1$ and $B=1$, then $R^{\lambda}(A,B,\alpha)=R^{\lambda}(-1,1,\alpha)=R^{\lambda}(\alpha)\ (0 \le \alpha < 1)$.(Refer Kanas and Sri-vastava[14]);
2) If $\alpha=0$, then $R^{\lambda}(A,B,\alpha)=R^{\lambda}(A,B,0)=R^{\lambda}(A,B)(-1 \le A < B \le 1, |\lambda| < \frac{\pi}{2})$. (Refer Shukla and Dashrath[23];
3) $R^{0}(-\beta,\beta,0)=D(\beta)$ the class of functions $f(z) \in A$ such that

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \quad (0 < \beta \le 1; z \in U)$$

introduced and studied by Padamanabhan[22] and followed by Caplinger and Causey [8].

4) $R^{0}(-\beta,\beta,\alpha)=R(\alpha,\beta)$ the class of functions $f(z) \in A$ satisfying the condition.

$$\left|\frac{f'(z)-1}{f'(z)+1-2\alpha}\right| < \beta \quad (0 \le \alpha < 1; 0 < \beta \le 1; z \in U)$$

studied by Junenja and Mogra[13].

The Condition that are Sufficient for function *f* to belong to the two classes $S^{*}(\alpha,\beta,\gamma)$ and $R^{\lambda}(A,B,\alpha)$ respectively are state below.

Theorem 1.1(See [24]).
A function $f(z)$ of the form (1) is in $S^{*}(\alpha,\beta,\gamma)$ if

$$\sum_{n=2}^{\infty}\left[(n-1)+\beta(n+1-2\gamma n-2\alpha\gamma)\right]|a_n| \le 2\beta\gamma(1-\alpha). (4)$$

$0 \le \alpha < 1, 0 < \beta \le 1$ *and* $0 < \gamma \le \frac{1}{2}$.

Theorem1.2[[1],Theorem 4, with $p=1$]
A function $f(z)$ of the form (1) is in $R^{\lambda}(A,B,\alpha)$ if

$$\sum_{n=2}^{\infty} n(1+|B|)|a_n| \le (B-A)(1-\alpha)\cos\lambda. (5)$$

$(-1 \le A < B \le 1, |\lambda| < \frac{\pi}{2}, 0 \le \alpha < 1)$.

Many Special functions appear as solutions of differential equations or

integrals of elementary functions. There is a rich literature on the geometric properties of different types of special functions. [See [4]-[6], [9], [10], [16], [17], [18], [25], [28]]
Struve functions (see [[19], [20], [29]]) are particular solutions of the non- homogeneous Bessel's differential equation

$$z^2 w''(z) + z w'(z) + (z^2 - p^2) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \quad (6)$$

where p is unrestricted real (or complex) number.
Struve functions of order p denoted by $\mathfrak{H}_p$ are given by

$$\mathfrak{H}_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\frac{3}{2})\Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1} \quad, \forall z \in \mathbb{C}. (7)$$

The solution of the non-homogeneous differential equation

$$z^2 w''(z) + z w'(z) - (z^2 + p^2) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \quad (8)$$

is called modified Struve function of order p and is defined by the formula

$$\mathfrak{L}_p(z) = -ie^{-ip\pi/2}\mathfrak{H}_p(iz) = \sum_{n=0}^{\infty} \frac{1}{\Gamma\left(n+\frac{1}{3}\right)\Gamma\left(p+n+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+p+1} \quad, \forall z \in \mathbb{C}$$

The generalized Struve function of order p given by

$$\mathfrak{w}_{p,b,c}(z) \sum_{n=0}^{\infty} \frac{(-1)^n (c)^n}{\Gamma\left(n+\frac{1}{3}\right)\Gamma\left(p+n+\frac{b+2}{2}\right)} \left(\frac{z}{2}\right)^{2n+p+1} \quad, \forall z \in \mathbb{C}.$$

is the particular solution of the second order non-homogeneous linear differential equation

$$z^2 w''(z) + bzw'(z) + \left[cz^2 - p^2 + (1-b)p\right] w(z) = \frac{4\left(z/2\right)^{p+1}}{\sqrt{\pi}\,\Gamma\left(p+\frac{b}{2}\right)} \quad (9)$$

where $b, p, c \in \mathbb{C}$ [see [20], [29] and references cited there]. Though the series defined above is convergent everywhere, the function $\mathfrak{w}_{p,b,c}(z)$ is generally not univalent in U . For $b = c = 1$, we get the Struve function (17) and for $c = -1, b = 1$, the modified Struve function (18).

Now, consider the function $\mathfrak{u}_{p,b,c}$ defined by the transformation

$$\mathfrak{u}_{p,b,c} = 2^p \sqrt{\pi}\,\Gamma\left(p+\frac{b+2}{2}\right) z^{\frac{-p-1}{2}} \mathfrak{w}_{p,b,c}(\sqrt{z}), \quad \sqrt{1} = 1.$$

We can also express $\mathfrak{u}_{p,b,c}$ as

$$\mathfrak{u}_{p,b,c} = \sum_{n=0}^{\infty} \frac{\left(-c/4\right)^n}{(m)_n \left(3/2\right)_n} z^n = b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots$$

where $m = \left(p+\frac{b+2}{2}\right) \neq 0, -1, -2, \cdots$ using the well knownPochhammer symbol (or the shifted factorial).
This function is analytic on $\mathbb{C}$ and satisfies the second-order non-homogeneous linear differential equation

$$4z^2\mathfrak{u}''(z) + 2(2p+b+3)z\mathfrak{u}'(z) + (cz+2p+b)\mathfrak{u}(z) = 2p+b.$$

For convenience throughout in the sequel, we use the following notations $\mathfrak{w}_{p,b,c}(z)=\mathfrak{w}_p(z),\quad \mathfrak{u}_{p,b,c}(z)=\mathfrak{u}_p(z),\quad m=p+\frac{b+2}{2}$ and for $c<0, m>0\ (m\neq 0,-1,-2,\cdots)$.

Let

$$z\mathfrak{u}_p(z)=z+\sum_{n=2}^{\infty}\frac{\left(-c/4\right)^{n-1}}{(m)_{n-1}\left(3/2\right)_{n-1}}z^n=z+\sum_{n=2}^{\infty}b_{n-1}z^n \quad (10)$$

and

$$\Psi(z)=z(2-\mathfrak{u}_p(z))=z-\sum_{n=2}^{\infty}\frac{\left(-c/4\right)^{n-1}}{(m)_{n-1}\left(3/2\right)_{n-1}}z^n \quad (11)$$

Lemma 1.1

If $b,p,c\in\mathbb{C}$ and $m\neq 0,-1,-2,\cdots$ then the function $\mathfrak{u}_p$ satisfies the recursive relation

$$2z\mathfrak{u}_p'(z)+\mathfrak{u}_p(z)+\frac{cz}{2m}\mathfrak{u}_{p+1}(z)=1\quad \forall z\in\mathbb{C}$$

II. MAIN RESULTS

Theorem 2.1

If $c<0, m>0\, m\neq 0,-1,-2,\cdots$, then the sufficient condition for $z\mathfrak{u}_p(z)\in S^*(\alpha,\beta,\gamma)$ is

$$\left[1+\beta(1-2\gamma)\right]\mathfrak{u}_p'(1)+2\beta\left[1-\gamma(1-\alpha)\right]\mathfrak{u}_p(1)\leq 2\beta \quad (12)$$

Moreover (12) is necessary and sufficient for $\psi(z)$, given by (11) to be in $S^*(\alpha,\beta,\gamma)$.

Proof

Theorem 1.1 states that

$$\sum_{n=2}^{\infty}\left[(n-1)+\beta(n+1-2\gamma n-2\alpha\gamma)\right]\frac{\left(-c/4\right)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}\leq 2\beta\gamma(1-\alpha) \quad (13)$$

Consider

$$\sum_{n=2}^{\infty}\left[(n-1)(1+\beta(1-2\gamma))+2\beta(1-\gamma(1+\alpha))\right]\frac{\left(-c/4\right)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}$$

$$=(1+\beta(1-2\gamma))\sum_{n=2}^{\infty}\frac{(n-1)\left(-c/4\right)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}+2\beta(1-\gamma(1+\alpha))\sum_{n=2}^{\infty}\frac{\left(-c/4\right)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}$$

$$=(1+\beta(1-2\gamma))\mathfrak{u}_p'(1)+2\beta(1-\gamma(1+\alpha))[\mathfrak{u}_p(1)-1]$$

This expression is bounded above by $2\beta\gamma(1-\alpha)$ if and only if (2.12) holds. As

$$z(2-\mathfrak{u}_p(z))=z-\sum_{n=2}^{\infty}\frac{\left(-c/4\right)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}z^n \quad (14)$$

Proof of theorem 2.1 follows from theorem 1.1.

Corollary 2.1

If $c<0, m>0\, m\neq 0,-1,-2,\cdots$, then the sufficient condition for $z\mathfrak{u}_p(z)\in S^*(\alpha,1,1/2)$ is

$$u'_p(1)+(1-\alpha)u_p(1)\le 2. (15)$$

Condition (15) is necessary and sufficient for $\psi(z)=z(2-u_p(z))$ to be in $S^*(\alpha,1,1/2)$.

Corollary 2.2

If $c<0, m>0\, m\neq 0,-1,-2,\cdots,$ then the sufficient condition for $zu_p(z)\in S^*(0,1,(2\gamma-1)/2\gamma)$, where $\gamma>1/2$ is

$$u'_p(1)+u_p(1)\le 2\gamma. (16)$$

Condition (16) is the necessary and sufficient condition for $\psi(z)=z(2-u_p(z))$ to be in $S^*(0,1,(2\gamma-1)/2\gamma)$, where $\gamma>1/2$.

Corollary 2.3

If $c<0, m>0\, m\neq 0,-1,-2,\cdots,$thenthe sufficient condition for $zu_p(z)\in S^*(1-\gamma/1+\gamma,\ 1,\ 1+\gamma/2)$, where $\gamma>1/2$ is

$$(1-\gamma)u'_p(1)+(1-\gamma)u_p(1)\le 2. (17)$$

It is well known that (17) becomes the necessary and sufficient for $\psi(z)=z(2-u_p(z))$ to belong to $S^*(1-\gamma/1+\gamma,\ 1,\ 1+\gamma/2)$ where $\gamma>1/2$.

Corollary2.4

If $c<0, m>0\, m\neq 0,-1,-2,\cdots,$ then the sufficient condition for $zu_p(z)\in S^*(1-\alpha,1,1/2)$ is

$$u'_p(1)+\alpha u_p(1)\le 2. (18)$$

Inequality (18) becomes the necessary and sufficient condition for $\psi(z)=z(2-u_p(z))$ to be in $S^*(1-\alpha,1,1/2)$.

Theorem 2.2

If $c<0, m>0\, m\neq 0,-1,-2,\cdots,$ then the sufficient condition for $zu_p(z)\in R^\lambda(A,B,\alpha)$ is

$$\left[u'_p(1)+u_p(1)\right](|B|+1)\le (B-A)(1-\alpha)cos\lambda+(|B|+1)\ (19)$$

Moreover (2.19) is necessary and sufficient for $\psi(z)$, given by (11) to be in $R^\lambda(A,B,\alpha)$.

Proof

According to Theorem 1.2, we must show that

$$\sum_{n=2}^{\infty} n(1+|B|)\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}\le (B-A)(1-\alpha)cos\lambda.$$

Now

$$\begin{aligned}&\sum_{n=2}^{\infty}((n-1)+1)(1+|B|)\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}\\ &=\left[\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}\right](1+|B|)\\ &=\left[u'_p(1)+u_p(1)-1\right](1+|B|)\end{aligned}$$

But the last expression is bounded above by $(B-A)(1-\alpha)cos\lambda$ if and only if (19) holds.

Since

$$z(2-\mathfrak{u}_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^n \quad (20)$$

the necessity of (19) for $z(2-\mathfrak{u}_p(z))$ to be in $R^{\lambda}(A,B,\alpha)$ follows from Theorem 1.2.

Corollary 2.5

If $c<0, m>0\, m\neq 0,-1,-2,\cdots$, then the sufficient condition for $z\mathfrak{u}_p(z)\in R^{\lambda}(-1,1,\alpha)=R^{\lambda}(\alpha)$ is

$$\mathfrak{u}'_p(1)+\mathfrak{u}_p(1)\le(1-\alpha)cos\lambda+1. \quad (21)$$

The result stated in inequality (21) is necessary and sufficient for $\psi(z)=z(2-\mathfrak{u}_p(z))$ to be in $R^{\lambda}(-1,1,\alpha)=R^{\lambda}(\alpha)$.

Corollary 2.6

If $c<0, m>0\; m\neq 0,-1,-2,\cdots$, then the sufficient condition for $z\mathfrak{u}_p(z)\in R^{\lambda}(A,B)$ is

$$\mathfrak{u}'_p(1)+\mathfrak{u}_p(1)\le(B-A)cos\lambda+(|B|+1). \quad (22)$$

We also state that (22) is necessary and sufficient for $\psi(z)=z(2-\mathfrak{u}_p(z))$ to be in $R^{\lambda}(A,B)$.

Corollary 2.7

If $c<0, m>0\; m\neq 0,-1,-2,\cdots$, then the sufficient condition for $z\mathfrak{u}_p(z)\in R^{0}(-\beta,\beta,0)=D(\beta)$ is

$$\mathfrak{u}'_p(1)+\mathfrak{u}_p(1)\le\frac{2\beta}{\beta+1}cos\lambda+1. \quad (23)$$

Moreover (23) is necessary and sufficient for $\psi(z)=z(2-\mathfrak{u}_p(z))$ to be in $R^{0}(-\beta,\beta,0)=D(\beta)$.

Corollary 2.8

If $c<0, m>0\, m\neq 0,-1,-2,\cdots$, then the sufficient condition for $z\mathfrak{u}_p(z)\in R^{0}(-\beta,\beta,\alpha)=R(\alpha,\beta)$ is

$$\mathfrak{u}'_p(1)+\mathfrak{u}_p(1)\le\frac{2\beta}{\beta+1}(1-\alpha)cos\lambda+1. \quad (24)$$

Condition (24) is necessary and sufficient for $\psi(z)=z(2-\mathfrak{u}_p(z))$ to be in $R^{0}(-\beta,\beta,\alpha)=R(\alpha,\beta)$.

III. INCLUSION PROPERTIES

For functions $f,g\in A$ given by $f(z)=z+\sum_{n=2}^{\infty}a_n z^n, g(z)=z+\sum_{n=2}^{\infty}b_n z^n$, the HadamardProduct(or) Convolution is defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_n b_n z^n, \quad z\in U.$$

The linear operator

$$\Im(c,m): A\to A$$

defined by

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$$\Im(c,m)f(z)=z\mathrm{u}_{p,b,c}(z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}z^n$$

where $m=p+\frac{b+2}{2}\neq 0$. A function $f\in A$ is said to be in the class $R^{\tau}(A,B),\tau\in\mathbb{C}-\{0\},-1\le A<B\le 1$ if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(B-A)\tau-B[f'(z)-1]}\right|<1 \quad z\in U.$$

The class $R^{\lambda}(A,B)$ was introduced earlier by Dixit and Pal [11]. If we put

$$\tau=1, B=\beta \quad and \quad A=-\beta \quad (0<\beta\le 1).$$

we obtain the class of functions $f\in A$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right|<\beta \quad (z\in\mathbb{C};0<\beta\le 1).$$

which was studied by (among others) Padmanabhan [22] and Caplinger and Causey [8]. Making use of the following lemma, we will study the effect of the Struve function on the class $R^{\lambda}(A,B,\alpha)$.

Lemma 3.2

If $f\in R^{\tau}(A,B)$ is of form (1), then

$$|a_n|\le(B-A)\frac{|\tau|}{n}, \quad n\in N\setminus\{1\} \quad (25)$$

The bound given in (25) is sharp.

Theorem 3.1

Let $c<0, m>0\; m\neq 0,-1,-2,\cdots$. If $f\in R^{\tau}(A,B)$ and if the inequality

$$|\tau|(1+|B|)(\mathrm{u}_p(1)-1)\le(1-\alpha)cos\lambda. \quad (26)$$

holds, then $\Im(c,m)(f)\in R^{\lambda}(A,B,\alpha)$.

Proof

Let f of the form (1) belong to the class $R^{\tau}(A,B)$. By virtue of Theorem 1.2 if suffices to show that

$$\mathfrak{L}(A,B,\alpha)=\sum_{n=2}^{\infty}n(1+|B|)\left[\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}\right]|a_n|\le(B-A)(1-\alpha)cos\lambda$$

Since $f\in R^{\tau}(A,B)$ then by Lemma 3.2 we have,

$$|a_n|\le(B-A)\frac{|\tau|}{n}.$$

Hence

$$\begin{aligned}\mathfrak{L}(A,B,\alpha)&=\sum_{n=2}^{\infty}n(1+|B|)\left[\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}\right]|a_n|\\&\le(B-A)|\tau|\sum_{n=2}^{\infty}(1+|B|)\left[\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}\right]\end{aligned} \quad (27)$$

Further, proceeding as in Theorem 2.1, we get

$$\mathfrak{L}(A,B,\alpha)\le|\tau|(1+|B|)\left(\mathrm{u}_p(1)-1\right).$$

Which is bounded above $(1-\alpha)cos\lambda$ if and only if (27) holds.

Theorem 3.2

Let $c < 0, m > 0\, m \neq 0, -1, -2, \cdots$, then

$$\mathfrak{L}(m,c,z) = \int_0^z \left(2 - \mathfrak{u}_p(t)\right) dt$$

is in $R^{\lambda}(A,B,\alpha)$ if and only if

$$(1+|\beta|)\left(\mathfrak{u}_p(1) - 1\right) \leq (B-A)(1-\alpha)cos\lambda \ (28)$$

Proof

Since

$$\mathfrak{L}(m,c,z) = z - \sum_{n=2}^{\infty} \frac{\left(-c/4\right)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \frac{z^n}{n}$$

By theorem 1.2 we need to show that

$$\sum_{n=2}^{\infty} n(1+|B|) \frac{\left(-c/4\right)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \leq (B-A)(1-\alpha)cos\lambda \ (29)$$

That is, let

$$\mathfrak{P}(m,c,z) = \left[(1+|B|)\left(\mathfrak{u}_p(1) - 1\right)\right].$$

which is bounded above by $(B-A)(1-\alpha)cos\lambda$ if and only if (29) holds.

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