Approximate Singularly Perturbed Boundary Value Problems Using G-Spline Based Differential Quadrature Method

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Abstract: The objective of this paper is to obtain an approximate solution to a singularly flustered boundary worth issues involving differential equation using differential construction technique. The specific procedure of the weight coefficients for estimation of derivatives are obtained by means of g-spline interpolation method. An illustrative example have been analyzed and compared with the precise resolution to determine the certainty, and capability, of the proposed technique.

Keywords: Differential Construction technique, g-spline interpolation, Singularly perturbed boundary worth issues.

I. INTRODUCTION

The worth of differential quadrature (DQ) escalated after being extended to determine the derivatives of numerous orders of a abundantly bland action [1-3]. In the added words, by-product of a bland action is almost accurate with sum of method characters at a cluster of nodes [4].

The significant method within the application of DQ lies in determining the coefficients. Originally, for the first order by-product, Bellman and Casti[1] appropriate two coefficients calculative strategies. The 1st is predicted on ill-conditioned pure mathematics equation method, whereas the 2nd employs easy pure mathematics formulation, however the grid points coordinates are mounted by the roots of the shifted Legendre polynomial [5].

Due to the pioneering efforts by the authors [6-13] within the computation of the coefficients, DQ technique and its applications developed speedly, and appeared as a prestigious numerical discretization tool. DQ will accomplish extremely precise results utilizing a considerably smaller variety of grid points, and thus demands comparatively very little process efforts, when put next to the standard low order finite distinction and finite part strategies. So far, DQ technique has been utilized with efficiency in a very style of issues in engineering and physical sciences. A comprehensive review has been given by Bert and Malik [10]. In 1968, Schoenberg [14] had presented the topic on spline function.

Since then splines have incontestible to be vastly vital in varied divisions of arithmetic, like mathematical analysis, mathematical treatment of normal, basic, limited mathematical equations, and approximation theory. Schoenberg [14] expanded a thought of Hermite for splines to mention which the order of derivatives specific might differ from node to node.

The g-spline is utilized to add the Hermite-Birkhoff problem (HB-problem). Information of this drawback are the values of the strategies and its by-product however while not Hermite's rule which the sole sequence be employed at every node. Moreover, Schoenberg [14] outlined g-spline as sleek partwise polynomials, their sleekness dominated by the occurrence matrix, so showed it fulfill the "minimum norm property", utilized to an ideal of the g-spline technique.

Define of this paper is as follows. Section two are going to be groundwork for the succeeding sections. Section three shows the approximation of definite practical with the knowledge of g-spline formula. Moreover, details of g-spline based mostly differential construction technique (GDQM) are presented in Section 4. In Section 5 the outline of singularly discomposed boundary worth problem is given. In Section 6 the capabilities of the developed GDQM is illustrated through two examples, and eventually the conclusion of this work is given in Section 7.

II. PRELIMINARIES

This section provides the mandatory definitions and notations that will be employed in the following sections.

Definition 2.1 [14]**.** To define The HB-problem, examine the node points arranged in ascending order of magnitude and represented as $x_1 < x_2 < \cdots < x_k$. Let α be the outmost of the orders of the by-product is described at the nodes. Outline an occurrence matrix $E = [a_{ij}]$, where $i = 1,2,\cdots,k$, $j = 0,1,\ldots,\alpha$, $e = \{(i,j) | i = 1,2,\cdots,k;\ j = 0,1,\ldots,\alpha$, and

$$a_{ij} = \begin{cases} 1 & (i,j) \in e, \\ 0 & (i,j) \notin e. \end{cases}$$

e is chosen as how that i takes the characters $1,2,\cdots,k$; one or additional times, while $j \in \{0,1,\ldots,\alpha\}$ and $j = \alpha$ is earned a minimum of one part (i,j) of e. Assume that every row of the occurrence matrix E and last column of E ought to contain some components upto 1. Let $y_i^{(j)}$ given real number for every $(i,j) \in e$. The HB-problem is to determine $f(x) \in C^{\alpha}$, which fulfills the interpolatory condition,

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$f^{(j)}(x_i) = y_i^{(j)}$ for $(i,j) \in e$ (1)

The matrix E will likewise outline the set of equations (1). If we have a tendency to outline $e = \{(i,j) | a_{ij} = 1\}$ then the whole number $n = \sum_{i,j} a_{ij}$ is actually the amount of interpolatory conditions required to represent the system (1).

Definition 2.2 [15]**.** Let m is a range, then the HB-problem (1) is claimed to be m-poised on condition that if $p(x) \epsilon \Pi_{m-1}$, $p^{(j)}(x_i) = 0$ if $(i,j) \in e$ then $p(x) = 0$. (Π_{m-1} is that the category of polynomials of degree m–1 or less). At this time, a g-spline interpolant in form m to f will granted to agreement of the basic g-spline strategies $L_{ij}(x)$,

$S_m(x) = \sum_{(i,j)\in e} L_{ij}(x)\, y_i^{(j)}$, (2)

where

$$L_{ij}^{(s)}(x_r) = \begin{cases} 0 & (r,s) \neq (i,j), \\ 1 & (r,s) = (i,j). \end{cases}$$

The explanation of g-spline is expedited using process matrix E^* acquired with the occurrence matrix E by including $m - \alpha - 1$ columns of zeros to the matrix E. Let $E^* = [a_{ij}^*]$ where $i = 1,2,\cdots,k;\ j = 0,1,\ldots,m-1$, and

$$a_{ij}^* = \begin{cases} a_{ij} & j \leq \alpha, \\ 0 & j = \alpha+1, \alpha+2, \ldots, m-1. \end{cases}$$

If $j = \alpha + 1$, then $E^* = E$.

Definition 2.3 [16]**.** A method $S(x)$ is accepted as common g-spline to nodes $x_1, x_2, \cdots, x_k$ and also the matrix E^* of form m if it fulfills the afterward conditions,

1. $S(x) \epsilon \Pi_{2m-1}$ in (x_i, x_{i+1}), $i = 1,2,\cdots,k-1$.
2. $S(x) \epsilon \Pi_{m-1}$ in $(-\infty, x_1)$, and in $(x_k, \infty,)$.
3. $S(x) \epsilon C^{m-1}(-\infty, \infty)$.
4. If $a_{ij}^* = 0$, then $S^{(2m-j-1)}(x)$ is constant at $x = x_i$.

Also, $S(E^*, x_1, x_2, \cdots, x_k)$ signifies the category of all g-spline of order m.

Definition 2.4 [17]**.** An n-th degree limited power basis with N knots $\chi_1, \chi_2, \ldots, \chi_N$ are $1, x, \ldots, x^n$, $(x-\chi_1)_+^n$, …, $(x - \chi N + n$, wherever $\tau + n = \tau + n$ denotes power n of the positive a part of τ with $\tau_+ = max(0, \tau)$.

One ought to notice that, the primary $(n+1)$ basis functions of the on top limited power basis are polynomials of degree up to n, and also the others are all the limited power functions of degree n. Conventionally, the premise of degree n = Zero, 1,2, 3 is named constant, linear, quadratic, three-dimensional limited power basis, respectively.

III. THE INTERPOLATION WITH G-SPLINE

Let $I = [\alpha, \beta]$ be a finite interval containing the knots $x_1, x_2, \cdots, x_k$ and consider a linear functional $\mathcal{L}f: C^{\ell}[\alpha,\beta] \longrightarrow R$ of the form,

$$\mathcal{L}f = \sum_{j=0}^{\ell} \int_{\alpha}^{\beta} \alpha_j(x) f^{(j)}(x)\, dx + \sum_{j=0}^{\ell} \sum_{i=1}^{\eta_j} \beta_{ji}\, f^{(j)}(x_{ji}). \quad (3)$$

Here $\alpha_j(x)$ are partwise continuous methods in I, $x_{ji} \in I$ and β_{ji} are real constants, then (3) could also be approximated as follows,

$\mathcal{L}f = \sum_{(i,j\in e)} \mathrm{B}_{ij} f^{(j)}(x_i) + Rf$ (4)

where Rf represents the rest. Hence, therefore on notify the approximation low frequency like (4), that's finest in some sense, the real's B_{ij} ought to be determined. Schoenberg [14] proposed Sard procedure to find B_{ij}.If $\ell < m < \eta$ and thus HB-problem (1) is m-poised, and Sard's perfect approximation (4) to low frequency in form m is acquired then performing $\mathcal{L}$ of either part within a g-spline interpolation formula (2) of form m. The coefficients $B_{ij} = \mathcal{L}L_{ij}(x)$ and $L_{ij}(x)$ are the basic methods.

IV. THE G-SPLINE BASED DIFFERENTIAL QUADRATURE METHOD (GDQM)

Assume $f(x)$ is abundantly swish method on the time $[x_1, x_N]$, and allow us to determine an m-poised Hermite-Birkoff problem $f^{(j)}(x_i) = y_i^{(j)}$, $(i,j) \in e$ on the N decided nodes $x_1 < x_2 < \cdots < x_N$ supported differential quadrature, the primary and second order by-product on every of these nodes are specified by the subsequent formulas,

$$\left.\frac{df}{dx}\right|_{x=x_k} = \sum_{(i,j)\in e} \alpha_{ki}^{(j)} f_i^{(j)}(x_i),$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=x_k} = \sum\nolimits_{(i,j)\in e} \beta_{ki}^{(j)} f_i^{(j)}(x_i),$$

where $k = 1,2,\cdots,N$.The coefficients $\alpha_{ki}^{(j)}$and $\beta_{ki}^{(j)}$ are the coefficients. In order to determine the weights $\alpha_{ki}^{(j)}$ and $\beta_{ki}^{(j)}$, we to think about associate m-poised Hermite-Birkhoff drawback to almost accurate the method f and our reason to be form a indeterminate of x, in that shape:

$f(x) = \sum_{(i,j)\in e} L_{ij}(x) f_i^{(j)}$,

which satisfy

$$L_{ij}^{(s)}(x_r) = \begin{cases} 1 & (i,j) = (r,s), \\ 0 & (i,j) \neq (r,s). \end{cases}$$

then, the approximated derivatives at any grid points is given by the formulas,

$$\left.\frac{df}{dx}\right|_{x=x_k} = \sum_{(i,j)\in e} \left.\frac{dL_{ij}(x)}{dx}\right|_{x=x_k} f_i^{(j)},$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x=x_k} = \sum_{(i,j)\in e} \left.\frac{d^2 L_{ij}(x)}{dx^2}\right|_{x=x_k} f_i^{(j)}.$$

Consequently, $\alpha_{ki}^{(j)}$, and $\beta_{ki}^{(j)}$ are the coefficients of the primary and second order derivatives given by

$$\alpha_{ki}^{(j)} = \left.\frac{dL_{ij}(x)}{dx}\right|_{x=x_k},$$

$$\beta_{ki}^{(j)} = \left.\frac{d^2 L_{ij}(x)}{dx^2}\right|_{x=x_k}.$$

and so on so forth. One will acquire formulae for upper form by-product by utilizing upper form coefficients, that are convey as $e_{ki}^{(j,m)}$. That represented using succeeding repeat formula,

$$e_{ki}^{(j,m)} = \left.\frac{d^m L_{ij}(x)}{dx^m}\right|_{x=x_k}.$$

where $(i,j) \in e$, $k = 1,2,\cdots,N$, $m = 1,2,\cdots,N-1$ and we can adopt $\alpha_{ki}^{(j)} = e_{ki}^{(j,1)}$ and $\beta_{ki}^{(j)} = e_{ki}^{(j,2)}$.

V. SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

The behaviors of various systems arise in different fields of sciences such as fluid dynamics, fluid motion, quantum mechanics, control, aerodynamics, elasticity, can be modeled by singularly perturbed differential equation [18-19] which depends on a small physical parameter ε with bi-

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point boundary worth problem,

$$\varepsilon\acute{\acute{\varphi}}(x) + q(x)\acute{\varphi}(x) + p(x)\varphi(x) = f(x) \qquad (5)$$

with the boundary state $\varphi(0) = a, \varphi(1) = b$, $a, b \in \mathbb{R}$, and ε could be a valued factor $0 < \varepsilon \leq 1$. We note that $q(x)$, $p(x)$, and $f(x)$ are assumed to be abundantly swish real method, cited in[20-21]. In various applications, (5) owns boundary surface, that are areas of speedy modification in the resolution close to the destinations.

VI. RESULTS

To indicate the productivity and correctness of the GDQM, we have exposed two problems of singularly perturbed differential equations, whose exact solutions known. To build the approximate solution via GDQM examine the 5-poised HB- problem, $e = \{(1,0), (2,0), (3,0), (4,0), (5,0), (6,0)\}$, which will be taken for both problems. Furthermore, $S_5(x) \in S(E^*, x_1, x_2, x_3, x_4, x_5, x_6)$ is pursued, where E is given as

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},$$

and for which $S_5(x_i) = \varphi_i^{(j)}$, $(i,j) \in e$. The fundamental g-spline functions $L_{10}(x)$, $L_{20}(x)$, $L_{30}(x)$, $L_{40}(x)$, $L_{50}(x)$, $L_{60}(x)$ are as follows

$$\begin{aligned} L_{10}(x) = 1 &- 11.23962x + 44.23179x^2 - 74.45954x^3 \\ &+ 46.45547x^4 - 14.02146(x-0)_+^9 \\ &+ 70.06572(x-0.2)_+^9 \\ &- 140.10244(x-0.4)_+^9 + 140.09067(x \\ &- 0.6)_+^9 - 70.03924(x \\ &- 0.8)_+^9 + 14.00675(x-1)_+^9 \end{aligned}$$

$$\begin{aligned} L_{20}(x) = 24.11477x &- 147.20064x^2 + 299.38104x^3 \\ &- 206.23569x^4 + 70.10731(x-0)_+^9 \\ &- 350.3286(x-0.2)_+^9 \\ &+ 700.51218(x-0.4)_+^9 - 700.45334(x \\ &- 0.6)_+^9 + 350.1962(x \\ &- 0.8)_+^9 - 70.03375(x-1)_+^9 \end{aligned}$$

$$\begin{aligned} L_{30}(x) = -23.22954x &+ 196.4846x^2 - 473.76208x^3 \\ &+ 360.38806x^4 - 140.21462(x-0)_+^9 \\ &+ 700.6572(x-0.2)_+^9 \\ &- 1401.02436(x \\ &- 0.4)_+^9 + 1400.90668(x \\ &- 0.6)_+^9 - 700.3924(x \\ &- 0.8)_+^9 + 140.06749(x-1)_+^9 \end{aligned}$$

$$\begin{aligned} L_{40}(x) = 14.89621x &- 136.06794x^2 + 369.59541x^3 \\ &- 308.30473x^4 + 140.21462(x-0)_+^9 \\ &- 700.6572(x-0.2)_+^9 \\ &+ 1401.02436(x \\ &- 0.4)_+^9 - 1400.90668(x \\ &- 0.6)_+^9 + 700.3924(x \\ &- 0.8)_+^9 - 140.06749(x-1)_+^9 \end{aligned}$$

$$\begin{aligned} L_{50}(x) = -5.36477x &+ 50.32564x^2 - 143.13104x^3 \\ &+ 128.11069x^4 - 70.10731(x-0)_+^9 \\ &+ 350.3286(x-0.2)_+^9 \\ &- 700.51218(x-0.4)_+^9 + 700.45334(x \\ &- 0.6)_+^9 - 350.1962(x \\ &- 0.8)_+^9 + 70.03375(x-1)_+^9 \end{aligned}$$

$$\begin{aligned} L_{60}(x) = 0.822954x &- 7.77346x^2 + 22.37621x^3 \\ &- 20.41381x^4 + 14.02146(x-0)_+^9 \\ &- 70.06572(x-0.2)_+^9 \\ &+ 140.10244(x-0.4)_+^9 - 140.09067(x \\ &- 0.6)_+^9 + 70.03924(x \\ &- 0.8)_+^9 - 14.00675(x-1)_+^9 \end{aligned}$$

Problem6.1: Examine the singularly perturbed differential equation with bi-point boundary worth problem,

$$\varepsilon\acute{\acute{\varphi}}(x) + \acute{\varphi}(x) = 1 + 2x, \qquad (6)$$

with $q(x) = 1$, $p(x) = 0$, $f(x) = 1 + 2x$, and the boundary conditions $\varphi(0) = 0$, $\varphi(1) = 1$ and its exact solution.

$$\varphi(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)\frac{1-e^{-\frac{x}{\varepsilon}}}{1-e^{-\frac{1}{\varepsilon}}}.$$

Additionally, the solution absolute error can be given by $\text{Error} = |\text{Exact} - \text{GDQM}|$. The exact and GDQM solutions along with the error of problem 6.1 are shown in table 1 to 3, for three values of ε and some values of $0 \leq x \leq 1$.

Table. 1 The exact solution and GDQM solution of Problem 6.1 at $\varepsilon = 2^{-1}$

x	Exact	GDQM	Error
0	0	0	0
0.2	0.04	0.04	2.99746×10^{-11}
0.4	0.16	0.16	6.89255×10^{-11}
0.6	0.36	0.36	1.12121×10^{-10}
0.8	0.64	0.64	1.31989×10^{-10}
1	1	1	0

Table. 2 The exact solution and GDQM solution of Problem 6.1at $\varepsilon = 2^{-4}$

x	Exact	GDQM	Error
0	0	0	0
0.2	-0.6243332	-0.5291176	9.52156×10^{-2}
0.4	-0.3635462	-0.2722937	9.12525×10^{-2}
0.6	0.0100592	0.1002787	9.02196×10^{-2}
0.8	0.4650023	0.5560835	9.10812×10^{-2}
1	1	1	0

Table. 3 The exact solution and GDQM solution of Problem 6.1at $\varepsilon = 2^{-8}$

x	Exact	GDQM	Error
0	0	0	0
0.2	-0.75375	-0.327302	4.26448×10^{-1}
0.4	-0.4353125	0.0367033	4.72016×10^{-1}
0.6	-0.036875	0.4165316	4.53407×10^{-1}
0.8	0.4415625	0.920459	4.78897×10^{-1}
1	1	1	0

Problem6.2: Examine the singularly perturbed differential equation with bi-point boundary value problem,

$$\varepsilon\acute{\varphi}(x) + \varphi(x) = 0 \quad (7)$$

with $q(x) = 0$, $p(x) = 1$, $f(x) = 0$, and the boundary conditions $\varphi(0) = 0$, $\varphi(1) = 0$and its correct result is $\varphi(x) = \sin\left(\frac{x}{\sqrt{\varepsilon}}\right)/\sin\left(\frac{1}{\sqrt{\varepsilon}}\right)$.

Additionally, the solution absolute error can be given by $\text{Error} = |\text{Exact} - \text{GDQM}|$.The exact and GDQM solutions along with the error of problem 6.1 are shown in table 4 to 6, for three values of ε and some values of $0 \le x \le 1$.

Table. 4 The exact solution and GDQM solution of Problem 6.2$\varepsilon = 2^{-1}$

x	Exact	GDQM	Error
0	0	0	0
0.2	0.2824827	0.2825432	6.04611×10^{-5}
0.4	0.5425972	0.5426332	3.59542×10^{-5}
0.6	0.759577	0.7596012	2.42038×10^{-5}
0.8	0.9162222	0.9162051	1.71074×10^{-5}
1	0	0	0

Table. 5 The exact solution and GDQM solution of Problem 6.2 $\varepsilon = 2^{-4}$

x	Exact	GDQM	Error
0	0	0	0
0.2	0.9689145	0.9478775	2.1037×10^{-2}
0.4	1.3688063	1.3207853	4.8021×10^{-2}
0.6	0.9386671	0.8925224	4.61447×10^{-2}
0.8	0.0651339	0.0771326	1.19987×10^{-2}
1	0	0	0

Table. 6 The exact solution and GDQM solution of Problem 6.2 $\varepsilon = 2^{-8}$

x	Exact	GDQM	Error
0	0	0	0
0.2	-0.0155346	0.2027561	2.18291×10^{-1}
0.4	-0.0106818	-0.4048206	3.94139×10^{-1}
0.6	0.0394291	0.6055046	5.66076×10^{-1}
0.8	-0.1005004	-0.8041235	7.03623×10^{-1}
1	0	0	0

VII. CONCLUSIONS

The GDQM have been found to be a successful generalization to the usual DQM. In addition, the results of the two problems shown in Tables 1 to 6, illustrates the potential of the strategy to provide smart results despite the tiny variety of node points used.

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