On Square Binormal and square n-binormal Operators

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Abstract: In this paper, a new class of operator called square binormal and square n-binormal are introduced. The objective of this paper is to study conditions on B which imply square binormal [sqBN] and square n-binormal [sqnBN]. A result is proved stating that every binormal operator is square binormal and an example is given stating that the converse is not true. It is also proved that square n-binormal operator and n-isometry operator are independent classes.

Index Terms: Hilbertspace, Normal operator, Binormal operator, Isometry operator.

I. INTRODUCTION

Throughout this paper H is a Hilbert space and L(H) is defined as the algebra of all bounded linear operators acting on a Hilbert space H. An operator $B \in L(H)$ is called normal if $B^*B = BB^*$. In [4] Panayappan S and Sivamani N introduced a new class of operators called binormal and defined as $B^*B = B = BB^*$. Meenambika K et al.,[1] have defined and analysed some of the properties of a new class of operator called Skew binormal and defined as $(B^*B)^{1/2}B = B(B^*B)^{1/2}$. In [8] Mahmood Kamil Shihab defined an operator called square normal operator defined as $B^2(B^2)^* = (B^2)^2B^2$ and an example is given to show that the square normal operator is not normal operator. In this paper, we investigate some basic properties of square n-binormal operator.

II. SQUARE BINORMAL AND SQUARE N-BINORMAL OPERATOR


Theorem 1.1

If B is a square n-binormal operator and $\beta$ is any scalar which is real then $\beta B$ is also square n-binormal operator.

Proof:

Since B is a square n-binormal operator, we have

\[(B^*)^2 B^n B^2 (B^*)^2 = B^n (B^*)^2 B^2 \quad (1.1)\]

If $\beta$ is any scalar, then $(\beta B)^* = \beta B$ also we have,

\[((\beta B)^*)^2 = \beta^2 (B^*)^2 \quad (1.2)\]

Using the results in (1.1),

\[[(\beta B)^*]^2 = \beta^2 B^n B^2 \beta^2 B^n B^2 \beta^2 B^2 \beta^2 (B^*)^2 \]

\[ = \beta^4 B^n B^2 \beta^2 B^n B^2 \beta^2 B^2 \beta^2 (B^*)^2 \quad (1.3)\]

From (1.1),(1.2) and (1.3) we see that $\beta B$ is also square n-binormal operator.

Theorem 1.2

Every binormal operator is square binormal.

Proof:


Hence B is square binormal operator.

The following example shows that square binormal operator need not be binormal.

Example 1.3

Let $B = \begin{pmatrix} i & i \\ 0 & -i \end{pmatrix}$ be an operator acting on a two dimensional Hilbert space. Then B is square binormal but not binormal.

Theorem 1.4

If B is a square n-binormal operator which is a self-adjoint operator, then $B^*$ is also square n-binormal operator.

Proof:
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If B is a square n-binormal operator, we have
\[(B^*)^2 B^{2n} = B^{2n} (B^*)^2 \] (1.4)

Since B is self-adjoint, \( B^* = B \)

Replace T by \( T^* \) in (6.4), we get
\[
\left( (B^*)^2 \right) (B^*)^{2n} (B^*)^2 = (B^{2n})^2 (B^*)^2 (B^*)^2 = (B^*)^2 (B^*)^{2n} (B^*)^2
\]

Also
\[
(B^*)^{2n} = (B^*)^{2n} (B^*) (B^*)^{2n} = (B^*)^{2n} (B^*) (B^*)^{2n}
\]

From (1.5) and (1.6), \( B^* \) is also square n-binormal.

Theorem 1.5:
If B is self-adjoint operator, then B is square n-binormal operator.

Proof:
Since B is self-adjoint, \( B^* = B \)

Now,
\[
(B^*)^2 B^{2n} (B^*)^2 = B^2 B^{2n} B^2
\]

\[
= B^2 B^{2n n}
\]

\[
= B^{2n+4n}
\]

(1.7)

\[
B^{2n} (B^*)^2 = B^2 B^2 B^2
\]

\[
= B^2 B^{2n+4n}
\]

(1.8)

\[
\therefore \text{B is square n-binormal operator.}
\]

Theorem 1.6
If B is any operator on a Hilbert space H, then
i) \( (B + B^*) \) is square n-binormal.
ii) \( B^* B \) is square n-binormal.
iii) \( B B^\ast \) is square n-binormal.

iv) \( I + B^* B \) are square n-binormal.

Proof:
i) Let \( N = B + B^* \)
\[
N^* = (B + B^*)^* = B^* + B = N
\]

Hence, \( N \) is a self-adjoint operator.

We know that every self-adjoint operator is square n-binormal.

Therefore, \( N = B + B^* \) is square n-binormal operator.

ii) \( (BB^\ast)^2 = BB^\ast BB^\ast = BB^\ast \)

Hence, \( BB^\ast \) is self-adjoint operator, so \( BB^\ast \) is square n-binormal operator.

iii) \( (BB^\ast)^2 = BB^\ast BB^\ast = BB^\ast B \)

Hence, \( B^\ast B \) is self-adjoint operator, so \( B^\ast B \) is square n-binormal operator.

iv) \( I + B^\ast B \) are self-adjoint operators, so \( I + B^\ast B, I + B^\ast B \) are square n-binormal operators.

Theorem 1.7
Let B be a square n-binormal operator on a Hilbert space H. Let L be self-adjoint operator for which B and L commute. Then L and B are also square n-binormal operators.

Proof:
Since L is self-adjoint operator, we have \( L^* = L \) and also since L and B are doubly commuting, we get
\[
LB = BL
\]

\[
L^* B = BL^*
\]

\[
B^* L = L^* B
\]

Hence \( (BL)^* = (BL)^* \)

Since B is a square n-binormal operator, we have
\[
(B^*)^2 B^{2n} (B^*)^2 = B^{2n} (B^*)^2 (B^*)^2
\]

From LB=BL it is easily seen that
\[
BL^n = L^n B
\]

\[
L^n L^* = L^* L^n
\]

\[
B^* L^n = L^n B^*
\]

\[
(\text{Consider}, \ \text{BL}^n = \text{BL}^n)
\]

\[
\therefore \text{L is sqBN}
\]
If B is a self-adjoint operator, then $B^{-1}$ is also square n-binormal operator.

Proof:
Since B is considered to be a self-adjoint operator, $B^* = B$

$\Rightarrow (B^{-1})^* = (B^*)^{-1} = B^{-1}$

$\Rightarrow B^{-1}$ is self-adjoint

$\Rightarrow B^{-1}$ is square n-binormal.

Proof verification:
Since $B^{-1}$ is self-adjoint, $(B^{-1})^* = B^{-1}$

Now consider:

$$
[B^{-1}] = (B^{-1})^* - (B^{-1})^* = (B^{-1})^* - (B^{-1})^* = (B^{-1})^* - (B^{-1})^*.
$$

From (1.9) and (1.10) $B^{-1}$ is also square n-binormal operator.

Theorem 1.9
Let B be a self-adjoint operator on a Hilbert space H and L be any operator on H, then $L^*BL$ is square n-binormal operator.

Proof:
Since B is considered to be a self-adjoint operator, $B^* = B$

Consider $(L^*BL)^* = L^*B^*L = L^*BL$

Therefore $L^*BL$ self-adjoint which implies $L^*BL$ is square n-binormal.

Proof verification:
Consider:

$$
[L^*BL] = [L^*BL] - [L^*BL] = [L^*BL] - [L^*BL] = [L^*BL] - [L^*BL].
$$

From (1.11) and (1.12), $L^*BL$ is square n-binormal.

Theorem 1.10
If B is a square binormal operator if and only if $B^2$ is binormal.

Proof:
Let B be a square binormal operator, then

$$
(B^2)^*B^2(B^2)^* = B^2(B^2)^*B^2
$$

$\Rightarrow (B^2)^*B^2(B^2)^* = B^2(B^2)^*B^2
$

$\Rightarrow B^2$ is binormal.

Theorem 1.11
If $B \in sqBN$, then so are,

i) any $L \in L(H)$ that is unitarily equivalent to B.

ii) the restriction $B/M$ of B to any closed subspace M of H that reduces B.

Proof of (i):
Let $L \in L(H)$ be unitarily equivalent to B then there is a unitary operator $U \in L(H)$ such that

$$
L^2 = U^* B^2 U, \quad L^* = U^* B^* U
$$

Consider,

$$
$$

Therefore,

$$
(B^*)^*B^2B^* = B^2(B^*)^*B^2
$$

From (1.13) and (1.14) $L \in sqBN$.

ii) If B is square binormal operator, then

$$
(B^*)^*B^2B^* = B^2(B^*)^*B^2
$$

Consider,

$$
$$

The following examples show that square n-binormal and n-isometry operators are independent classes.

Example 1.12
Consider the operator $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ on $R^2$, which is square 3-binormal but not 3-isometry.

Example 1.13
Consider the operator $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $R^2$, which is square 3-isometry but not square 3-binormal.

Theorem 1.14
Let $B_1, B_2 \ldots B_m$ be n-binormal operators in L(H). Then

$$
\forall B_i \otimes B_j \otimes \ldots \otimes B_m
$$

are square n-binormal operators.

Proof:
On Square Binormal and square n-binormal Operators

\[ \left( B_1 \oplus B_2 \oplus \ldots \oplus B_m \right) \]

Hence \( (B_1 \oplus B_2 \oplus \ldots \oplus B_m) \) is square n-binormal operator.

Now \( x_1, x_2, \ldots, x_m \in H \),

\[ \{a_{11}a_{22}a_{33} \ldots a_{nn} \}^2 \{ a_{11}a_{22}a_{33} \ldots a_{nn} \} \]

Hence \( (B_1 \ominus B_2 \ominus \ldots \ominus B_m) \) is square n-binormal operator.

III. RESULTS

The following results are proved in this article.

1. If B is a square n-binormal operator and \( \beta \) is a scalar which is real then \( \beta B \) is also square n-binormal operator.
2. Every binormal operator is square binormal.
3. Let \( B = \begin{pmatrix} i & i \\ 0 & -i \end{pmatrix} \) be an operator acting on a two dimensional Hilbert space. Then B is square binormal but not binormal.
4. If B is a square n-binormal operator which is a self-adjoint operator, then \( B^* \) is also square n-binormal operator.
5. If B is self-adjoint operator, then B is square n-binormal operator.
6. If B is any operator on a Hilbert space H, then
   \[ i(B + B^*) \]
   i) is square n-binormal.
   ii) \( BB^* \) is square n-binormal.
   iii) \( B^*B \) is square n-binormal.
   iv) \( I + B^*B, I + BB^* \) are square n-binormal.
7. Let B be a square n-binormal operator on a Hilbert space H. Let L be self-adjoint operator for which B and L commute, then LB is also square n-binormal operator.

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