An Algebraic Procedure for the Design of Linear Time Invariant Discrete and Continuous Systems Employing Lower Order Model

N. Malathi, N. Devarajan

Abstract: ***A simple algebraic procedure for model reduction of Linear Time Invariant Discrete Systems (LTIDS) is formulated. For the given original higher order system, a second order reduced model is assumed with unknown parameters. These parameters are determined by matching the selected amplitudes (including the steady state and dominant dynamics) from the plot of original system response with the Laurent series terms of reduced second order unit step response, which are the expressions in terms of unknown parameters. The responses of original and the determined second order systems are compared. The proposed reduced order system can retain the stability, steady state and the peak amplitudes of the original higher order system response. However, if the dynamics of resultant reduced order system response diverge, then the sample time is tuned to an appropriate value to attain the time match. The proposed model reduction method is extended for Linear Time Invariant Continuous Systems (LTICS). By employing the proposed second order reduced order model, the Proportional Integral Derivative (PID) controller is designed and then attached to the original higher order system for stabilization of the output response. The results for LTIDS and LTICS are shown with few examples.***

Keywords: Model order reduction, identification, step response, Laurent series, amplitude matching, sample time, LTIDS, LTICS, PID.

I. INTRODUCTION

Reduced order systems are used to avoid the computational complexity in higher order plant models. In general, the reduced order model is obtained from either the system function or the step response data. Plenty of works are going on to develop a perfect reduced order system from the transfer function model including the classical and intelligent based techniques. However, the response matching technique is also considered in past works to obtain the reduced order model for both continuous and discrete systems. The amplitude matching technique which is proposed by N.Devarajan in 1998 [1] is the basis of the presented technique. In which the first three amplitudes of the step response are calculated from the original order transfer function and then matched to the reduced order time indexes. Selected amplitudes are used for obtaining the algebraic equations to determine the reduced order system [2]. Recently, a collection of reduced order by moment matching techniques are discussed with examples in [3]. Response matchings for pole evaluation of the reduced order models are developed [4-7] as well. In the presented method, the amplitudes including peaks and steady state value are taken from the step response plot of the original system and are directly matched with the amplitudes of the second order reduced system whose unknown parameters are to be determined. All the resultant reduced second order responses can strictly retain the stability and the values of peaks and steady state. However, some of them may deviate from the original in terms of sample time while retaining the above mentioned dynamics. In that case, a simple sample time tuning is implemented.

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II. METHODOLOGY–LTIDS ORDER REDUCTION

Consider the given stable original nth order original LTIDS as

$$G(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{n} b_i z^i} = \frac{a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0}{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0} \quad (1)$$

where m<n; ai ($0 \leq i \leq m$) and bi ($0 \leq i \leq n$) are scalar values; N(z) is the numerator polynomial; D(z) is the denominator polynomial. The reduced second-order LTIDS to be derived is

$$R(z) = \frac{k_1 z + k_0}{z^2 + l_1 z + l_0} \quad (2)$$

where k_1, k_0, l_1, and l_0 are scalar constants.

A successful reduced order system response should possess the same amplitudes as that of its original higher order system response in all the time indexes. In the proposed method, the values of unknown parameters in (2) are obtained by assuming that the reduced order response amplitudes should have at least a few selected original step response amplitudes. Therefore, the original step response amplitudes are equated the reduced order system response amplitudes. If needed the time match is obtained by sample time adjustment. The amplitudes of the original higher order response are chosen from the plot. While the amplitudes of lower order model response are obtainedfrom the Laurent series expressions in terms of unknown parameters.

A Laurent series is a power series with both positive and negative power of $(z-z_0)$. It gives the analytic expression of a function on an annular region { $r_1 < | z-z_0| < r_2$. If f(z) is analytic in the annular region between and on assumed two concentric circles n_1 and n_2 cantered at $z=z_0$ and of radii $r_1$and

$r_2<r_1$ respectively, then there exists a unique series. It is known that long division of function f(z) yields the Laurent series of that function. To formulate the model reduction of LTIDS given in (1), consider the second-order stable reduced order model R(z) as in (2), with the region of convergence $0 < |z|<1$. Laurent series of the step response of R(z) becomes, $U(R(z)) = \sum_{n=0}^{\infty} U(r_n)\, z^{-n}$ gives the step response magnitudes $U[r_n]$. This is true when it is related to the definition of the z transform. The unknown parameters of reduced order are determined by matching the selected amplitudes (including the steady-state and peak values) from the plot of original system step response with the Laurent series magnitudes of reduced model step response U(R(z)).

Unit step response of the reduced order system is given as,

$$U(R(z)) = \left(\frac{z}{z-1}\right) R(z) = \frac{k_1 z^2 + k_0 z}{z^3+(l_1-1)z^2+(l_0-l_1)z-l_0} \qquad (3)$$

Laurent series of (3) gives

$$\frac{k_1 z^2 + k_0 z}{z^3+(l_1-1)z^2+(l_0-l_1)z-l_0} = [k_1 z^{-1} + (k_0 - k_1 l_1 + k_1)z^{-2} + (-k_1 l_0 + k_1 l_1^2 - k_0 l_1 + k_0 - k_1 l_1 + k_1)z^{-3} + \ldots] \qquad (4)$$

From (3) and (4),

$$\sum_{n=0}^{\infty} U(r_n)\, z^{-n} = [k_1 z^{-1} + (k_0 - k_1 l_1 + k_1)z^{-2} + (-k_1 l_0 + k_1 l_1^2 - k_0 l_1 + k_0 - k_1 l_1 + k_1)z^{-3} + \ldots \qquad (5)$$

$$U(r_n) = [k_1, (k_0 - k_1 l_1 + k_1), (-k_1 l_0 + k_1 l_1^2 - k_0 l_1 + k_0 - k_1 l_1 + k_1 \ldots] \qquad (6)$$

Which is nothing but the amplitudes of reduced order model step response.

The steady-state value of the reduced order model is

$$R(z)|_{z=1} = ss = \frac{k_1+k_0}{1+l_1+l_0} \qquad (7)$$

Consider the step response of the original higher order system as shown in Fig.1. To determine the second-order system, select any four α_i amplitudes from the step response graph (including peaks and steady-state) = [α_1, α_2, α_3, α_{ss}]. In Fig.1, α_1, α_2, and α_3 values are the amplitudes of the first three peaks and α_{ss} value is always the steady-state value of the response. The first three α_i values are chosen with equal time space which is going to influence the sampling rate of the reduced order system.

Now to match the amplitudes of original and reduced order models, four amplitudes obtained from (6) & (7) are equated to [α_1, α_2, α_3, α_{ss}].

$$\alpha_1 = k_1 \qquad (8)$$

$$\alpha_2 = k_0 - k_1 l_1 + k_1 \qquad (9)$$

$$\alpha_3 = -k_1 l_0 + k_1 l_1^2 - k_0 l_1 + k_0 - k_1 l_1 + k_1 \qquad (10)$$

$$\alpha_{ss} = ss = \frac{k_1+k_0}{1+l_1+l_0} \qquad (11)$$

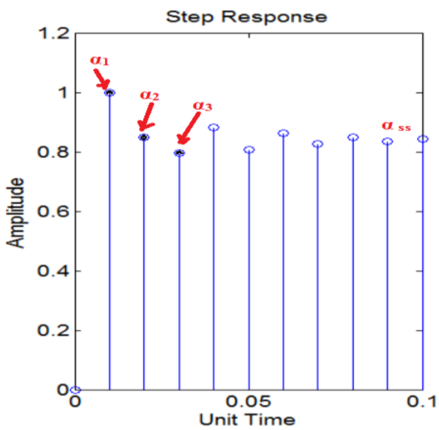

Fig. 1 Step Response of a System for Amplitude Selection.

The unknown parameters of the second order reduced system are determined by solving (8) - (11). Hence the four unknown parameters of the second order system to be designed (2) are obtained. Then the reduced order response is plotted and compared with the original response. The random dynamics-pick may end up in sample time tuning of formulated reduced order model to ensure the time match. The diverged sample rate of resultant reduced order model might influence the system output in specific applications like process industries. However, if needed the number of samples of the reduced order system can be improved by interpolation [8] or resampling. In the presented method, the sample rate is adjusted by resampling using zero-order hold method in MATLAB environment. In some cases, the order of the reduced order model may get altered while tuning the sample time.

The general way of comparing the step responses is to find the Cumulative Error Index or ISE,

$$J=\sum_{i=0}^{N}(y_i - y_{ri})^2 \qquad (12)$$

Where, y_i is the output of the original system; y_{ri} is the output of the reduced order system. Also, the reduced order model is validated by designing the Proportional Integral and Derivative (PID) controller.

III. EXTENSION TO LTICS ORDER REDUCTION

As per Shamash [9], the continuous system can also be approximated to a lower order model by the proposed method after the linear transformation without any change in the system order and accuracy of the reduction.

The general form of the nth order stable single input single output LTICS is represented as

$$G(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{n-1} a_i s^i}{\sum_{i=0}^{n} b_i s^i} \qquad (13)$$

where a_i and b_i are scalar constants.

The continuous system transfer function is transformed to discrete transfer function by linear transformation as suggested by Shamash in [9], s=z-1.

Then the transformed continuous system in the z plane becomes,

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$$G(z) = G(s)|_{s=z-1} = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{n-1} a_i' z^i}{\sum_{i=0}^{n} b_i' z^i} \quad (14)$$

By using an exclusive software, the step response of the transformed G(z) is observed. Now by applying the proposed methodology to G(z) as explained in section II, the reduced order model of order 2 is obtained in the form of

$$R(z) = \frac{N_r(z)}{D_r(z)} = \frac{k_1 z + k_0}{z^2 + l_1 z + l_0} \quad (15)$$

The inverse transformation, z=s+1 is applied to (15) to get the corresponding reduced second order continuous model

$$R(s) = R(z)|_{z=s+1} = \frac{N_r(s)}{D_r(s)} = \frac{k_1' s + k_0'}{s^2 + l_1' s + l_0'} \quad (16)$$

IV. PID CONTROLLER DESIGN

PID is the simplest controller widely used in the industries to improve the response of the system. The term PID stands for Proportional, Integral, and Derivative. The basic parameters of the PID controller design are K_p, K_i, and K_d. PID is the closed-loop feedback mechanism that continuously monitors the error and tunes its parameters to an optimum value to minimize the output error. The transfer function of a digital parallel PID controller is

$$Gc(z) = Kp + Ki\frac{Tz}{z-1} + Kd\frac{z-1}{Tz} \quad (17)$$

Where K_p is the proportional gain, K_i is the integral gain K_d is the derivative gain and T is the sampling time. To reduce the complexity of computation, the PID controller is initially designed for the reduced order system and then the same controller is employed to the original higher order system. As this paper aims to show the satisfactory adoption of the proposed reduced order PID controller to the original system. A simple graphical based tuning method is utilized to design the PID parameters K_p, K_i, and K_d. The MATLAB – Simulink - interactive tuning is used for the balanced performance and robustness. In the Simulink environment, the proposed reduced system is connected to the PID controller block as in Fig.2.

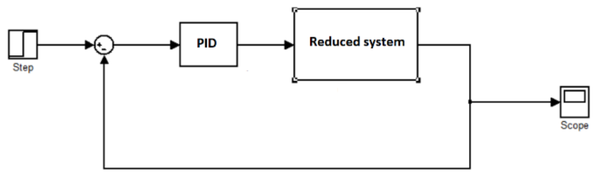

Fig. 2 MATLAB Simulink – Reduced order system with PID controller

This PID controller block automatically provides the plant model computation and initial PI controller design during launching. Then the controller avails the graphical interactive tuning by reference tracking method. In which the design specifications can be manually adjusted and the corresponding PID controller parameters K_p, K_i and K_d are generated by the software itself which can be exported for further process. The general desired design specifications to be met by the system are: Over shoot less than 3%; settling time less than 3 seconds; Steady state error less than 2%. The designed PID controller is then applied to the original system and the results are compared.

V. ILLUSTRATIONS

The proposed methodology is explained and validated for LTIDS and LTICS with the following illustrations.

A. *Example 1 -LTIDS*

Consider the higher order discrete time system from [9] $G(z) = \frac{z^2+0.9z+0.08}{z^3+1.05z^2+0.29z+0.012}$

Fig.3 shows the step response of the given system. From the plot, it is inferred that the response dynamics are placed at 0.01, 0.02, 0.03…etc. Hence the amplitudes α_1, α_2 and α_3 are taken at 0.01, 0.02 and 0.03. Steady-state value is taken as α_{ss}. Hence $\alpha_1 = 1$; $\alpha_2 = 0.85$; $\alpha_3 = 0.7975$; $\alpha_{ss} = 0.8418$; By substituting the α values in (8)-(11) and solving these equations we get, $k_1=1$; $k_0=0.97729$; $l_1= 1.1272$; $l_0= 0.2215$. Hence the reduced order system determined by the proposed method is $R(z) = \frac{z+0.97729}{z^2+1.1272z+0.2215}$.

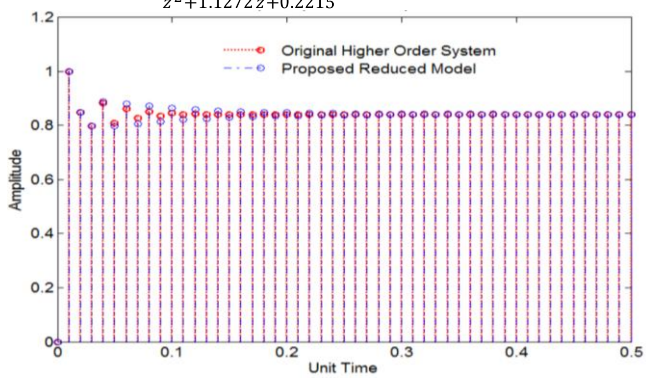

Fig. 3 Comparison of Step Responses - Example 1

Fig.3 shows the step responses of original and reduced order systems. It is found that they are well associated. And the ISE calculated for various methods are shown in tableI.

Table.1 ISE Comparison of Various Methods of Example 1

Sl.No.	Methods	ISE
1.	Shamash model [1]	0.320591
2.	N.Devarajan Method [1]	0.2006
3.	Proposed Method	0.0038

PID controller is designed for the reduced order system that is determined by the proposed method. The PID parameters tuned are: The PID parameters tuned are: Kp = 9.6642; Ki = 199.9049; Kd = 0.0454;Now this controller is connected to the original higher order system.

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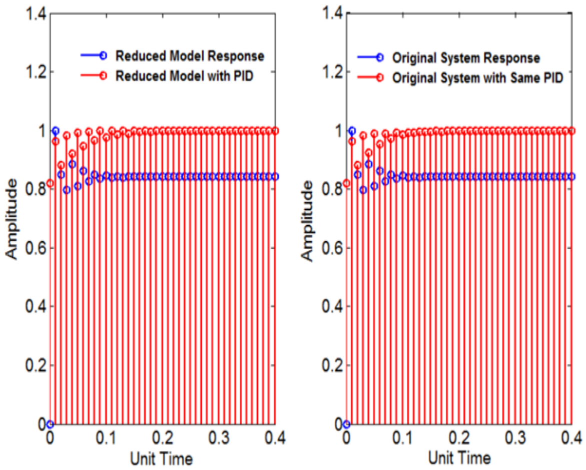

Fig.4 Step Responses with and without PID controller – Example 1

The responses with PID controller are compared in Fig.4. The performance factors of original and reduced order systems are in table II.

Table.2 Performance Companison of G(z) and R(z) With PID

Sl.No	Specifications	R(z) with PID	G(z) with same PID
1.	Rise Time	0.0200	0.0400
2.	Overshoot	0.0130	0
3.	Settling Time	0.1700	0.1600
4.	Steady State Error	4.4409×10^{-16}	-2.2204×10^{-16}

Example 2 – LTIDS – Sample Time Adjustment

Consider the higher order discrete time system [10]

$$G(z) = \frac{1.682z^7 + 1.116z^6 - 0.21z^5 + 0.152z^4 - 0.516z^3 - 0.262z^2 + 0.044z - 0.018}{8z^8 - 5.046z^7 - 3.348z^6 + 0.63z^5 - 0.456z^4 + 1.548z^3 + 0.786z^2 - 0.132z + 0.018}$$

From Fig.5. the amplitudes selected at peaks are: α_1=1.4893; α_2= 0.82135; α_3 =1.0544; and the steady state α_{ss}= ss = 0.9940. Reduced order system derived by the proposed method is,$R(z) = \frac{1.4893\ z - 0.15319}{z^2 + 0.34563\ z - 0.00146}$. It was observed that the above formulated reduced order model is in need of a sample time tuning to ensure the time match. The amplitudeswere chosenwith 0.1sample time. Hence the resultant reduced order sample time has become 0.1unit time, whereas the original system is 0.01unit time. The proposed reduced order model after sample time tuning is:$R(z) = \frac{0.0271\ z^2 + 0.1191\ z - 0.1051}{z^2 - 2.291\ z^2 + 1.801\ z - 0.4688}$.

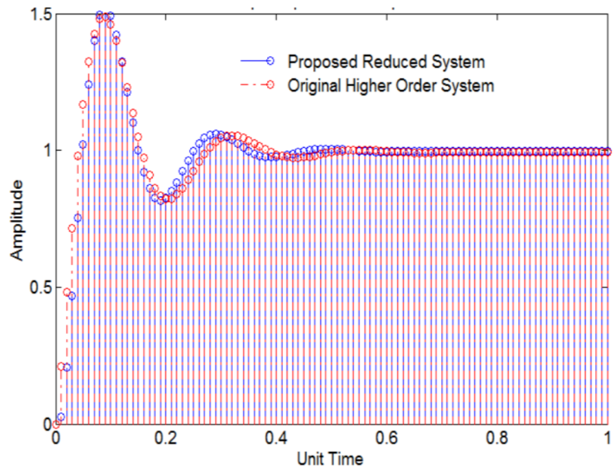

Fig. 5 Responses of original and reduced order systems – Example 2

From the Fig.5, it is inferred that the proposed reduced order model response is in good agreement with the original higher order system. And the ISE calculated for various methods are shown in table III.

Table. 3 ISE Comparison of Various methods of Example 2

Sl.No.	Methods	Reduced Model	ISE
1.	Prasad Model[10]	$R(z) = \frac{0.08516\ z - 0.03123}{z^2 - 1.730351\ z + 0.78428}$	2.5716
2.	SastryModel [11]	$R(z) = \frac{0.3166654281\ z - 0.2773}{z^2 - 1.68317616\ z + 0.722}$	0.7591
3.	Proposed Method	$R(z) = \frac{0.0271\ z^2 + 0.1191\ z - 0.}{z^2 - 2.291\ z^2 + 1.801\ z - (}$	0.2999

PID Design: Tuned PID parameters for the reduced order system are: Kp = 5.5686; Ki = 99.6024; Kd = 0.1087; This controller is then connected to the original system. Table IV compares the performance factors of G(z) and R(z) with the PID controller that is designed for the proposed reduced order system.

Table. 4 Comparison of System Performance – Example 2

Sl.No	Specifications	R(z) with PID	G(z) with same PID
1.	Rise Time	0.07	0.02
2.	Overshoot	0.00002%	0.21%
3.	Settling Time	0.18	0.4865
4.	Steady State Error	0	0

Example 3 – LTICS

Let's consider the eighth order system from [12].

$$G(s) = \frac{35s^7 + 1086\,s^6 + 13285s^5 + 82402s^4 + 278376s^3 + 511812\,s^2 + 482964s + 194480}{s^8 + 33s^7 + 437s^6 + 3017s^5 + 11870s^4 + 27470\,s^3 + 37492s^2 + 28880s + 9600}$$

By substituting s=z-1, G(s) is transformed into G(z)

$$G(z) = \frac{35z^7 + 841z^6 + 7504\,z^5 + 31042\,z^4 + 61123\,z^3 + 53801\,z^2 + 25014z + 15120}{z^8 + 25z^7 + 234\,z^6 + 1032z^5 + 2255z^4 + 2519z^3 + 2022z^2 + 1512z}$$

And the simulated step response of the discrete transformed LTICS, G(z) is given in Fig.6. (transient response alone is shown for accuracy).

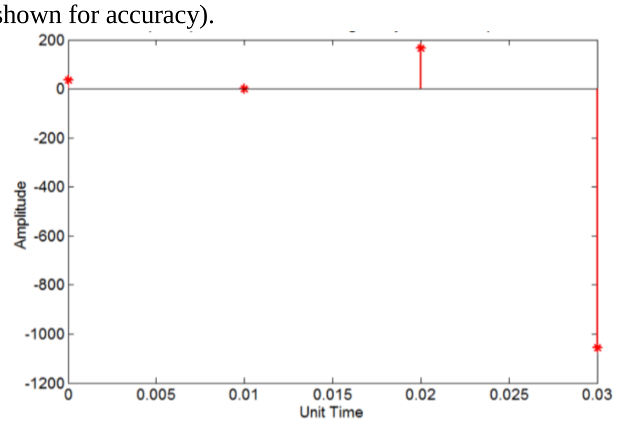

Fig. 6 Response of discrete transformed G(s)

From the above figure the peaks that are chosen as α_1 =35, α_2 = 1, α_3 =165 and steady-state α_{ss} =20.2583. Substituting and solving for equation from (8) - (11), the reduced order system determined by the proposed method is $R(z) = \frac{35\,z+502.3}{z^2+15.32\,z+10.2}$. Transforming back to obtain R(s), and $R(s) = R(z)|_{z=s+1} = \frac{35\,s+537.3}{s^2+17.32\,s+26.52}$. The responses of higher order continuous time system and the reduced order continuous time system are obtained and compared in Fig.7. From which, it is found that the proposed model response is close to the original system response.

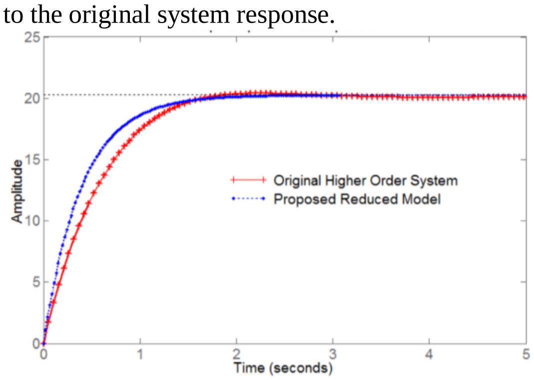

Fig.7 Step Responses of G(s) and R(s)Exampe 3

The ISE values obtained in various methods are shown in table V.

Table. 5 ISE Comparison of R(S)-Example3

Sl.No.	Methods	Reduced Model	ISE
1.	Gutman Method [12]	$\boldsymbol{R(s) = \frac{51.52715\,s + 145.24}{s^2 + 5.39208\,s + 7.16}}$	482.3756
2.	Bhagat Method [13]	$\boldsymbol{R(s) = \frac{17\,s + 6.855}{s^2 + 1.018\,s + 0.3384}}$	3.6199x103
3.	Manigandan Method[14]	$\boldsymbol{R(s) = \frac{35\,s + 547.03}{s^2 + 17.59\,s + 27.03}}$	130.2349
	Proposed Method	$\boldsymbol{R(s) = \frac{35\,s + 537.3}{s^2 + 17.32\,s + 26.52}}$	127.4646

PID Design: K_p=10; K_i= 10; and K_d= 0.0980; The responses of reduced and original system with PID controller designed are shown in Fig. 8. Which infers that the PID controller designed for the proposed reduced order system is appropriately implemented for the original system. The performance of the system with PID is analyzed in table VI.

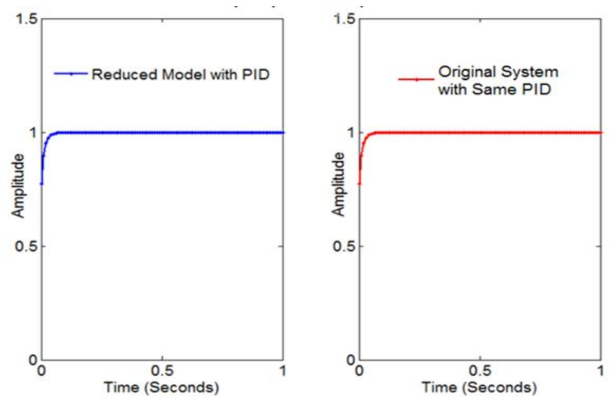

Fig.8 G(S) and R(S) with PID Controller- Example 3

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Table.6 G(S) And R(S) with PID Controller

Sl.No	Specifications	R(s) with PID	G(s) with same PID
1.	Rise Time	0.0294	0.0294
2.	Overshoot	0	0
3.	Settling Time	0.0589	0.0586
4.	Steady State Error	0	0

VI. CONCLUSION

With the simple procedure, the presented technique could model the reduced second order system from any higher order system. While adopting the proposed amplitude matching method, some systems may end up in need of time match which can be done by sample time adjustment nevertheless they could retain the stability, peaks and steady state value. Resampling is adopted to compensate the sample rate in such systems. The PID controller which is designed for the proposed lower order system works well when applied to the original system in all the cases. The proposed method is validated for both continuous and discrete systems with illustrations.

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