Properties of Multisets

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Abstract: The combinatorial properties of multisets are characterized. Some concepts in multiset topological spaces are discussed.

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I. INTRODUCTION AND PRELIMINARIES

Multiset theory was introduced by Yager and Wayne D. Blizard in [1],[2]. Multiset topological spaces are studied by Girish& Sunil Jacob John in [3],[4]. Functions between multiset topological spaces have been extensively studied in [6]. In this paper sub msets of a mset and sub msets of mset topological spaces are characterized.

Definition 1.1: An mset M is a pair (X, C) where X is a non empty set and C: X → \{0,1,2,3,...\} is a function known as the count function of M.

Some authors use the terminology to identify the count function C with the mset M. That is C(x) is identified with M(x). The elements having zero count need not be written in the above representation of a mset. The notation a"b" is also used to represent an mset over \{a, b\}.

Notations 1.2: Let X be a set from which msets are constructed.

[X]^w = the collection of all msets over X.

[X]^w = the collection of all msets over X with maximum count w.

For every x∈X if M'(x) = w − M(x) then M' is the complement of M. The notations \(\phi_x\) and 1x denote the empty mset and the absolute mset respectively in \(X)^w\).

Definition 1.3: Let M and N be two msets over X. Then

(i). M = N if M(x) = N(x),

(ii). M is a sub mset of N if M(x) ≤ N(x),

(iii). (M ∪ N)(x) = Max{M(x), N(x)},

(iv). (M ∩ N)(x) = Max{M(x)−N(x), 0} and

(vi). If N is a sub mset of M then (M\N)(x) = M(x) − N(x) for x in X.

Lemma 1.4: M\N = M\N if and only if N is a sub mset of M.

Definition 1.5: If \(\{M_i : i\in\Omega\}\) is a family of msets over X then

\((\cup\{M_i : i\in\Omega\})(x) = \text{Sup}\{M_i(x) : i\in\Omega\}\) and \((\cap\{M_i : i\in\Omega\})(x) = \text{Inf}\{M_i(x) : i\in\Omega\}\).

Example 1.6: Let R be the set of all real numbers. Define \(M_i(x) = [x] + 1\) for every i=1,2,3,... Then we see that \(M_1(0) = M_2(1) = \cdots\) does not exist. Therefore we restrict our study to msets in \(X)^w\) for some appropriate positive integer w.

Definition 1.7: The ordinary set N*={x∈X: N(x) > 0} is called the support of N. N* is also called the root set of N.

The following lemma is due to Padmapriya[5].

Lemma 1.8: For A, B∈\(X)^w\), the following properties hold.

(i) If A ⊆ B then A* ⊆ B*,

(ii) (A ∩ B)* = A* ∩ B*,

(iii) (A ∪ B)* = A* ∪ B*.

Remark 1.9: The results (ii) and (iii) of Lemma 1.8 are also valid for arbitrary union and intersection of msets.

Throughout this paper N is an mset in \(X)^w\).

Definition 1.10: A collection τ of msubsets of N is a multiset topology on N or (N, τ) is a multiset topological space if

(i) N and \(\phi_X\) are in τ,

(ii) τ is closed under finite intersection.

(iii) τ is closed under arbitrary union.

The members of τ are called open msets in (N, τ). A sub mset B of N is said to be a closed mset if NB is an open mset. The interior of B denoted by IntB and the closure of B denoted by ClB of a sub mset B of N can be defined in the usual manner.

II. GENERAL PROPERTIES

Proposition 2.1: If A, B ∈ \(X)^w\) and A ∩ B = \(\phi_X\) then

(i). A* ⊆ (B*)* and B* ⊆ (A*)*,

(ii) for every x∈X, B(x) ≤ A'(x) or A(x) ≤ B'(x)

Proof: Let A, B be any two multisets over X with maximum multiplicity w, such that A ∩ B = \(\phi_X\). Then A* ∩ B* = (A ∩ B)* = \(\phi\) that implies A* ⊆ X\B* and B* ⊆ X\A*. If A* ⊆
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Let $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{X}$ be multisets. Then $A^* \subseteq B^* = \{x: B(x) = 0\} = \{x: B'(x) = w\} = \{x: B(x)^* = 0\}$ implies $A^* \subseteq (B')^*$. Again if $B^* \subseteq A^*$ then $B^* \subseteq (A')^*$. This proves (i).

$(A \cap B)(x) = \text{Min}\{A(x), B(x)\} = 0 \Rightarrow A(x) = 0$ or $B(x) = 0$ \Rightarrow $A'(x) = w$ or $B'(x) = w \Rightarrow B(x) \leq A'(x)$. This proves (ii).

**Proposition 2.2:** If $A$ and $B$ are multisets of $X$ such that $A \cap B = \emptyset$ then

(i). $A^* \subseteq (N \cap B)^*$ and $B^* \subseteq (N \cap A)^*$

(ii). for every $x \in X$, $B(x) \leq (N \cap A)(x)$ or $A(x) \leq (N \cap B)(x)$.

**Proof:** Suppose $A$ and $B$ are multisets of $X$ such that $A \cap B = \emptyset$. Then $N \cap B = N \cap B$ and $N \cap A = N \cap A$. Also $A^* \subseteq N^* A^*$ and $B^* \subseteq N^* B^*$. Thus $A^* \subseteq N^* A^*$ and $B^* \subseteq N^* B^*$.

If $A^* \subseteq N^* A^*$ then $A^* \subseteq N^* B^* = \{x \in N^*: B(x) = 0\} = \{x \in N^*: (N \cap B)(x) \leq N^*: (N \cap B)(x) > 0\}$ implies $A^* \subseteq (N \cap B)^*$. Again if $B^* \subseteq N^* A^*$ then $B^* \subseteq (N \cap A)^*$.

This proves (i).

$(A \cap B)(x) = \text{Min}\{A(x), B(x)\} = 0 \Rightarrow A(x) = 0$ or $B(x) = 0$ \Rightarrow $(N \cap A)(x) = N(x)$ or

$(N \cap B)(x) = N(x)$

$(N \cap A)(x) = N(x)$ and $B(x) \leq (N \cap A)(x)$ or $A(x) \leq (N \cap B)(x)$.

**Proposition 2.3:** If $A, B \subseteq [X]^n$ and $A \cup B = 1_1$ then

(i). $X \setminus A^* \subseteq B^*$ and $X \setminus B^* \subseteq A^*$

(ii). for every $x \in X$, $B(x) \leq A(x)$ and $A'(x) \leq B(x)$.

**Proof:** Let $A, B$ be any two multisets over $X$ with maximum multiplicity $w$, such that $A \cup B = 1_1$. Since $A \cup B = 1_1$, $(A \cup B)^* = X$ that implies $A^* \cup B^* = X$ so that $(X \setminus A^*) \cap (X \setminus B^*) = \emptyset$. Therefore $X \setminus A^* \subseteq B^*$ and $X \setminus B^* \subseteq A^*$ which proves (i).

Now for $x \in X$,

$(A \cup B)(x) = \text{Max}\{A(x), B(x)\} = w \Rightarrow A(x) = w$ or $B(x) = w$.

If $A(x) = w$ then $B'(x) = 0 \Rightarrow A'(x) = 0 \leq B(x)$.

If $B(x) = w$ then $B'(x) = 0 \leq A(x)$ and $A'(x) \leq B(x)$.

This proves (ii).

**III. COMBINATORIAL PROPERTIES**

We begin this section with the following lemma which will be useful.

**Lemma 3.1:** For any natural number $r$, \[
\left\lfloor \frac{r}{2} \right\rfloor \leq r - \left\lfloor \frac{r}{2} \right\rfloor \leq \left\lfloor \frac{r}{2} \right\rfloor + 1
\]

where $[s]$ denotes the integral part of a real number $s$.

**Proposition 3.2:** Let $A$ be a multiset of a set $N$ over $X$ such that $A(x) \leq \left\lfloor \frac{N(x)}{2} \right\rfloor$ for every $x \in X$. Then $A \subseteq N \Theta A = N \setminus A$.

The inclusion may be proper.

**Proof:**

$N \Theta A(x) = \text{Max}\{N(x) - A(x), 0\} = N(x) - A(x) \geq N(x) - \left\lfloor \frac{N(x)}{2} \right\rfloor \geq A(x)$.

This shows that $A \subseteq N \Theta A$. If $N = a^b c^2$ and $A = a^b c^2$ then the condition $A(x) \leq \left\lfloor \frac{N(x)}{2} \right\rfloor$ is satisfied and $A = a^b c^2 \subseteq a^b c = N \Theta A$ so that $A$ is a part of $N \Theta A$. This proves the proposition.

**Proposition 3.3:** Let $A$ be a multiset of a set $N$ over $X$ such that $A(x) > \left\lfloor \frac{N(x)}{2} \right\rfloor$ for every $x \in X$. Then $N \setminus A = N \Theta A \subseteq A$. The inclusion may be proper.

**Proof:**

$N \Theta A(x) = \text{Max}\{N(x) - A(x), 0\} = N(x) - A(x) \leq N(x) - \left\lfloor \frac{N(x)}{2} \right\rfloor < N(x) - \left\lfloor \frac{N(x)}{2} \right\rfloor + 1 = A(x) + 1$. Therefore $N \Theta A(x)$ is satisfied and $N \Theta A = a^b c^2 \subseteq a^b c = A$ so that $N \Theta A$ is a part of $A$. This proves the proposition.

**Proposition 3.4:** Let $A$ be a multiple set of a set $N$ over $X$ such that $A(x) = \left\lfloor \frac{N(x)}{2} \right\rfloor$ on $X_1$, $A(x) < \left\lfloor \frac{N(x)}{2} \right\rfloor$ on $X_2$, and $A(x) > \left\lfloor \frac{N(x)}{2} \right\rfloor$ on $X_3$, where $X = X_1 \cup X_2 \cup X_3$. Then $A \subseteq N \Theta A$ on $X_1 \cup X_2$ and $N \Theta A \subseteq A$ on $X_3$.

**Proof:** If $A(x) = \left\lfloor \frac{N(x)}{2} \right\rfloor$ on $X_1$ and $A(x) < \left\lfloor \frac{N(x)}{2} \right\rfloor$ on $X_2$ then $A(x) = \left\lfloor \frac{N(x)}{2} \right\rfloor$ on $X_1 \cup X_2$ that implies by applying Proposition 3.2 we have $A \subseteq N \Theta A$ on $X_1 \cup X_2$.

Again by using Proposition 3.3., we get $N \Theta A \subseteq A$ on $X_3$.
Proposition 3.5: Let \( A \subseteq [X]^w \) be a mset such that 
\[
A(x) \leq \left[ \frac{W}{2} \right]
\]
for every \( x \in X \). Then \( A \subseteq A' \). The inclusion may be proper.

Proof: \( A'(x) = w - A(x) \geq w - \left[ \frac{W}{2} \right] \geq \left[ \frac{W}{2} \right] \geq A(x) \). This shows that \( A \subseteq A' \). If \( w=5 \) and if \( A = ab^2c \) then the condition 
\[
A(x) \leq \left[ \frac{W}{2} \right]
\]
is satisfied and \( A = ab^2c \subseteq a^2b^2c^2 = A' \) so that \( A \) is a part of \( A' \). This proves the proposition.

Proposition 3.6: Let \( A \subseteq [X]^{**} \) be a mset such that \( A(x) > \left[ \frac{W}{2} \right] \) for every \( x \in X \). Then \( A' \subseteq A \). The inclusion may be proper.

Proof: \( A'(x) = w - A(x) < w - \left[ \frac{W}{2} \right] < \left[ \frac{W}{2} \right] + 1 < A(x) + 1 \).
Therefore \( A'(x) \leq A(x) \) that implies \( A' \subseteq A \). If \( w=5 \) and \( A = a^2b^2c^2 \) then the condition 
\[
A(x) > \left[ \frac{W}{2} \right]
\]
is satisfied and \( A' = a^2b^2c^2 \subseteq a^2b^2c^2 = A \) so that \( A' \) is a part of \( A \). This proves the proposition.

Proposition 3.7: Let \( A \subseteq [X]^{***} \) be a mset such that 
\[
A(x) = \left[ \frac{W}{2} \right] \] on \( X_1 \), \( A(x) < \left[ \frac{W}{2} \right] \) on \( X_2 \) and \( A(x) > \left[ \frac{W}{2} \right] \) on \( X_3 \)
where \( X = X_1 \cup X_2 \cup X_3 \). Then \( A \subseteq A' \) on \( X_1 \cup X_2 \) and \( A' \subseteq A \) on \( X_3 \).

Proof: If \( A(x) = \left[ \frac{W}{2} \right] \) on \( X_1 \) and if \( A(x) < \left[ \frac{W}{2} \right] \) on \( X_2 \), then 
\[
A(x) \leq \left[ \frac{W}{2} \right] \] on \( X_1 \cup X_2 \) that implies by applying Proposition 2.2.5 we have \( A \subseteq A' \) on \( X_1 \cup X_2 \). Again by using Proposition 2.2.6., we get \( A' \subseteq A \) on \( X_3 \).

IV. PROPERTIES RELATED TO TOPOLOGY

Proposition 4.1: Every topology on \( X \) can be considered as a \( m \)-topology over \( X \).

Proof: Follows from the fact every subset \( A \) of \( X \) can be considered as a mset by defining \( A(x) = 1 \) for every \( x \in A \) and \( = 0 \) for every \( x \in X \setminus A \).

Proposition 4.2: If \( \tau \) is a \( m \)-topology on a mset \( N \) over \( X \) then \( \tau^* = \{ A^*: A \in \tau \} \) is a topology on \( N^* \).

Proof: Follows from the following facts \( (\mathcal{G})^* = \mathcal{G}, (A \cap B)^* = A^* \cap B^* \) and \( (\cup \{ A: A \in \Omega \})^* = \cup \{ A^*: A \in \Omega \} \).

Proposition 4.3: If \( \tau \) is a \( m \)-topology on a mset \( N \) over \( X \) then \( |\tau^*| \leq |\tau| \) and strict inequality will hold in certain cases.

Proof: For each open set \( G \) in \( \tau \), there is an open mset \( A \in \tau \) with \( G = A^* \). Therefore \( |\tau^*| \leq |\tau| \). However the inequality is strict as shown below.

Let \( N = a^2b^2c^2 \) and \( \tau = \{ X, N, ab, a^2b^3 \} \). Then \( \tau \) is a \( m \)-topology on \( N \). Let \( \tau^* = \{ X, N, a^2, a^2b \} \) that implies \( |\tau^*| < |\tau| \). Again if \( \tau = \{ X, N, a^2, a^2b \} \) then \( \tau^* = \{ X, N, a^2, a^2b \} \) so that \( |\tau^*| = |\tau| \).

Proposition 4.4: Let \( \eta \) be a topology on \( X \). Then there is a \( m \)-topology \( \tau \) over \( X \) such that \( \tau^* = \eta \).

Proof: Fix a natural number \( w \). For each \( A \in \eta \) with \( A \neq \emptyset \) or \( A = X \), the msets \( M_{A\Omega i}, i=1,2,3,\ldots,w \) are defined as \( M_{A\Omega i}(x) = i \) when \( x \in A \) and zero otherwise. Then it is easy to see that \( (M_{A\Omega i})^* = A \).

Let \( M_{S*} \) be defined by \( M_{S*}(x) = w \) for every \( x \in X \) that implies \( (M_{S*})^* = X \). Let \( S \) be a collection of msets formed by choosing exactly one mset from \( \{ M_{A\Omega i}, i=1,2,3,\ldots,w \} \) for every \( A \in \eta \) with \( A \neq \emptyset, A \neq X \). Clearly each member of \( S \) is a mset of \( M_{S*} \). Then \( S \cup (M_{S*}) \) will generate a \( m \)-topology \( \tau \subseteq [X]^* \). Clearly \( \tau^* = \eta \).

Corollary 4.5: Let \( \eta \) be a topology on \( X \). Then there are infinitely many \( m \)-topologies whose support topologies are equal to \( \eta \).

Proof: Follows from Proposition 4.4.

Proposition 4.6: Let \( \tau \) be a \( m \)-topology on a mset \( N \) over \( X \) and \( A \) be a msubset of \( N \). Then \( (Int)\) \( \ast = IntA^* \). (\( Cl \) \( \ast \) \( \neq ClA^* \)) the inclusion in \( (ii) \) can be proper.

Proof: It follows from the definition of the interior that \( (IntA)\ast \subseteq IntA^* \). To prove the reverse inclusion, let \( x \in IntA^* \). Then \( x \in IntA^* \) for some \( U \in \tau \). Therefore \( U \in B^* \) for some \( B \in \tau \) that implies \( x \in B^* \subseteq A^* \). This proves that \( x \in B \subseteq A^{**} \) which means \( x \in (IntA)^* \). This proves (i). Now \( (ClA)^* = (N \setminus IntA) \ast \subseteq N^* \setminus IntA^* \). If \( (ClA)^* \neq ClA^* \) then \( Int(N \setminus IntA) \ast \subseteq N^* \setminus Int\{N \setminus IntA\} \ast \). Replacing \( A \) by \( N \) and \( (ClA)^* \neq N^* \setminus IntA^* \). The inclusion in (ii) is proper as shown below.

Let \( N = a^2b^2c^2 \) be a mset over \( X = \{ a, b, c \} \). Then \( \tau = \{ X, N, b^2c^2 \} \) is an \( m \)-topology on \( N \) and \( \tau^* = \{ X, \{ a \} \} \). This is the corresponding root topology on \( N^* \).

Let \( A = a^2b^2c^2 \). Then it is easy to see that \( (ClN)\ast = \{ a, b, c \} \) and \( N^* \setminus IntA^* = \{ a \} \). This proves (ii) and (iv).

Clearly \( (ClA)^* = \{ a, b, c \} \) and \( ClA^* = \{ a \} \) if \( B = a^2 \) then \( (ClB)^* = \{ a, b, c \} \) and \( ClB^* = \{ a \} \). This proves (iii).

V. CONCLUSION

The properties that are related to combinatorics and topology are investigated in the domain of multisets.
REFERENCES