Some constraints for the Struve function to belong to certain subclasses of Analytic functions

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Abstract: The objective of the present article is to obtain some constraints that are sufficient for the generalized Struve functions of first kind to belong to the subclasses $S^*(\alpha, \beta, \gamma)$, $R^*(A, B, \alpha)$ and to study the inclusion properties.

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I. INTRODUCTION

The class of all normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in U$$

be denoted by $A$.

Let $S$ be the subclass of $A$, which consists of all univalent functions in the open unit disk $U$.

**Definition 1.1** [24]

Let $f \in A$. Then $f \in S^*(\alpha, \beta, \gamma)$ if for $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $0 < \gamma \leq 1$.

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \beta \quad (2)$$

For Suitable values of $\alpha$, $\beta$ and $\gamma$, this class reduces to the following subclasses.

1) If $\beta = 1$, $\gamma = \frac{1}{2}$, then $S^*(\alpha, \beta, \gamma) = S^*(\alpha, 1, 1/2)$.

2) If $\alpha = 0$, $\beta = 1$ and $\gamma = \frac{2\gamma - 1}{2\gamma}$, $(\gamma > 1/2)$, then $S^*(\alpha, \beta, \gamma) = S^*(0, 1, \frac{2\gamma - 1}{2\gamma})$

3) If $\alpha = \frac{1-\gamma}{1+\gamma}$, $\beta = 1$, $\gamma = \frac{1+\gamma}{2}$, then $S^*(\alpha, \beta, \gamma) = S^*(\frac{1-\gamma}{1+\gamma}, 1, \frac{1+\gamma}{2})$

4) If $\alpha = 1-\alpha$, $\beta = 1$ and $\gamma = 1/2$, then $S^*(\alpha, \beta, \gamma) = S^*(1-\alpha, 1, 1/2)$.

These classes were studied by C.P. McCarty [15], R. Singh [26], K. S. Padmanabhan[21] and P.J. Eenigenburg[12].

**Definition 1.2** [1, with p=1]
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For $-1 \leq A < B \leq 1, \left| \lambda \right| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, we say that a function $f(z) \in A$ is in the class $R^\lambda (A, B, \alpha)$ if it satisfies

$$e^{\lambda z} f'(z) < \cos \lambda \left[ (1 - \alpha) \frac{1 + A z}{1 + B z} + \alpha \right] + \text{isin}\lambda.$$  

According to Principle of sub-ordination, $f(z) \in R^\lambda (A, B, \alpha)$ iff there exists a function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1(z \in U)$, such that

$$e^{\lambda z} f'(z) = \cos \lambda \left[ (1 - \alpha) \frac{1 + A w(z)}{1 + B w(z)} + \alpha \right] + \text{isin}\lambda.$$  

(or) equivalently,

$$\left| \frac{e^{\lambda (f'(z)-1)}}{Be^{\lambda f'(z)}+(A-B)(1-\alpha)\cos\lambda} \right| < 1 \quad (z \in U), (3)$$

Choosing $A$, $B$ and $\alpha$ suitably, we obtain the following subclasses:

1) If $A = -1$ and $B = 1$, then $R^\lambda (A, B, \alpha) = R^\lambda (-1, 1, \alpha) = R^\lambda (\alpha) \ (0 \leq \alpha < 1).$ (Refer Kanas and Sri-vastava[14]);

2) If $\alpha = 0$, then $R^\lambda (A, B, \alpha) = R^\lambda (A, B, 0) = R^\lambda (A, B) (-1 \leq A < B \leq 1, |\lambda| < \frac{\pi}{2})$. (Refer Shukla and Dashrath[23];

3) $R^0 (\beta, \beta, 0) = D(\beta)$ the class of functions $f(z) \in A$ such that

$$\left| \frac{f'(z)-1}{f'(z)+1} \right| < \beta \quad (0 < \beta \leq 1; z \in U)$$

introduced and studied by Padamanabhan[22] and followed by Caplinger and Causey [8].

4) $R^0 (\beta, \beta, \alpha) = R(\alpha, \beta)$ the class of functions $f(z) \in A$ satisfying the condition.

$$\left| \frac{f'(z)-1}{f'(z)+1-2\alpha} \right| < \beta \quad (0 \leq \alpha < 1; 0 < \beta \leq 1; z \in U)$$

studied by Junenja and Mogra[13].

The Condition that are Sufficient for function $f$ to belong to the two classes $S^\alpha (\alpha, \beta, \gamma)$ and $R^\lambda (A, B, \alpha)$ respectively are state below.

**Theorem 1.1** (See [24]).

A function $f(z)$ of the form (1) is in $S^\alpha (\alpha, \beta, \gamma)$ if

$$\sum_{n=2}^{\infty} \left| (n-1) + \beta(n+1-2\gamma n-2\alpha \gamma) \right| |a_n| \leq 2\beta\gamma(1-\alpha). \quad (4)$$

$0 \leq \alpha < 1, 0 < \beta \leq 1$ and $0 < \gamma \leq \frac{1}{2}$.

**Theorem 1.2** [1], Theorem 4, with $p = 1$

A function $f(z)$ of the form (1) is in $R^\lambda (A, B, \alpha)$ if

$$\sum_{n=2}^{\infty} n(1+|B|)|a_n| \leq (B-A)(1-\alpha)\cos\lambda. \quad (5)$$

$(-1 \leq A < B \leq 1, |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1)$.

Many Special functions appear as solutions of differential equations or
integrals of elementary functions. There is a rich literature on the geometric properties of different types of special functions. [See [4]-[6], [9], [10], [16], [17], [18], [25], [28]] Struve functions (see [19], [20], [29]) are particular solutions of the non-homogeneous Bessel's differential equation

\[
\frac{d^2}{dz^2} w(z) + \frac{2}{z} \frac{d}{dz} w(z) + \left(\frac{z^2}{2} - \left(\frac{p^2}{4} + \frac{1}{4}\right)\right) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi} \Gamma\left(\frac{p+1}{2}\right)}
\]

where \( p \) is unrestricted real (or complex) number.

Struve functions of order \( p \) denoted by \( \mathcal{H}_p(z) \) are given by

\[
\mathcal{H}_p(z) = \frac{1}{\sqrt{\pi} \Gamma\left(\frac{3}{2}\right)} \Gamma\left(\frac{p + \frac{1}{2}}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\left(n + \frac{1}{2}\right) \Gamma\left(n + p + \frac{1}{2}\right)} \left(\frac{z^2}{4}\right)^n , \forall z \in \mathbb{C}, (7)
\]

The solution of the non-homogeneous differential equation

\[
\frac{d^2}{dz^2} w(z) + \frac{2}{z} \frac{d}{dz} w(z) - \left(z^2 - \left(p^2 + 1\right)\right) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi} \Gamma\left(\frac{p+1}{2}\right)}
\]

is called modified Struve function of order \( p \) and is defined by the formula

\[
\mathfrak{L}_p(z) = -ie^{-i\pi/2} \mathcal{H}_p(iz) = \frac{1}{\sqrt{\pi} \Gamma\left(\frac{3}{2}\right)} \Gamma\left(\frac{p + \frac{1}{2}}{2}\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\left(n + \frac{1}{2}\right) \Gamma\left(n + p + \frac{1}{2}\right)} \left(\frac{z^2}{4}\right)^n , \forall z \in \mathbb{C}
\]

The generalized Struve function of order \( p \) given by

\[
\mathfrak{m}_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (e)^n}{\Gamma\left(n + \frac{1}{2}\right) \Gamma\left(n + p + b + \frac{1}{2}\right)} \left(\frac{z^2}{4}\right)^n , \forall z \in \mathbb{C}
\]

is the particular solution of the second order non-homogeneous linear differential equation

\[
\frac{d^2}{dz^2} w(z) + bw'(z) + \left[cz^2 - p^2 + (1-b)p\right] w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi} \Gamma\left(\frac{p + b}{2}\right)}
\]

where \( b, p, c \in \mathbb{C} \) [see [20], [29] and references cited there]. Though the series defined above is convergent everywhere, the function \( \mathfrak{m}_{p,b,c}(z) \) is generally not univalent in \( U \). For \( b = c = 1 \), we get the Struve function (17) and for \( c = -1, b = 1 \), the modified Struve function (18).

Now, consider the function \( \mathfrak{u}_{p,b,c}(z) \) defined by the transformation

\[
\mathfrak{u}_{p,b,c} = 2^p \sqrt{\pi} \Gamma\left(\frac{p + b + 2}{2}\right) \mathfrak{m}_{p,b,c}(\sqrt{z}) , \sqrt{1} = 1.
\]

We can also express \( \mathfrak{u}_{p,b,c} \) as

\[
\mathfrak{u}_{p,b,c} = \sum_{m=0}^{\infty} \frac{(-c/4)^m}{(m)!} z^m = b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots
\]

where \( m = \left(\frac{p + b + 2}{2}\right) \neq 0, -1, -2, \cdots \) using the well known Pochhammer symbol (or the shifted factorial).

This function is analytic on \( \mathbb{C} \) and satisfies the second-order non-homogeneous linear differential equation

\[
4z^2 \mathfrak{u}'(z) + 2(2p + b + 3)z \mathfrak{u}'(z) + (cz + 2p + b) \mathfrak{u}(z) = 2p + b.
\]
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For convenience throughout in the sequel, we use the following notations
\[ w_{p,b,c}(z) = w_p(z), \quad u_{p,b,c}(z) = u_p(z), \quad m = p + \frac{b+2}{2} \] and for \( c < 0, m > 0 \) \( (m \neq 0, -1, -2, \cdots) \).

Let
\[ z u_p(z) = z + \sum_{n=2}^{\infty} \left( \frac{-c/4}{m}_{n-1} (3/2)_{n-1} \right) z^n = z + \sum_{n=2}^{\infty} b_n z^n \] (10)
and
\[ \Psi(z) = z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \left( \frac{-c/4}{m}_{n-1} (3/2)_{n-1} \right) z^n \] (11)

Lemma 1.1

If \( b, p, c \in \mathbb{C} \) and \( m \neq 0, -1, -2, \cdots \) then the function \( u_p \) satisfies the recursive relation
\[ 2 z u'_p(z) + u_p(z) + \frac{cz}{2m} u_{p+1}(z) = 1 \quad \forall z \in \mathbb{C} \]

II. MAIN RESULTS

Theorem 2.1

If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \cdots \) then the sufficient condition for \( z u_p(z) \in S^*(\alpha, \beta, \gamma) \) is
\[ [1 + \beta(1 - 2\gamma)] u'_p(1) + 2\beta [1 - \gamma(1 - \alpha)] u_p(1) \leq 2\beta \] (12)

Moreover (12) is necessary and sufficient for \( \psi(z) \), given by (11) to be in \( S^*(\alpha, \beta, \gamma) \).

Proof

Theorem 1.1 states that
\[ \sum_{n=2}^{\infty} \left( (n-1) + \beta(n+1 - 2\alpha - 2\gamma) \right) \left( \frac{-c/4}{m}_{n-1} (3/2)_{n-1} \right) \leq 2\beta \]
(13)

Consider
\[ \sum_{n=2}^{\infty} \left( (n-1)(1 + \beta(1 - 2\gamma)) + 2\beta(1 - \gamma(1 + \alpha)) \right) \left( \frac{-c/4}{m}_{n-1} (3/2)_{n-1} \right) \]
\[ = (1 + \beta(1 - 2\gamma)) \sum_{n=2}^{\infty} \left( \frac{c}{m}_{n-1} (3/2)_{n-1} \right) + 2\beta(1 - \gamma(1 + \alpha)) \sum_{n=2}^{\infty} \left( \frac{c}{m}_{n-1} (3/2)_{n-1} \right) \]
\[ = (1 + \beta(1 - 2\gamma)) u'_p(1) + 2\beta(1 - \gamma(1 + \alpha)) [u_p(1) - 1] \]
This expression is bounded above by \( 2\beta \) if and only if (2.12) holds. As
\[ z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \left( \frac{-c/4}{m}_{n-1} (3/2)_{n-1} \right) z^n \] (14)

Proof of theorem 2.1 follows from theorem 1.1.

Corollary 2.1

If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \cdots \) then the sufficient condition for \( z u_p(z) \in S^*(\alpha, 1, 1/2) \) is
\[ u'_p (1) + (1 - \alpha) u_p (1) \leq 2. \tag{15} \]

Condition (15) is necessary and sufficient for \( \psi(z) = z(2 - u_p(z)) \) to be in \( S'(\alpha, 1, 1/2) \).

**Corollary 2.2**

If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \ldots \), then the sufficient condition for \( z u_p(z) \in S'(0, 1, (2\gamma - 1)/2\gamma) \), where \( \gamma > 1/2 \) is

\[ u'_p (1) + u_p (1) \leq 2\gamma. \tag{16} \]

Condition (16) is the necessary and sufficient condition for \( \psi(z) = z(2 - u_p(z)) \) to be in \( S'(0, 1, (2\gamma - 1)/2\gamma) \), where \( \gamma > 1/2 \).

**Corollary 2.3**

If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \ldots \), then the sufficient condition for \( z u_p(z) \in S'(1 - \gamma/1 + \gamma, 1, 1 + \gamma/2) \), where \( \gamma > 1/2 \) is

\[ (1 - \gamma) u'_p (1) + (1 - \gamma) u_p (1) \leq 2. \tag{17} \]

It is well known that (17) becomes the necessary and sufficient for \( \psi(z) = z(2 - u_p(z)) \) to belong to \( S'(1 - \gamma/1 + \gamma, 1, 1 + \gamma/2) \) where \( \gamma > 1/2 \).

**Corollary 2.4**

If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \ldots \), then the sufficient condition for \( z u_p(z) \in S'(1 - \alpha, 1, 1/2) \) is

\[ u'_p (1) + \alpha u_p (1) \leq 2. \tag{18} \]

Inequality (18) becomes the necessary and sufficient condition for \( \psi(z) = z(2 - u_p(z)) \) to be in \( S'(1 - \alpha, 1, 1/2) \).

**Theorem 2.2**

If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \ldots \), then the sufficient condition for \( z u_p(z) \in R^L (A, B, \alpha) \) is

\[ \left[ u'_p (1) + u_p (1) \right] (|B| + 1) \leq (B - A)(1 - \alpha) \cos \lambda + (|B| + 1) \tag{19} \]

Moreover (2.19) is necessary and sufficient for \( \psi(z) \), given by (11) to be in \( R^L (A, B, \alpha) \).

**Proof**

According to Theorem 1.2, we must show that

\[
\sum_{n=2}^{\infty} n (1+|B|) \left( \frac{-c/4}{(m)_{n-1}(3/2)_{n-1}} \right) \leq (B-A)(1-\alpha) \cos \lambda.
\]

Now

\[
\sum_{n=2}^{\infty} (n-1) (1+|B|) \left( \frac{-c/4}{(m)_{n-1}(3/2)_{n-1}} \right) = \left[ u'_p (1) + u_p (1) - 1 \right] (1+|B|)
\]

But the last expression is bounded above by \((B-A)(1-\alpha) \cos \lambda\) if and only if (19) holds.
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Since

\[ z(2 - u_\rho(z)) = z - \sum_{n=2}^{\infty} \left( \frac{-c}{4} \right)^{n-1} \frac{z^n}{(m)_{n-1}(3/2)_{n-1}} \]  \tag{20} \]

the necessity of (19) for \( z(2 - u_\rho(z)) \) to be in \( R^j(A, B, \alpha) \) follows from Theorem 1.2.

**Corollary 2.5**
If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \ldots \), then the sufficient condition for \( z u_\rho(z) \in R^j(-1, 1, \alpha) = R^j(\alpha) \) is

\[ u'_\rho(1) + u_\rho(1) \leq (1 - \alpha) \cos \lambda + 1. \]  \tag{21} \]

The result stated in inequality (21) is necessary and sufficient for \( \psi(z) = z(2 - u_\rho(z)) \) to be in \( R^j(-1, 1, \alpha) = R^j(\alpha) \).

**Corollary 2.6**
If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \ldots \), then the sufficient condition for \( z u_\rho(z) \in R^j(A, B) \) is

\[ u'_\rho(1) + u_\rho(1) \leq (B - A) \cos \lambda + (|B| + 1). \]  \tag{22} \]

We also state that (22) is necessary and sufficient for \( \psi(z) = z(2 - u_\rho(z)) \) to be in \( R^j(A, B) \).

**Corollary 2.7**
If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \ldots \), then the sufficient condition for \( z u_\rho(z) \in R^0(-\beta, \beta, 0) = D(\beta) \) is

\[ u'_\rho(1) + u_\rho(1) \leq \frac{2\beta}{\beta + 1} \cos \lambda + 1. \]  \tag{23} \]

Moreover (23) is necessary and sufficient for \( \psi(z) = z(2 - u_\rho(z)) \) to be in \( R^0(-\beta, \beta, 0) = D(\beta) \).

**Corollary 2.8**
If \( c < 0, m > 0 \) \( m \neq 0, -1, -2, \ldots \), then the sufficient condition for \( z u_\rho(z) \in R^0(-\beta, \beta, \alpha) = R(\alpha, \beta) \) is

\[ u'_\rho(1) + u_\rho(1) \leq \frac{2\beta}{\beta + 1} (1 - \alpha) \cos \lambda + 1. \]  \tag{24} \]

Condition (24) is necessary and sufficient for \( \psi(z) = z(2 - u_\rho(z)) \) to be in \( R^0(-\beta, \beta, \alpha) = R(\alpha, \beta) \).

**III. INCLUSION PROPERTIES**

For functions \( f, g \in A \) given by \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = z + \sum_{n=2}^{\infty} b_n z^n \), the HadamardProduct(or) Convolution is defined by

\[ (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \]

The linear operator

\[ \mathcal{I}(c, m) : A \rightarrow A \]

defined by
\( \mathcal{Z}(c, m) f(z) = zu_{p,b,c}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^n \)

where \( m = p + \frac{b+2}{2} \neq 0 \). A function \( f \in A \) is said to be in the class \( R^\tau(A,B), \tau \in \mathbb{C} - \{0\}, -1 \leq A < B \leq 1 \) if it satisfies the inequality

\[
\left| \frac{f'(z) - 1}{(B - A)\tau - B[f'(z) - 1]} \right| < 1 \quad z \in U.
\]

The class \( R^\tau(A,B) \) was introduced earlier by Dixit and Pal [11]. If we put \( \tau = 1, B = \beta \) and \( A = -\beta \) \((0 < \beta \leq 1)\), we obtain the class of functions \( f \in A \) satisfying the inequality

\[
\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in \mathbb{C}; 0 < \beta \leq 1).
\]

which was studied by (among others) Padmanabhan [22] and Caplinger and Causey [8]. Making use of the following lemma, we will study the effect of the Struve function on the class \( R^\tau(A,B,\alpha) \).

**Lemma 3.2**
If \( f \in R^\tau(A,B) \) is of form (1), then

\[
|a_n| \leq (B - A) \left| \frac{\tau}{n} \right| , \quad n \in \mathbb{N} \setminus \{1\} \quad (25)
\]

The bound given in (25) is sharp.

**Theorem 3.1**
Let \( c < 0, m > 0 m \neq 0, -1, -2, \ldots \). If \( f \in R^\tau(A,B) \) and if the inequality

\[
|\tau| (1+|B|)(u_{p,1} - 1) \leq (1 - \alpha)\cos \lambda \quad (26)
\]

holds, then \( \mathcal{Z}(c, m)(f) \in R^\tau(A,B,\alpha) \).

**Proof**
Let \( f \) of the form (1) belong to the class \( R^\tau(A,B) \). By virtue of Theorem 1.2 if suffices to show that

\[
\mathcal{S}(A,B,\alpha) = \sum_{n=2}^{\infty} n(1+|B|) \left[ \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \right] |a_n| \leq (B - A)(1 - \alpha)\cos \lambda.
\]

Since \( f \in R^\tau(A,B) \) then by Lemma 3.2 we have,

\[
|a_n| \leq (B - A) \left| \frac{\tau}{n} \right| .
\]

Hence

\[
\mathcal{S}(A,B,\alpha) = \sum_{n=2}^{\infty} n(1+|B|) \left[ \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \right] |a_n| \leq (B - A) |\tau| \sum_{n=2}^{\infty} (1+|B|) \left[ \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} \right] \quad (27)
\]

Further, proceeding as in Theorem 2.1, we get

\[
\mathcal{S}(A,B,\alpha) \leq |\tau| (1+|B|)(u_{p,1} - 1).
\]

Which is bounded above \((1 - \alpha)\cos \lambda\) if and only if (27) holds.
Theorem 3.2
Let \( c < 0, m > 0 \) m \( \neq 0, -1, -2, \ldots \), then
\[
\mathcal{L}(m,c,z) = \int_0^\infty \left( 2 - u_p(t) \right) dt
\]
is in \( R^2(A,B,\alpha) \) if and only if
\[
(1 + |\beta|) \left( u_p(t) - 1 \right) \leq (B - A)(1 - \alpha)cos\lambda \quad (28)
\]

Proof
Since
\[
\mathcal{L}(m,c,z) = z - \sum_{n=2}^{\infty} \left( -c / 4 \right)^{n-1} \frac{z^n}{n}
\]
By theorem 1.2 we need to show that
\[
\sum_{n=2}^{\infty} p(1 + |B|) \left( -c / 4 \right)^{n-1} \frac{1}{(m,n-1)(3/2)_{n-1}} \leq (B - A)(1 - \alpha)cos\lambda \quad (29)
\]
That is, let
\[
\mathfrak{U}(m,c,z) = \left[ (1 + |B|) \left( u_p(t) - 1 \right) \right].
\]
which is bounded above by \( (B - A)(1 - \alpha)cos\lambda \) if and only if \( (29) \) holds.

REFERENCES