International Journal of Innovative Technology and Exploring Engineering (IJITEE)
ISSN: 2278-3075, Volume-8 Issue-4, February 2019

Zariski Topology on L-Slices

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Abstract: The action of a locale L on a join semilattice J gives us the newly introduced notion of L-slices (σ,J) .We have tried to extend the idea of Zariski topology on modules to L-slices. Given a locale L and a L-slice (σ,J) , for $m \in (\sigma,J)$ and $r \in L$,we have constructed (σ,J) ideals $[r ↳ m] = \{n \in (\sigma,J) | \sigma(r,n) \leq m\}$. Their properties and characteristics are studied. Similarly, for a given L-slice (σ,J) and $n,m \in (\sigma,J)$, we examine the properties of L-ideals $[r ↳ n] = \{r \in L | \sigma(r,n) \leq m\}$. We introduce the notion of L-prime elements on (σ,J) and their properties are discussed. The collection of L-prime elements is defined as Spec((σ,J) and we examine the possibility of existence of Zariski topology on it.

Keywrods: join semilattice, L-slices, Zariski topology, L-ideals

I. INTRODUCTION

Vector spaces can be viewed as the action of a field over an abelian group. The generalisation of this concept led to the development of module theory. Modules are the action of a ring on an abelian group. Locales can be viewed as a commutative idempotent semiring .The concept of modules is extended to that of the action of a locale on a join semilattice through the concept of L-slices. Through the action of the locale ,the join semilattice will adopt some topological properties of the locale. In literature, we can find many different generalisation of Zariski topology for modules over commutative rings. We developed a Zariski topology on a more generalised concept of modules.

This article is divided into five sections. In the first section we give some preliminaries on locales and L –slices. In the second section, we define implicative ideals on a locale L and on the L -slice (σ,J) and some of their properties are studied . Third section, introduces the concept of L-prime elements and Spec(σ,C) and their properties. In the fourth section we investigate the possibility of a Zariski topology on Spec(σ,C)

II. PRELIMINARIES

Definition 1.1 [1] A join semilattice is a poset $(J,\leq)$ in which every finite subset has a join.

Example: The set of all natural numbers with the partial order less than or equal to .

Definition1.2 [1] A lattice is a poset in which every finite subset has both a join and meet.

Definition 1.3 [1] A subset I of a lattice J is said to be an ideal if

i) I is a sub –join semilattice of J ; i.e. 0 ∈ I , and a ∈ I ,b∈ I imply a ∨ b ∈ I ; and

ii)I is a lower set ;i.e. a ∈ I and b≤ a imply b∈ I.

Definition 1.4 [1] A subset F of a lattice A is a set which satisfies the axioms dual to those defining an ideal ,is called filter of A.

Proposition [1] Let I be an ideal of a lattice A .The following conditions are equivalent :

i) The complement of I in A is a filter

ii) 1 ∉ I , and (a ∧ b) ∈ I implies either a ∈ I or b ∈ I

iii) I is the kernel of a lattice homomorphism f: A ⟶ 2, where 2 denotes the two element lattice {0,1}.

Definition 1.5 [1] An ideal satisfying the equivalent conditions of the above proposition is called a prime ideal; its complement is called a prime filter.

Definition 1.6 [2] A poset is a complete lattice if every subset has a join and meet.

Definition 1.7 [2] A frame is a complete lattice L satisfying the distributivity law

(⋁ A) ∧ b = ⋁ { a ∧ b | a ∈ A } , for any subset A ⊆ L and any b ∈ L.

Definition 1.8 [2] Frame homomorphism h:L ⟶ M between frames L and M are maps L⟶ M preserving all joins and all finite meets .

The resulting category will be denoted as Frm.

The dual category Frm^{op} is called the category of locales ,and this category can be viewed as an extension of the category of sober spaces and hence the locales can be viewed as generalized spaces.

Examples 1.9[2]

i)The lattice of open sets of topological space.

ii)The Boolean algebra B of all open sets U of real line such that U= *int*(*cl*(U)).

Definition 1.10 [1] In a locale L an element p is said to be meet irreducible if whenever

$a \wedge b \leq p$ implies either $a \leq p$ or $b \leq p$.

Definition 1.11 L-slices [3]

Let L be a locale with bottom element 0_L, top element 1_L and (σ,J) be a L - slice with bottom element 0_J . By the"action of L on J"we mean a function $\sigma: L \times J \to J$ such that the following conditions are satisfied .

1. $\sigma(a, x_1 \vee x_2) = \sigma(a,x_1) \vee \sigma(a,x_2)$ for all $a \epsilon L$ and for all $x_1, x_2 \epsilon J$.
2. .$\sigma\left(a, 0_J\right) = 0_J$ for all $a \epsilon L$
3. $\sigma(a \sqcap b, x) = \sigma(a, \sigma(b,x)) = \sigma(b, \sigma(a,x))$ for all ,$b \epsilon L$, $x \epsilon J$.
4. $\sigma(1_L, x) = x$ and $\sigma(0_L, x) = 0_J$ for all $x \epsilon J$.
5. $\sigma(a \sqcup b, x) = \sigma(a,x) \vee \sigma(b,x)$, for ,$b \epsilon L$, $x \epsilon J$.

If σ is an action of the locale L on a join semilattice , then we call (σ,J) as L slice.

Revised Manuscript Received on February 05, 2019.

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Examples 1.12[3]

1. Let L be a locale and I be any ideal of L .Consider each $x\epsilon\, I$ and define $\sigma: L \times I \to I$ as $\sigma(a, x) = a \wedge x,\ a \in L$. It can be easily seen that (σ, I) is a L –slice.
2. Let L be a chain with top and bottom elements and J be any join semilattice with bottom element 0_J .Define $\sigma: L \times J \to J$ by $\sigma(a, j) = j$, for every $a \neq 0_L$ and $\sigma(0_L, j) = 0_J$. This is called a trivial L-slice.
3. Any locale L can be viewed as the meet L –slice$(\sqcap, L)$ where the action σ is defined as $\sigma(a, x) = a \sqcap x$

Proposition 1.13

Here we give some properties of the action $\sigma: L \times J \to J$

1. For every $a \in L$ and $\in (\sigma, J)$, $\sigma(a, x) \leq x$

Proof: $x = \sigma(1_L, x) = \sigma(a \sqcup 1_L, x) = \sigma(a, x) \vee \sigma(1_L, x) = \sigma(a, x) \vee x$.

Therefore , $\sigma(a, x) \leq x$.

1. For $n, l\ \in (\sigma, J)$,if $l \leq n$,then $\sigma(a, l) \leq\ \sigma(a, n)$, for every $a \in L$

Proof: $l \leq n$ implies that $l \vee n = n$.

Then, for $a \in L$ $\sigma(a, n) = \sigma(a, l \vee n) = \sigma(a, l) \vee \sigma(a, n)$.

Hence $\sigma(a, l) \leq \sigma(a, n)$.

2. If $a \leq b$, for $a, b\, \epsilon\, L$, then $\sigma(a, x) \leq\ \sigma(b, x)$ for every $x\ \in (\sigma, J)$.

Proof: We have $\sqcup\, b = b$. Then , $\sigma(b, x) =\ \sigma(a \sqcup b, x) =\ \sigma(a, x) \vee\ \sigma(b, x)$.

Thus, $\sigma(a, x) \leq \sigma(b, x)$

III. IMPLICATIVE IDEALS OF L

Let L be a locale and (σ, J) be a L - slice with bottom element 0_J.

Definition 2.1 Let $n, l\ \in (\sigma, J)$ then we define a set $[l ↳ n]_L = \{r \in L\, |\sigma(r, l) \leq n\}$.

Proposition 2.2 For $n, l\ \in (\sigma, J)$, then $[l ↳ n]_L$ is an ideal of L.

Proof: We know that $(0_L, l) \leq n, \forall\, l\ \in (\sigma, J)$. Hence $[l ↳ n]_L$ is non empty. Let $a, b\ \in\ [l ↳ n]_L$ then $\sigma(a, l) \leq n$ and $(b, l) \leq n$, implies $\sigma(a \vee b, l) \leq n$.

Therefore , $a \vee b \in\ [l ↳ n]_L$.

Let $c\ \in\ L$ and $\leq a$, then $\sigma(c, l) \leq \sigma(a, l) \leq n$,

which implies that $c \in\ [l ↳ n]_L$.

Hence , $[l ↳ n]_L$ is an ideal of L.

Definition 2.3 For $, l\ \in (\sigma, J)$, $[l ↳ n]_L$ is called the implicative ideal of the locale L .

Proposition 2.4 i)For $n, l\ \in (\sigma, J)$ and $n \leq l$, $[x ↳ n]_L \subseteq [x ↳ l]_L , \forall\, x \in (\sigma, J)$

ii)For $n, l\, , k\ \in (\sigma, J)$ and $n \leq l$, $[l ↳ k]_L \subseteq [n ↳ k]_L$

Proof: i) Let $\in\ [x ↳ n]_L$, then $\sigma(r, x) \leq n \leq l$, implies $r \in\ [x ↳ l]_L$

Therefore, $[x ↳ n]_L \subseteq [x ↳ l]_L$.

ii) $s \in\ [l ↳ k]_L \Longrightarrow \sigma(s, l) \leq k$. Since , $n \leq l$, $\sigma(s, n) \leq \sigma(s, l) \leq k$. Therefore , $s \in\ [n ↳ k]_L$. Thus, $[l ↳ k]_L \subseteq [n ↳ k]_L$

Proposition 2.5 For $n, l\, , k\ \in (\sigma, J)$, $[l ↳ k]_L \cap [n ↳ k]_L =\ [l \vee n ↳ k]_L$.

Proof: Let $\in\ [l ↳ k]_L \cap [n ↳ k]_L$, then $\sigma(r, l) \leq k$ and $\sigma(r, n) \leq k$.

$\Longrightarrow\ \sigma(r, l) \vee \sigma(r, n) \leq k\ \Longrightarrow\ \sigma(r, l \vee n) \leq k \qquad \Longrightarrow r \in\ [l \vee n ↳ k]_L$.

$\therefore\ [l ↳ k]_L \cap [n ↳ k]_L \subseteq\ [l \vee n ↳ k]_L$

Now , let $s \in\ [l \vee n ↳ k]_L$, then $\sigma(s, l \vee n) \leq k$. We have , $\sigma(s, l) \vee \sigma(s, n) \leq k$

$\Longrightarrow\ \sigma(s, l) \leq k$ and $\sigma(s, n) \leq k$

$\Longrightarrow s \in\ [l ↳ k]_L$ and $s \in\ [n ↳ k]_L$.

$\therefore\ [l \vee n ↳ k]_L \subseteq [l ↳ k]_L \cap [n ↳ k]_L$.

Hence the proof.

Proposition 2.6 For $n, l\, , k\ \in (\sigma, J)$, $[k ↳ l]_L \cap [k ↳ n]_L \subseteq\ [k ↳ l \vee n]_L$

Proof: I t can be easily verified, hence we omit the proof .

IV. IMPLICATIVE IDEALS OF (σ, J)

Definition 3.1

Let $n \in (\sigma, J)$ and $\in L$, then we define a set $[r ↳ n]_{(\sigma, J)} = \{l \in (\sigma, J) / \sigma(r, l) \leq n\}$

Proposition .3.2 For $r \in L\, , n \in (\sigma, J)$, $[r ↳ n]_{(\sigma, J)}$ is an ideal of (σ, J)

Proof: $0_J\ \in\ [r ↳ n]_{(\sigma, J)}$ and hence it is non empty. Let $l\, , m \in\ [r ↳ n]_{(\sigma, J)}$ then $\sigma(r, l) \leq n$ and $\sigma(r, m) \leq n$ implies $\sigma(r, l \vee m) \leq n$, i.e. $l \vee m \in\ [r ↳ n]_{(\sigma, J)}$. Also,let $x \leq l$, for some $x\ \in (\sigma, J)$,then $\sigma(r, x) \leq \sigma(r, l) \leq n$,and hence $x\ \in [r ↳ n]_{(\sigma, J)}$. If $x\ \in [r ↳ n]_{(\sigma, J)}$ and $a \in L$,then $\sigma(r, x) \leq n$ implies $\sigma(r, \sigma(a, x)) = \sigma(a, \sigma(r, x)) \leq \sigma(r, x) \leq n$.

Thus, $[r ↳ n]_{(\sigma, J)}$ is an ideal of (σ, J).

Definition 3.3 The ideal $[r ↳ n]_{(\sigma, J)}$ is called the implicative ideals of (σ, J).

We have the following properties for the implicative ideals of (σ, J)

Propositon 3.4 i)For $r, s\epsilon\, L$, if $r \leq s$, then $[r ↳ n]_{(\sigma, J)} \supseteq [s ↳ n]_{(\sigma, J)}$

ii) For $, m\, \epsilon (\sigma, J)$, if $n \leq m$,then $[r ↳ n]_{(\sigma, J)} \subseteq [r ↳ m]_{(\sigma, J)}$

iii) $[r ↳ n]_{(\sigma, J)} \cup [r ↳ m]_{(\sigma, J)}\ \subseteq\ [r ↳ n \vee m]_{(\sigma, J)}$

iv) $[r ↳ n]_{(\sigma, J)} \cup [s ↳ n]_{(\sigma, J)} \subseteq\ [r \sqcup s ↳ n]_{(\sigma, J)}$

Proof: Since it can be easily verified , we omit the proof

Lemma 3.5 Let $n\, \epsilon (\sigma, J)$ and $r, s\epsilon\, L$

i) $[0_L ↳ n]_{(\sigma, J)} = (\sigma, J)$

ii) $[1_L ↳ n]_{(\sigma, J)} = \downarrow n$

iii) $[r ↳ n]_{(\sigma, J)} \cap [s ↳ n]_{(\sigma, J)} = [r \sqcup s ↳ n]_{(\sigma, J)}$, where $r, s\epsilon\, L$

Proof: i , ii follows from definitions. To prove iii , let $l\, \epsilon\, [r ↳ n]_{(\sigma, J)} \cap [s ↳ n]_{(\sigma, J)}$,then $\sigma(r, l) \leq n$ and $\sigma(s, l) \leq n$ implies $\sigma(r \sqcup s, l)$ hence $l\, \epsilon\, [r \sqcup s ↳ n]_{(\sigma, J)}$. Therefore, $[r ↳ n]_{(\sigma, J)} \cap [s ↳ n]_{(\sigma, J)} \subseteq [r \sqcup s ↳ n]_{(\sigma, J)}$.

Now , let $m\, \epsilon\, [r \sqcup s ↳ n]_{(\sigma, J)}$ then $\sigma(r \sqcup s, m) \leq n$ implies $\sigma(r, m) \vee \sigma(s, m) \leq n$ $\Rightarrow \sigma(r, m) \leq n$ and $\sigma(s, m) \leq n$ $\Rightarrow$ $m\, \epsilon\, [r ↳ n]_{(\sigma, J)} \cap [s ↳ n]_{(\sigma, J)}$.

Thus $[r ↳ n]_{(\sigma, J)} \cap [s ↳ n]_{(\sigma, J)} \supseteq [r \sqcup s ↳ n]_{(\sigma, J)}$.

Hence $[r ↳ n]_{(\sigma, J)} \cap [s ↳ n]_{(\sigma, J)} = [r \sqcup s ↳ n]_{(\sigma, J)}$

Theorem.3.6 The collection $\{\ [r ↳ n]_{(\sigma, J)}$, for a $n\, \epsilon (\sigma, J)$ and $r\, \epsilon\, L\ \}$ forms a basis for a topology on the L-slice (σ, J).

Proof: Follows from the above lemma.

V. L-PRIME ELEMENTS AND SPEC(σ, C)

In this section, we introduce the concept of L-component . We construct a L-slice over a sup lattice . A complete L-slice is called a L-component , that is , it is the action of a locale on a sup lattice.

Definition 4.1 Let L be a locale and C be a sup lattice with bottom element 0_C and top element 1_C

L-component (σ, C) is the 'action of L on C' which is defined as a map $\sigma: L \times C \to C$ such that it satisfies the following conditions in addition to the definition of L-slice

i) $\sigma(a, \vee_i x_i) = \vee_i \sigma(a, x_i)$ for $\{x_i\}_{i\in I} \in (\sigma, C)$, for some indexed set I

ii) $\sigma(\sqcup_i a_i, x) = \vee_i \sigma(a_i, x)$ for $\{a_i\}_{i\in I} \in L$, for some indexed set I.

Now we investigate the possibility of existence of Zariski toplogy on L-component .

Definition 4.2 L-prime

An element $\neq 1_C$, of (σ, C) is said to be L-prime element if for every $r \in L$ and $n \in (\sigma, C)$, $\sigma(r, n) \leq p$ implies that either $r \in [1_C \Lsh p]_L$ or $n \leq p$.

Example : If we consider the L-slice $(\sqcap, L)$ then the L-prime elements are precisely the meet irreducible elements of L.

Now we discuss some properties of L-prime elements

Theorem 4.3 Let p be a L-prime element and $x \in (\sigma, C)$ then $[x \Lsh p]_L$ is a prime ideal of L.

Proof: Let $\wedge s \in [x \Lsh p]$, then $\sigma(r \wedge s, x) \leq p$ implies $\sigma(r, \sigma(s, x)) \leq p$ .Then, p being a L-prime element ,will satisfy $\sigma(s, x) \leq p$ or $r \in [1_C \Lsh p]_L$. That is , either $x \leq p$ or $s \in [x \Lsh p]_L$ or $r \in [1_C \Lsh p]_L$. We have from proposition 2.4 that $[1_C \Lsh p]_L \subseteq [x \Lsh p]_L$. Thus, $s \in [x \Lsh p]_L$ or $\in [x \Lsh p]_L$. Hence, $[x \Lsh p]_L$ is a prime ideal .

Corollary 4.4 If p is a L - prime element , the $[1_C \Lsh p]_L$ is a prime ideal of a locale L.

Proof: Follows from the above theorem.

Theorem 4.5 If p and q are any two L-prime elements then so is $p \wedge q$.

Proof: Let $(r, n) \leq p \wedge q$, for some $n \in (\sigma, C)$ and $r \in L$. Then $\sigma(r, n) \leq p$ and $\sigma(r, n) \leq q$ .Since p and q are L-prime elements ,we have the following statements:

1. either $n \leq p$ or $r \in [1_C \Lsh p]_L$ and
2. either $n \leq q$ or $r \in [1_C \Lsh q]_L$.

From these two statements the theorem follows.

Definition 4.6: The set of all L -prime elements of (σ, C) is called the spectrum of $(\sigma, C)$and is denoted by Spec(σ, C).

We now investigate the possibility of a Zariski toplogy on Spec(σ, C).

VI. AN INVESTIGATION ON THE EXISTENCE OF ZARISKI TOPOLOGY ON SPEC(σ, C)

Definition 5.1 For $\in (\sigma, C)$, define C(n)={ p ∈Spec(σ, C)| $n \leq p$ }.

Proposition 5.2 : For $n \in (\sigma, C)$ and $p \in$ Spec (σ, C) we have the following :

i) C(0_C) = Spec(σ, C)
ii) C(1_C) = ∅
iii) $\bigcap_{i\epsilon I}$ C(n_i) = C$(\vee_{i\epsilon I} n_i$) , for some indexed set I
iv) C(n)∪ C(l)⊆ C$(n \wedge l)$.

Proof: i) C(0_C) = { p ∈Spec(σ, C) | $0_C \leq p$ }.Since 0_C is the bottom element , $0_C \leq p$, for every p ∈Spec(σ, C). Hence C(0_C) = Spec(σ, C).

ii) C(1_C) = { p ∈Spec(σ, C) | $1_C \leq p$}. Since 1_C is the top element , no p ∈Spec(σ, C) belongs to C(1_C).Hence the latter is empty.

iii)Let $p \in \bigcap_{i\epsilon I}$ C(n_i) , implies $n_i \leq p$, for every $i \in I$. Then we have $\vee_{i\epsilon I} n_i \leq p$ .Therefore, $p \in$ C$(\vee_{i\epsilon I} n_i)$ .That is , $\bigcap_{i\epsilon I}$ C$(n_i) \subseteq$ C$(\vee_{i\epsilon I} n_i$)

Now, suppose $p \in$ C$(\vee_{i\epsilon I} n_i)$, then $\vee_{i\epsilon I} n_i \leq p$,

$\Rightarrow n_i \leq p$ $,\forall\, i \in I$

$\Rightarrow$ $p \in$ C(n_i) $,\forall\, i \in I$

Hence , C$(\vee_{i\epsilon I} n_i$) ⊆ $\bigcap_{i\epsilon I}$ C(n_i).

Thus , we have $\bigcap_{i\epsilon I}$ C(n_i) = C$(\vee_{i\epsilon I} n_i$) , for some indexed set I.

iv)Let $p \in$ C(n)∪ C(l) , that is, $p \in$ C(n) or $p \in$ C(l)

$\Rightarrow n \leq p$ or $l \leq p$

$\Rightarrow n \wedge l \leq p$

$\Rightarrow p \in$ C$(n \wedge l)$.

Hence , C(n)∪ C(l)⊆ C$(n \wedge l)$.

Theorem 5.3 : On Spec(σ, C) , Λ= { C(n) | $n \in (\sigma, C)$} forms a basis for some topology Ω (σ, C).

Proof : Follows directly from the previous theorem.

Definition 5.4 We define Spec$_\wedge(\sigma, C)$ = { p | p is meet irreducible element of C and is L-prime of (σ, C)}

Proposition 5.5 If $p \in$ Spec$_\wedge(\sigma, C)$, then C(n)∪ C(l)= C$(n \wedge l)$

Proof: We have C(n)∪ C(l)⊆ C$(n \wedge l)$. Now , let $p \in$ C$(n \wedge l)$ then $n \wedge l \leq p$.

p being meet irreducible,we have that either $n \leq p$ or $l \leq p$. That is, either $p \in$ C(n) or $p \in$ C(l).

Hence $p \in$ C(n)∪ C(l). Therefore C(n)∪ C(l)= C$(n \wedge l)$.

Theorem 5.6 The collection ϑ = { C(n) : $n \in (\sigma, C)$ } defined on Spec$_\wedge(\sigma, C)$ forms a family of closed sets for some topology on Spec$_\wedge(\sigma, C)$.

Proof: Follows from Proposition 5.2 and proposition 5.6

Definition 5.7 The topology Ψ generated by the family of closed sets ϑ is called the Zariski topology on Spec$_\wedge(\sigma, C)$.

VII. CONCLUSION

Further study on the topology Ψ on Spec$_\wedge(\sigma, C)$ can be done. As a sequel to this article we intend to investigate the properties of Ψ and Λ and relations between them.

REFERENCES

1. P.T.Johnstone, Stone Spaces
2. Jorge Picado and Ales pultr, Frames and locales , Topology without points.
3. Sabna K.S, Mangalambal N.R, An isomorphism theorem for L-slices of a locale (communicated)
4. M.F.Athiyah and I.G.Macdonald , Introduction to Commutative Algebra
5. C.P.Lu, The Zariski topology on the prime spectrum of a module,Houston J Math.25(3)(1999)417-432.

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