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Properties of Multisets

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***Abstract:** The combinatorial properties of multisets are characterized. Some concepts in multiset topological spaces are discussed.*

***Keywords:** Multisets, absolute mset, topology, multiset topology, open mset. MSC2010 : 54A05,54A10.*

I. INTRODUCTION AND PRELIMINARIES

Multiset theory was introduced by Yager and Wayne D. Blizard in [1],[2]. Multiset topological spaces are studied by Girish& Sunil Jacob John in [3],[4]. Functions between multiset topological spaces have been extensively studied in [6]. In this paper sub msets of a mset and sub msets of mset topological spaces are characterized.

Definition 1.1: An mset M is a pair (X, C) where X is a non empty set and C: X $\rightarrow$ {0,1,2,3,…} is a function known as the count function of M.

Some authors use the terminology to identify the count function C with the mset M. That is C(x) is identified with M(x).The elements having zero count need not be written in the above representation of a mset. The notation $a^m b^n$ is also used to represent an mset over {a, b}.

Notations 1.2: Let X be a set from which msets are constructed**.**

$[X]^\infty$ = the collection of all msets over X.
$[X]^w$ = the collection of all msets over X with maximum count w .
For every x$\in$X if $M'(x) = w - M(x)$ then M′ is the complement of M. The notations Φ_X **and** 1_X denote the empty mset and the absolute mset respectively in $[X]^w$.

Definition 1.3 : Let M and N be two msets over X. Then
(i). M = N if M(x) = N(x),
(ii). M is a sub mset of N if $M(x) \leq N(x)$,
(iii). $(M \cup N)(x)$ = Max{M(x), N(x)},
(v) $(M \ominus N)(x)$ = Max{M(x)−N(x), 0} and
(vi). If N is a sub mset of M then (M\N)(x) = M(x) – N(x) for x in X.

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Lemma 1.4: $M \ominus N$ = M\N if and only if N is a sub mset of M.

Definition 1.5: If $\{M_i : i\in\Omega\}$ is a family of msets over X then
$(\cup\{M_i : i\in\Omega\})(x) = \text{Sup}\{M_i(x) : i\in\Omega\}$ and $(\cap\{M_i : i\in\Omega\})(x) = \text{Inf}\{M_i(x) : i\in\Omega\}$.

Example 1.6 : Let R be the set of all real numbers . Define $M_i(x)$ = [x] +I for every i=1,2,3,…. Then we see that $(M_1\cup M_2\cup \ldots)(1.5)$ = Sup{2,3,4,5,….} does not exist.
Therefore we restrict our study to msets in $[X]^w$ for some appropriate positive integer w.

Definition 1.7: The ordinary set N*={x$\in$X: N(x) > 0} is called the support of N. N* is also called the root set of N.
The following lemma is due to Padmapriya[5].

Lemma 1.8: For A , B$\in[X]^w$, the following properties hold.
(i) If $A \subseteq B$ then $A^* \subseteq B^*$,
(ii) $(A\cap B)^* = A^* \cap B^*$
(iii) $(A\cap B)^* = A^* \cup B^*$

Remark 1.9: The results (ii) and (iii) of Lemma 1.8 are also valid for arbitrary union and intersection of msets.
Throughout this paper N is an mset in $[X]^w$.

Definition 1.10: A collection τ of msubsets of N is a multiset topology on N or (N, τ) is a multiset topological space if
(i) N and Φ_X are in τ,
(ii) τ is closed under finite intersection.
(iii) τ is closed under arbitrary union.

The members of τ are called open msets in (N,τ). A sub mset B of a N is said to be a closed mset if N\B is an open mset. The interior of B denoted by *Int*B and the closure of B denoted by *Cl*B of a sub mset B of N can be defined in the usual manner.

II. GENERAL PROPERTIES

Proposition 2.1: If A, B $\in [X]^w$ and $A \cap B = \varnothing_X$ then
(i). $A^* \subseteq (B')^*$ and $B^* \subseteq (A')^*$
(ii).for every x$\in$X, $B(x) \leq A'(x)$ or $A(x) \leq B'(x)$

Proof: Let A, B be any two multisets over X with maximum multiplicity w, such that $A \cap B = \varnothing_X$. Then $A^* \cap B^* = (A \cap B)^* = \varnothing$ that implies $A^* \subseteq X\backslash B^*$ and $B^* \subseteq X\backslash A^*$. If $A^* \subseteq$

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$X\backslash B^*$ then $A^* \subseteq X\backslash B^* = \{x: B(x) = 0\} = \{x: B'(x) = w\} \subseteq \{x: B'(x) > 0\}$ that implies $A^* \subseteq (B')^*$. Again if $B^* \subseteq X\backslash A^*$ then $B^* \subseteq (A')^*$. This proves (i).

$(A \cap B)(x) = \mathrm{Min}\{A(x), B(x)\} = 0 \Rightarrow A(x) = 0$ or $B(x) = 0$ $\Rightarrow A'(x) = w$ or $B'(x) = w \Rightarrow B(x) \le A'(x)$ or $A(x) \le B'(x)$

Proposition 2.2: If A and B are msubsets of N such that $A \cap B = \emptyset_X$ then

(i). $A^* \subseteq (N \ominus B)^*$ and $B^* \subseteq (N \ominus A)^*$

(ii). for every $x \in X$, $B(x) \le (N \ominus A)(x)$ or $A(x) \le (N \ominus B)(x)$.

Proof: Suppose A and B are msubsets of N such that $A \cap B = \emptyset_X$. Then $N \ominus B = N\backslash B$ and $N \ominus A = N\backslash A$. Also $A^* \subseteq N^*$, $B^* \subseteq N^*$ and $A^* \cap B^* = \emptyset$ that implies $A^* \subseteq N^*\backslash B^*$ and $B^* \subseteq N^*\backslash A^*$.

If $A^* \subseteq N^*\backslash B^*$ then $A^* \subseteq N^*\backslash B^* = \{x \in N^*: B(x) = 0\} = \{x \in N^*: (N\backslash B)(x) = N(x)\} \subseteq \{x \in N^*: (N\backslash B)(x) > 0\}$ that implies $A^* \subseteq (N \ominus B)^*$ .Again if $B^* \subseteq N^*\backslash A^*$ then $B^* \subseteq (N \ominus A)^*$.

This proves (i).

$(A \cap B)(x) = \mathrm{Min}\{A(x), B(x)\} = 0 \Rightarrow A(x) = 0$ or $B(x) = 0$

$\Rightarrow (N \ominus A)(x) = N(x)$ or $(N \ominus B)(x) = N(x)$

$\Rightarrow B(x) \le (N \ominus A)(x)$ or $A(x) \le (N \ominus B)(x)$

Proposition 2.3: If $A, B \in [X]^w$ and $A \cup B = 1_x$ then

(i).$X\backslash A^* \subseteq B^*$ and $X\backslash B^* \subseteq A^*$

(ii).for every $x \in X$, $B'(x) \le A(x)$ and $A'(x) \le B(x)$.

Proof: Let A, B be any two multisets over X with maximum multiplicity w, such that $A \cup B = 1_x$. Since $A \cup B = 1_x$, $(A \cup B)^* = X$ that implies $A^* \cup B^* = X$ so that $(X\backslash A^*) \cap (X\backslash B^*) = \emptyset$. Therefore $X\backslash A^* \subseteq B^*$ and $X\backslash B^* \subseteq A^*$ which proves (i). Now for $x \in X$,

$(A \cup B)(x) = \mathrm{Max}\{A(x), B(x)\} = w \Rightarrow A(x) = w$ or $B(x) = w$.

If $A(x) = w$ then $B'(x) \le w = A(x)$ and $A'(x) = 0 \le B(x)$.

If $B(x) = w$ then $B'(x) = 0 \le A(x)$ and and $A'(x) \le w = B(x)$. This proves (ii).

III. COMBINATORIAL PROPERTIES

We begin this section with the following lemma which will be useful.

Lemma 3.1: For any natural number r, $\left[\frac{r}{2}\right] \le r - \left[\frac{r}{2}\right] \le \left[\frac{r}{2}\right] + 1$ where [s] denotes the integral part of a real number s.

Proposition 3.2: Let A be a msubset of a mset N over X such that $A(x) \le \left[\frac{N(x)}{2}\right]$ for every x in X. Then $A \subseteq N \ominus A = N - A$. The inclusion may be proper.

Proof:

$(N \ominus A)(x) = \mathrm{Max}\{N(x) - A(x), 0\} = N(x) - A(x) \ge N(x) - \left[\frac{N(x)}{2}\right] \ge \left[\frac{N(x)}{2}\right] \ge A(x)$.

This shows that $A \subseteq N \ominus A$. If $N = a^4b^5c^2$ and $A = ab^2c$ then the condition $A(x) \le \left[\frac{N(x)}{2}\right]$ is satisfied and $A = ab^2c \subset a^3b^3c = N \ominus A$ so that A is a part of $N \ominus A$. This proves the proposition.

Proposition 3.3: Let A be a msubset of a mset N over X such that $A(x) > \left[\frac{N(x)}{2}\right]$ for every x in X . Then $N - A = N \ominus A \subseteq A$. The inclusion may be proper.

Proof: $(N \ominus A)(x) = \mathrm{Max}\{N(x) - A(x), 0\} = N(x) - A(x) < N(x) - \left[\frac{N(x)}{2}\right] < \left[\frac{N(x)}{2}\right] + 1 < A(x) + 1$. Therefore $(N \ominus A)(x) \le A(x)$ that implies $N \ominus A \subseteq A$. If $N = a^4b^5c^2$ and $A = a^3b^3c^2$ then the condition $A(x) > \left[\frac{N(x)}{2}\right]$ is satisfied and $N \ominus A = ab^2 \subset a^3b^3c^2 = A$ so that $N \ominus A$ is a part of A. This proves the proposition.

Proposition 3.4: Let A be a sub mset of a mset N over X such that $A(x) = \left[\frac{N(x)}{2}\right]$ on X_1, $A(x) < \left[\frac{N(x)}{2}\right]$ on X_2 and $A(x) > \left[\frac{N(x)}{2}\right]$ on X_3 where $X = X_1 \cup X_2 \cup X_3$. Then $A \subseteq N \ominus A$ on $X_1 \cup X_2$ and $N \ominus A \subseteq A$ on X_3.

Proof: If $A(x) = \left[\frac{N(x)}{2}\right]$ on X_1 and if $A(x) < \left[\frac{N(x)}{2}\right]$ on X_2 then $A(x) \le \left[\frac{N(x)}{2}\right]$ on $X_1 \cup X_2$ that implies by applying Proposition 3.2 we have $A \subseteq N \ominus A$ on $X_1 \cup X_2$. Again by using Proposition 3.3., we get $N \ominus A \subseteq A$ on X_3.

Proposition 3.5: Let $A\in[X]^w$ be a mset such that $A(x)\leq\left[\frac{w}{2}\right]$ for every x in X. Then $A\subseteq A'$. The inclusion may be proper.

Proof: $A'(x) = w - A(x) \geq w-\left[\frac{w}{2}\right]\geq\left[\frac{w}{2}\right]\geq A(x)$.This shows that $A\subseteq A'$. If w=5 and if $A = ab^2c$ then the condition $A(x)\leq\left[\frac{w}{2}\right]$ is satisfied and $A = ab^2c \subset a^4b^3c^4 = A'$ so that A is a part of A′. This proves the proposition.

Proposition 3.6: Let $A\in[X]^{w+}$ be a mset such that $A(x) > \left[\frac{w}{2}\right]$ for every x in X . Then $A' \subseteq A$. The inclusion may be proper.

Proof: $A'(x) = w-A(x) < w-\left[\frac{w}{2}\right]<\left[\frac{w}{2}\right]+1<A(x)+1$ .Therefore $A'(x)\leq A(x)$ that implies $A'\subseteq A$. If w=5 and $A = a^3b^3c^3$ then the condition $A(x)>\left[\frac{w}{2}\right]$ is satisfied and $A' = a^2b^2c^2 \subset a^3b^3c^3 = A$ so that A′ is a part of A. This proves the proposition.

Proposition 3.7: Let $A\in[X]^{w+}$ be a mset such that $A(x) = \left[\frac{w}{2}\right]$ on X_1, $A(x)<\left[\frac{w}{2}\right]$ on X_2 and $A(x)>\left[\frac{w}{2}\right]$ on X_3 where $X = X_1 \cup X_2\cup X_3$. Then $A \subseteq A'$ on $X_1 \cup X_2$ and $A' \subseteq A$ on X_3.

Proof: If $A(x) = \left[\frac{w}{2}\right]$ on X_1 and if $A(x)<\left[\frac{w}{2}\right]$ on X_2 then $A(x)\leq\left[\frac{w}{2}\right]$ on $X_1\cup X_2$ that implies by applying Proposition 2.2.5 we have $A \subseteq A'$ on $X_1 \cup X_2$. Again by using Proposition 2.2.6., we get $A' \subseteq A$ on X_3.

IV. PROPERTIES RELATED TO TOPOLOGY

Proposition 4.1: Every topology on X can be considered as a m-topology over X.

Proof: Follows from the fact every subset A of X can be considered as a mset by defining A(x) =1 for every $x\in A$ and =0 for every $x\in X\backslash A$.

Proposition 4.2: If τ is a m-topology on a mset N over X then $\tau^* = \{A^* : A\in\tau\}$ is a topology on N*.

Proof: Follows from the following facts $(\varnothing_X)^* = \varnothing$, $(A \cap B)^* = A^* \cap B^*$ and $(\cup\{A: A\in \Omega\})^* = \cup\{A^*: A\in \Omega\}$.

Proposition 4.3: If τ is a m-topology on a mset N over X then $|\tau^*| \leq|\tau|$ and strict inequality will hold in certain cases.

Proof: For each open set G in τ^*, there is an open mset $A\in\tau$ with $G = A^*$. Therefore $|\tau^*| \leq|\tau|$. However the inequality is strict as shown below.

Let $N = a^4b^3c^2$ and $\tau =\{ \varnothing_X, N, ab, a^2b^3\}$. Then τ is m-topology on N with $\tau^*= \{ \varnothing, \{a,b\}, \{a,b,c\} \}$ that implies $|\tau^*| <|\tau|$. Again if $\tau = \{ \varnothing_X, N, a^2, a^2b^3\}$ then $\tau^*= \{\varnothing, \{a\}, \{a,b\}, \{a,b,c\} \}$ so that $|\tau^*| = |\tau|$.

Proposition 4.4: Let η be a topology on X. Then there is a mset topology τ over X such that $\tau^* = \eta$.

Proof: Fix a natural number w. For each $A \in\eta$ with $A \neq\varnothing$, $A\neq X$, the msets $M_{A(i)}$, i=1,2,3,…,w are defined as $M_{A(i)}(x) = i$ when $x\in A$ and zero otherwise. Then it is easy to see that $(M_{A(i)})^* =A$. Let M_X be defined by $M_X(x) = w$ for every $x\in X$ that implies $(M_X)^* = X$. Let S be a collection of msets formed by choosing exactly one mset from $\{M_{A(i)}$, i=1,2,3,…,w$\}$ for every $A \in\eta$ with $A \neq\varnothing$, $A\neq X$. Clearly each member of S is a msubset of M_X. Then $S\cup\{M_X\}$will generate a mset topology $\tau\subseteq[X]^w$. Clearly $\tau^* = \eta$.

Corollary 4.5: Let η be a topology on X. Then there are infinitely many mset topologies whose support topologies are equal to η.

Proof: Follows from Proposition 4.4.

Proposition 4.6: Let τ be a m-topology on a mset N over X and A be a msubset of N. Then $(IntA)^* =IntA^*$. $(Cl (N\text{-}A))^* \supseteq N^*\backslash IntA^* = N^*\backslash (IntA)^*$. $(ClA)^* \neq ClA^*$ The inclusion in (ii) can be proper.

Proof: It follows from the definition of the interior that $(IntA)^*\subseteq IntA^*$. To prove the reverse inclusion, let $x\in IntA^*$. Then $x\in U\subseteq A^*$ for some $U \in\tau^*$. Therefore $U=B^*$ for some $B\in\tau$ that implies $x\in B^*\subseteq A^*$. This proves that $x\in B\subseteq A$ that implies $x\in IntA$ which means $x\in(IntA)^*$. This proves (i). Now $(ClA)^* = (N\backslash Int(N\backslash A))^* \supseteq N^* \backslash (Int(N\backslash A))^* = N^* \backslash Int(N\backslash A)^*$. Replacing A by N\A we get $(Cl (N\backslash A))^* \supseteq N^*\backslash IntA^*$. The inclusion in (ii) is proper as shown below.

Let $N = a^2b^3c^4$ be a multiset over X={a,b,c}. Then $\tau = \{\varnothing, N, b^2c^2 \}$ is an m-topology on N and $\tau^* = \{\varnothing, X, \{b, c\} \}$ is the corresponding root topology on N*.

Let $A = a^2b^2c^2$. Then it is easy to see that $(Cl (N\backslash A))^* = \{a,b,c\}$ and $N^*\backslash IntA^* = \{a\}$. This proves (ii) and (iv). Clearly $(ClA)^* = \{a,b,c) = ClA^*$. If $B= a^2$ then $(ClB)^* = \{a,b,c)$ and $ClB^* = \{a\}$. This proves (iii).

V. CONCLUSION

The properties that are related to combinatorics and topology are investigated in the domain of multisets.

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