On The Dynamical Behaviour of Three Species Food Chain Model with Time Delay

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Abstract: A three species model in an ecosystem involving a mutualist interaction among two species and a predator is considered across an autonomous system of intigro-ordinary delay differential equations. Due to the gestation of the predator, the delay term is proposed to the predator functional response in the model equations. The delay length is estimated how far the stability of the interior equilibrium continues to hold, if the interior equilibrium point is asymptotically stable with no delay term is under consideration. Further the local and global stabilities are discussed by perturbed method and a suitable Lyapunov technique. Also observed the increasing delay can cause a bifurcation of the stable equilibrium into periodic solutions. Finally, the numerical solutions are compared with theoretical results of the model at the end.

Index Terms: Delay, Equilibrium points, Local and global stability, Lyapunov technique, Mutual species, Numerical simulations, Predator.

I. INTRODUCTION

The discussion on local and global stability of ecological models on different types of interactions on species growth is very engrossing and demanding mathematically and biologically. The introduction to mathematical modeling in life sciences can be found in [19]. Ever since the stability of the complex ecological models were studied by [22, 23, 24]. The comprehensive report on theoretical ecology can be seen in [26]. [4, 15] have studied some applications on life and social sciences, which are connected to ordinary differential equations. A detailed discussion on more than one species population models can be observed in [3, 8, 16]. The numerical study of commensalism interaction between two species had been investigated by [25]. In particular, the more detailed discussion on models with mutualism interactions can be had from [1, 7, 9, 13, 28, 29, 31].

In biologically, the evaluation process of one species interaction with other species doesn’t respond instantaneously. For this the time delay has been implemented to the species in model equations to see how long the species will sustain in nature. The introduction of population growth models with time delay have been motivated by the influence of the past history as well as the present situations of the species in models. The impact of delay can change the stable equilibrium point to unstable and vice versa in the biological models. Particularly, the time delay plays a key role to marks out the stability of systems, which involves the growth response functions to the species. The effect of the time delay further stabilizes or destabilizes the system. Many authors have been included the time delays to the different species into their biological models. The general discussions on delayed biological systems can be had from the monographs of [2, 5, 6, 20] and also the detailed investigation on delayed predator-prey models can be found in [10, 11, 12, 14, 17, 18, 21, 27, 30].

The aim of this paper is to establish the local and global stabilities of a system involving three species of a predator, two species mutual interaction with continuous time delay. The mathematical model is formulated in section 2. We identified the possible equilibrium points of the system in section 3. The local stability is discussed through perturbed method in section 4. The global stability at the positive equilibrium point is investigated with a suitable Lyapunov function in section 5. We investigated the numerical results to support the theoretical results in section 6. At the end, we concluded with a short discussion in section 7.

II. MODEL FORMULATION

In this sub section, a mathematical model for a three species predator and the mutual interaction between two species with time delay is given here under in terms of intigro-differential equations:

\[
\begin{align*}
\frac{dx_1}{dt} &= a_1 x_1 - \alpha_1 x_1^2 + \alpha_1 x_2 x_1 - \alpha_1 x_1 y \\
\frac{dx_2}{dt} &= a_2 x_2 - \alpha_2 x_2^2 + \alpha_2 x_1 x_2 - \alpha_2 x_2^2 \int_{-\infty}^{t} K_1(t-u)y(u)du \\
\frac{dy}{dt} &= \alpha_3 y - \alpha_3 y^2 + \alpha_3 x_1 y + \alpha_3 x_2 y - \frac{\alpha_3 \beta_2}{xy} \int_{-\infty}^{t} K_2(t-u)x_2(u)du
\end{align*}
\]

(2.1)

Nomenclature:

\begin{itemize}
  \item \(x_1, x_2, y\) : The populations’ densities of mutual species and a predator.
  \item \(a_1, a_2, a_3\) : The natural growth rates of all three species.
\end{itemize}
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\(a_{11}, a_{22}, a_{33}\) : The decreasing rates of three species because of their limited resources available in nature.

\(a_{12}, a_{21}\) : The increasing rate for the mutual species due to their support.

\(a_{13}, a_{23}\) : The rate decrease due to predator feed on mutual species.

\(a_{31}, a_{32}\) : The rates of increase for the predator form the mutual species.

\(K_t(t-u), K_2(t-u)\) : The weight factors for the second mutual species and a predator.

Normalizing the kernels \(K_1\) and \(K_2\) in (2.1) with the following conditions

\[
\int_0^\infty K_1(z)dz = 1, \int_0^\infty K_2(z)dz = 1, \int_0^\infty zK_1(z)dz < \infty \text{ and } \int_0^\infty zK_2(z)dz < \infty \tag{2.2}
\]

By employing delay kernels conditions in (2.2), it becomes

\[
dx_1{dt} = a_1x_1 - a_{11}x_1^2 + a_{12}x_1x_2 - a_{13}x_1y
\]

\[
dx_2{dt} = a_2x_2 - a_{22}x_2^2 + a_{21}x_1x_2 - a_{23}x_2 \int_0^\infty K_1(z) y(t-z)dz
\]

\[
dy{dt} = a_3y - a_{33}y^2 + a_{31}x_1y + a_{32} \int_0^\infty K_2(z)x_2(t-z)dz
\]

(2.3)

with conditions: \(x_1(0) = 0, x_2(0) = 0, y(0) = 0\).

III. EQUILIBRIUM STATES

In this section, all positive equilibrium points of the system (2.3) are identified by setting \(\frac{dx_i}{dt} = 0\) (i = 1, 2) and \(\frac{dy}{dt} = 0\).

(2.3) \(E_1\) : All species extinct equilibrium state

\(\bar{x}_1 = 0; \bar{x}_2 = 0; \bar{y} = 0\)

\(E_2\) : Only the first mutual survival equilibrium state

\(\bar{x}_1 = \frac{a_1}{a_{11}}; \bar{x}_2 = 0; \bar{y} = 0\)

\(E_3\) : Only the second mutual species survival equilibrium state

\(x_1 = \bar{x}_1; x_2 = \frac{a_2}{a_{22}}; y = \bar{y}\)

\(E_4\) : Only predator survival equilibrium state

\(\bar{x}_1 = 0; \bar{x}_2 = 0; \bar{y} = \frac{a_3}{a_{33}}\)

\(E_5\) : The predator free equilibrium

\(E_6\) : Second mutual species washed out equilibrium

This case exists only when \(a_2a_{33} > a_3a_{23}\).

\(E_7\) : First mutual species washed out equilibrium

This is positive under the condition \(a_1a_{33} > a_3a_{13}\).

\(E_8\) : The coexistence state

with case exists the following four conditions hold

\(a_{12}a_{33}a_{31} + a_{13}a_{23}a_{21} + a_{23}a_{12}a_{13} + a_{13}a_{23}a_{12} + a_{13}a_{23}a_{12} > 0\)

\(a_{12}a_{33}a_{31} + a_{13}a_{23}a_{21} + a_{23}a_{12}a_{13} + a_{13}a_{23}a_{12} > 0\)

\(a_{12}a_{33}a_{31} + a_{13}a_{23}a_{21} + a_{23}a_{12}a_{13} + a_{13}a_{23}a_{12} > 0\)

We note that the equilibrium points of (2.1) and (2.3) are same with those same kernels.

IV. LOCAL STABILITY ANALYSIS

Local stability around the corresponding equilibrium points of the system can be observed based on the nature of eigen values of a Jacobian matrix for the linearized perturbed equations of the dynamical system. Let us...
take \( u_1 = A_1 e^{t/\alpha}, u_2 = A_2 e^{t/\alpha}, u_3 = A_3 e^{t/\alpha} \), the system (2.3) becomes
\[
\frac{du}{dt} = (a_1 - 2a_1 x_1 + a_2 x_2 - a_1 y)u_1 + a_2 x_2 u_2 - a_3 x_3 u_3
\]
\[
\frac{du_2}{dt} = a_2 x_2 u_1 + (a_2 - 2a_3 x_2 + a_1 x_1 - a_3 y)u_2 - a_3 x_3 k_2(\lambda)u_3
\]
\[
\frac{du_3}{dt} = a_3 y u_1 + a_3 x_2 k_1(\lambda)u_2 - (a_3 - 2a_3 y + a_1 x_1 + a_3 x_2)u_3
\]
(4.1)

where 
\[
k_1(\lambda) = \int_0^\infty k_1(z)\exp(-\lambda z)dz
\]
and 
\[
k_2(\lambda) = \int_0^\infty k_2(z)\exp(-\lambda z)dz
\]
are the Laplace Transforms of 
\( k_1(\lambda) \) and \( k_2(\lambda) \).

The associated Jacobian matrix for the system (4.1) after linearization is as follows
\[
\frac{dU}{dt} = J_E U
\]
where
\[
J_E = \begin{bmatrix}
a_1 - a_2 y_1 + a_2 y_2 - a_3 y & a_1 y_1 & -a_3 y_1 \\
a_2 y_2 & a_2 - 2a_3 y_2 + a_2 y_1 - a_2 y_2 & -a_3 x_2 k_2(\lambda) \\
a_3 y_1 & a_3 x_2 k_1(\lambda) & a_3 - 2a_3 y_1 + a_1 y_1 + a_3 y_2
\end{bmatrix}
\]
(4.3)

The system is stable when all eigen values of Jacobian matrix \( J_E \) are negative reals or negative real parts when the eiven values are complex at that particular equilibrium point.

In the following prepositions, the local stability nature of the dynamical system at existing eight equilibrium points is discussed.

**Proposition: 1** The extinct equilibrium state \( E_1 \) is always unstable.

**Proof:** The eigen values for the following Jacobian matrix at this extistent state are all positive (i.e. \( \lambda_1 = a_1, \lambda_2 = a_2 \) and \( \lambda_3 = a_3 \)) and hence the system is unstable at this \( E_1 \).

\[
J_{E_1} = \begin{bmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{bmatrix}
\]

**Proposition: 2** Only first mutual survival equilibrium state \( E_2 \) is a saddle node.

**Proof:** In this case the following is the Jacobian matrix and whose eigen values are \( \lambda_1 = a_1 - \frac{a_2 a_1}{\alpha_{22}}, \lambda_2 = -a_2 < 0 \) and \( \lambda_3 = a_3 + \frac{a_2 a_1}{\alpha_{22}} > 0 \), so this equilibrium point is a saddle point for any possible conditions of \( \lambda_4 \).

\[
J_{E_2} = \begin{bmatrix}
a_1 - \frac{a_2 a_1}{\alpha_{22}} & 0 & 0 \\
\frac{a_2 a_1}{\alpha_{22}} & -a_2 & -a_3 a_2 K_1(\lambda) \\
0 & 0 & a_3 + \frac{a_2 a_1}{\alpha_{22}}
\end{bmatrix}
\]

**Proposition: 3** The second mutual species survival equilibrium state \( E_3 \) is unstable.

**Proof:** Jacobian matrix in this case is

\[
J_{E_3} = \begin{bmatrix}
a_1 & \frac{a_2 a_1}{\alpha_{11}} & -\frac{a_2 a_3}{\alpha_{11}} \\
0 & \frac{a_2 a_1}{\alpha_{11}} & 0 \\
0 & 0 & a_3 + \frac{a_2 a_1}{\alpha_{11}}
\end{bmatrix}
\]

The corresponding eigen values of which are
\( \lambda_1 = -a_1 < 0, \lambda_2 = a_2 + \frac{a_2 a_1}{\alpha_{11}} > 0 \) and
\( \lambda_3 = a_3 + \frac{a_2 a_1}{\alpha_{11}} > 0 \). One can observe that the present steady state is unstable always.

**Proposition: 4** If \( a_1 < \frac{a_2 a_1}{\alpha_{13}} \) and \( a_2 < \frac{a_2 a_3}{\alpha_{33}} \) then the predator survival equilibrium state \( E_4 \) is stable otherwise is unstable.

**Proof:** The relevant Jacobian matrix at this state is

\[
J_{E_4} = \begin{bmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{bmatrix}
\]
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The secular equation of this matrix is

$$\lambda^2 - \lambda \left( a_1 + a_2 + a_3 \cdot \bar{x}_1 - a_{33} \bar{y} \right)$$

$$\lambda^2 - \lambda \left( a_1 + a_2 + a_3 \cdot \bar{x}_1 - a_{33} \bar{y} - 2a_1 \bar{x}_1 - 2a_3 \bar{y} \right) + \left( -a_1 a_3 \bar{x}_1^2 - a_{33} \bar{y} + 2a_3 a_3 \bar{y}^2 \right) = 0$$

The steady state is stable if satisfies the condition

$$a_2 + 2a_1 \bar{x}_1 = a_{33} \bar{y} < 0,$$

$$a_1 + a_2 + a_3 \cdot \bar{x}_1 + 2a_1 \bar{x}_1 + 2a_3 \bar{y} + 2a_3 a_3 \bar{y}^2 > (4.4)$$

Otherwise is unstable.

Proposition: 7 The first mutual species washed out equilibrium state \( E_7 \) is stable satisfying the condition (4.5), otherwise is unstable.

Proof: The Jacobian matrix for this state is

$$J_{E_7} = \begin{bmatrix}
    0 & 0 & 0 \\
    -a_{33} \bar{y} & a_{33} \bar{y} & a_{32} K'_2(\lambda) \\
    -a_{33} \bar{y} & a_{32} K'_2(\lambda) & a_3 - 2a_{33} \bar{y} + a_2 \bar{x}_2
\end{bmatrix}$$

The characteristic equation is

$$\lambda^2 - \lambda \left( a_1 + a_2 + a_3 \cdot \bar{x}_1 - a_{33} \bar{y} \right)$$

$$\lambda^2 - \lambda \left( a_1 + a_2 + a_3 \cdot \bar{x}_1 - a_{33} \bar{y} - 2a_1 \bar{x}_1 - 2a_3 \bar{y} \right) + \left( -a_1 a_3 \bar{x}_1^2 - a_{33} \bar{y} + 2a_3 a_3 \bar{y}^2 \right) = 0$$

From above one can observe the steady state is unstable.

Proposition: 6 Only the second mutual species washed out equilibrium state \( E_6 \) is stable for satisfying certain condition (4.4) otherwise is unstable.

Proof: The corresponding Jacobian matrix at this equilibrium point is

$$J_{E_6} = \begin{bmatrix}
    a_1 - 2a_1 \bar{x}_1 + a_2 \bar{x}_2 & a_1 \bar{x}_1 & -a_{31} \bar{y} \\
    -a_{21} \bar{x}_1 & a_2 - a x_1 & -a_{31} \bar{y} \\
    a_3 \bar{y} & a_{32} K'_2(\lambda) & a_3 - 2a_{33} \bar{y} + a_2 \bar{x}_2
\end{bmatrix}$$

The steady state is stable if satisfies the condition

$$a_2 + a_1 \bar{x}_1 - a_{33} \bar{y} < 0,$$

$$a_1 + a_2 + a_3 \cdot \bar{x}_1 < 2a_1 \bar{x}_1 + 2a_3 \bar{y} + 2a_3 a_3 \bar{y}^2$$

$$2a_1 a_3 \bar{x}_1 + 2a_1 a_1 \bar{x}_1 + a_3 a_3 \bar{y} + a_3 a_3 \bar{y}^2$$

Otherwise is unstable.

Proposition: 8 If \( b_1 > 0 \), \( b_3 > 0 \) and \( b_3 - b_3 > 0 \), then the positive equilibrium state \( E_8 \) is locally asymptotically stable, otherwise it is unstable.

Proof: For the coexistence point, the corresponding Jacobian matrix is
The characteristic equation of $J_{kq}$ is
\[ \lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = 0. \]
Here
\[
\begin{align*}
b_1 &= -(\alpha_1 + \alpha_2 + \alpha_3) \\
b_2 &= \alpha_4 \lambda_2 + \lambda_3 \lambda_4 - \alpha_1 \alpha_2 \lambda_1 \lambda_2 + \alpha_3 \lambda_2 \lambda_3 + \alpha_1 \lambda_3 \lambda_4 - \alpha_2 \alpha_3 \lambda_1 \lambda_3 \\
b_3 &= \alpha_3 \lambda_3 \lambda_4 - \alpha_1 \alpha_2 \lambda_1 \lambda_3 + \alpha_1 \lambda_3 \lambda_4 - \alpha_2 \alpha_3 \lambda_1 \lambda_3 \\
\end{align*}
\]
where
\[
\begin{align*}
\lambda_1 &= a_1 - 2a_1 \theta_1 + a_2 \theta_2 - a_3 \theta_3 \\
\lambda_2 &= a_2 - 2a_2 \theta_2 + a_1 \theta_1 - a_3 \theta_3 \\
\lambda_3 &= a_3 - 2a_3 \theta_3 + a_1 \theta_1 - a_2 \theta_2 \\
\end{align*}
\]
Now
\[
\begin{align*}
b_1 b_2 - b_3 &= -\lambda_1^2 (\lambda_2 + \lambda_3) - \lambda_2^2 (\lambda_1 + \lambda_3) - \lambda_3^2 (\lambda_1 + \lambda_2) - 2\lambda_1 \lambda_2 \lambda_3 \\
&\quad - (\lambda_1 + \lambda_2 + 2\lambda_3) a_1 a_3 \theta_1 \theta_3 \\
&\quad - (\lambda_2 + \lambda_3) a_2 a_3 \lambda_1 \theta_1 \\
&\quad + (\lambda_1 + \lambda_2) a_1 a_3 \lambda_2 \theta_2 \\
&\quad + (-a_1 a_3 a_2 \lambda_1 \lambda_2 + a_1 a_2 a_3 \lambda_2 \lambda_3) \theta_1 \theta_2 \theta_3
\end{align*}
\]
The positive equilibrium point is locally asymptotically, by Routh-Hurwitz criterion, if $b_1 > 0$, $b_2 > 0$ and $b_1 b_2 - b_3 > 0$.

V. GLOBAL STABILITY

The global stability of the system is discussed at the positive equilibrium point $E_k$ in this subsection with suitable Lyapunov's function.

**Theorem 1:** The positive equilibrium state $E_k (\theta_1, \theta_2, \theta_3)$ for the system (4.1) is globally asymptotically stable if it holds the condition.

\[ (a_{11} + a_{22} + a_{33} + \frac{1}{2} a_{13}) > \frac{1}{2} (a_{12} + a_{21} + a_{23} + a_{31} + a_{32} + a_{13}) \]

**Proof:** The suitable Lipunov function for the system (4.1) is as follows:

\[ V(x_1, x_2, y) = x_1 - \theta_1 \left[ 1 - \log \left( \frac{x_1}{\theta_1} \right) \right] \]
\[ + x_2 - \theta_2 \left[ 1 - \log \left( \frac{x_2}{\theta_2} \right) \right] \]
\[ + y - \theta_3 \left[ 1 - \log \left( \frac{y}{\theta_3} \right) \right] \]
\[ + \frac{1}{2} a_{23} \int_0^\infty k_1(z) (y(t) - \theta_3)^2 dz \]
\[ + \frac{1}{2} a_{32} \int_0^\infty k_2(z) (x_2(t) - \theta_2)^2 dz \]
\[ + \frac{1}{2} a_{13} \int_0^\infty k_3(z) (x_1(t) - \theta_1)^2 dz \]

(5.1)

For all positive values of $x_1, x_2, y$, the above function $V(x_1, x_2, y)$ in equation (5.1) is non zero at the positive equilibrium state. Differentiating (5.1) with respect to the time along the solution of (4.1) is

\[ \frac{dV}{dt} = \left( 1 - \frac{\theta_1}{x_1} \right) x_1' + \left( 1 - \frac{\theta_2}{x_2} \right) x_2' + \left( 1 - \frac{\theta_3}{y} \right) y' + \frac{1}{2} a_{23} \int_0^\infty k_1(z) (y(t) - \theta_3)^2 dz \\
- \frac{1}{2} a_{23} \int_0^\infty k_1(z) (y(t) - \theta_3)^2 dz \\
+ \frac{1}{2} a_{32} \int_0^\infty k_2(z) (x_2(t) - \theta_2)^2 dz \\
- \frac{1}{2} a_{32} \int_0^\infty k_2(z) (x_2(t) - \theta_2)^2 dz \\
+ \frac{1}{2} a_{13} \int_0^\infty k_3(z) (x_1(t) - \theta_1)^2 dz \\
- \frac{1}{2} a_{13} \int_0^\infty k_3(z) (x_1(t) - \theta_1)^2 dz \\
\]

(5.2)

\[ \frac{dV}{dt} = (x_1 - \theta_1) (a_{11} x_1 + a_{12} x_2 + a_{13} y - a_{13} y) \\
+ (x_2 - \theta_2) (a_{21} x_1 + a_{22} x_2 + a_{23} y - a_{23} y) \\
+ (y - \theta_3) (a_{31} x_1 + a_{32} x_2 + a_{33} y - a_{33} y) \\
+ \frac{1}{2} a_{23} \int_0^\infty k_1(z) (y(t) - \theta_3)^2 dz \\
- \frac{1}{2} a_{23} \int_0^\infty k_1(z) (y(t) - \theta_3)^2 dz \\
+ \frac{1}{2} a_{32} \int_0^\infty k_2(z) (x_2(t) - \theta_2)^2 dz \\
- \frac{1}{2} a_{32} \int_0^\infty k_2(z) (x_2(t) - \theta_2)^2 dz \\
+ \frac{1}{2} a_{13} \int_0^\infty k_3(z) (x_1(t) - \theta_1)^2 dz \\
- \frac{1}{2} a_{13} \int_0^\infty k_3(z) (x_1(t) - \theta_1)^2 dz \\
\]

(5.3)

The equation (5.3) becomes
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\[ \frac{dV}{dt} = -a_1(x_1 - \bar{x}_1)^2 + a_2(x_2 - \bar{x}_2)^2 + a_3(x_3 - \bar{x}_3)^2 \]

...more mathematical content...

by the following parameters values of \( a_1, a_2, a_3 \):

\[ a_1 = a_{11}\bar{x}_1 - a_{12}\bar{x}_2 + a_{13}\bar{x}_3 \]
\[ a_2 = a_{22}\bar{x}_2 - a_{21}\bar{x}_1 + a_{23}\int_0^\infty k_1(z)\left(y(t-z) - \bar{y}\right)dz \]
\[ a_3 = a_{33}\bar{x}_3 - a_{31}\bar{x}_1 + a_{32}\int_0^\infty k_2(z)x_2(t-z)dz \]

Using the inequalities

\[ ab \leq \frac{a^2 + b^2}{2}, \int_0^\infty k_1(z)[y(t-z) - \bar{y}]^2dz \leq \int_0^\infty k_1(z)dz = 1 \]

and

\[ \int_0^\infty k_2(z)[x_2(t-z) - \bar{x}_2]^2dz \leq \int_0^\infty k_2(z)dz = 1 \]

Equation (5.4) becomes

\[ \frac{dV}{dt} \leq -a_1(x_1 - \bar{x}_1)^2 + \frac{1}{2}\left[a_{12}(x_2 - \bar{x}_2)^2 + a_{13}(x_3 - \bar{x}_3)^2\right] \]

where

\[ n = \min\left(\frac{1}{2}(a_{12} + a_{13} + a_{21} + a_{31} + a_{32} + a_{23} + a_{33})\right) \]

This shows that \( \frac{dV}{dt} < 0 \) for all positive values of \( x_1, x_2 \)

and \( y \) . Hence by well known stability theorem the function \( V(x_1, x_2, y) \) is asymptotically globally stable at the positive equilibrium state \( E_0 \) . Therefore the dynamical model (2.1) is asymptotically stable at the same positive equilibrium state satisfying the above condition.

VI. Numerical Simulation

Using transformation method, the system of equations (4.1) is converted the following fifth order of ordinary differential equations:

\[ \frac{dx}{dt} = a_1x_1 - a_{21}x_1x_2 - a_{31}x_1x_3 \]
\[ \frac{dx_2}{dt} = a_{22}x_2 - a_{21}x_1 + a_{23}\int_0^\infty k_1(z)\left(y(t-z) - \bar{y}\right)dz \]
\[ \frac{dx_3}{dt} = a_{33}x_3 - a_{31}x_1 + a_{32}\int_0^\infty k_2(z)x_2(t-z)dz \]

Defining the kernels as follows

\[ w_1 = \int_{-\infty}^t k_1(t-u)y(u)du \]
\[ w_2 = \int_{-\infty}^t k_2(t-u)x_2(u)du \]

Now the system of equations (6.1) is solved numerically to observe its behavior using MATLAB in both the cases of the absence and the presence of time delay in dynamical system as follows:

A. Simulation in the absence of a time delay

System of equations without delay is the following form and is solved numerically for suitable parametric values in the system with same package; we get the following results illustrated by the Figures 1 and 2.
\[
\begin{align*}
\frac{dx_1}{dt} &= \alpha_1 x_1 - \alpha_{11} x_1^2 + \alpha_{12} x_1 x_2 - \alpha_{13} x_1 y \\
\frac{dx_2}{dt} &= \alpha_2 x_2 - \alpha_{22} x_2^2 + \alpha_{21} x_1 x_2 - \alpha_{23} x_2 y \\
\frac{dy}{dt} &= \alpha_3 y - \alpha_{33} y^2 + \alpha_{32} x_2 y + \alpha_{31} x_1 y
\end{align*}
\] (6.2)

\[a_1 = 12.52, \ a_{11} = 12.48, \ a_{12} = 22.96, \ a_{13} = 16.96,\]
\[a_2 = 18.56, \ a_{21} = 12.48, \ a_{22} = 9.52, \ a_{23} = 21.44, \ a_{31} = 2.88, \ a_{32} = 2.64, \ a_{33} = 14.8, \ x_1(0) = 10, \ x_2(0) = 10, \ y(0) = 5\]

The solution curves for system (6.2) are shown below for the above parameters.

B. Simulation in the presence of a time delay

Delay effect for the system (6.1) for different values of \(\alpha\) and \(\beta\) are illustrated with same above parameter values.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Figures</th>
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<tbody>
<tr>
<td>1</td>
<td>Odd number figured from 3 to 13.</td>
<td>Shows the variation of (x_1, x_2) and (y) with time to the system (4.1).</td>
</tr>
<tr>
<td>2</td>
<td>Even number figured from 4 to 14.</td>
<td>The phase portrait of (x_1, x_2) and (y) for the system (4.1).</td>
</tr>
</tbody>
</table>

The delay kernels for different values \(\alpha\) and \(\beta\) for the system of equations are plotted as follows:

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Parameter values of (\alpha) and (\beta) &amp; Converging Equilibrium point</th>
<th>Nature of the system</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\alpha = 0.01; \ \beta = 0.01) (E(0.595, 0.301))</td>
<td>The second mutual species extinct first and both the predator and the first mutual population are increasing from initial population to certain extent and stabilize at fixed population exhibits linear growth dynamics.</td>
</tr>
<tr>
<td>2</td>
<td>(\alpha = 0.6; \ \beta = 0.6) (E(1.75, 1.35, 0.854))</td>
<td>The mutual species populations are dominating the predator species due to the less attack rate of the predator over the mutual species.</td>
</tr>
<tr>
<td>3</td>
<td>(\alpha = 0.9; \ \beta = 0.9) (E(5.567, 4.025, 2.088))</td>
<td>All three species exists and there will be no considerable growth rates after the time 2 units in the interval ([0, 20]).</td>
</tr>
</tbody>
</table>

The solutions of system (6.2) shown in the above figures are asymptotically stable in the absence of delay arguments. Time series analysis shows periodic monotonic decay oscillations’ converging to the equilibrium point \(E(9.009, 6.677, 3.147)\) is shown in phase portrait. Hence the system is asymptotically stable.
On The Dynamical Behaviour of Three Species Food Chain Model with Time Delay

<p>| | | |</p>
<table>
<thead>
<tr>
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<th></th>
<th></th>
</tr>
</thead>
</table>
| 4 | $\alpha = 0.95; \beta = 0.95$  
  $E(7, 5.127, 2.529)$ | With this time delay terms all three species gradually increasing with low growth rate. In this case the mutual species dominate the predator initially and then converging to fixed equilibrium point, and make stable. |
| 5 | $\alpha = 1.01; \beta = 1.01$  
  $E(9.754, 7.24, 3.373)$ | The periodic solutions for all three species in the interval [0, 20] exhibit the instability of the system. |
| 6 | $\alpha = 1.02; \beta = 1.02$  
  $E(4.301, 3.201, 2.172)$ | The oscillatory population growths with high intensity in mutual species comparing the predator on the time interval [0, 20] with cyclic nature in the dynamical system indicate the instability at this equilibrium point. |

$\alpha = 0.95; \beta = 0.95$

$\alpha = 1.01; \beta = 1.01$

$\alpha = 1.02; \beta = 1.02$

$\alpha = 0.6, \beta = 0.6, E(1.75, 1.35, 0.854)$

$\alpha = 0.01; \beta = 0.01, E(0.595, 0, 0.301)$
\[ \alpha = 0.9, \beta = 0.9, \ E(5.567, 4.025, 2.088) \]

\[ \alpha = 0.95, \beta = 0.95, \ E(7.5127, 2.529) \]

\[ \alpha = 1.01, \beta = 1.01, \ E(9.754, 7.24, 3.373) \]
VII. CONCLUSIONS

In the present paper, we explore an ecological model of a predator and two mutual species for investigation. The time delay is imposed on the predator - second mutual species and all three species are within the limited resources. We identified the positive equilibrium points for the model satisfying certain conditions, and the local stability is discussed around those of the dynamical system. Further, we also studied the system global stability at coexistence state using suitable Lapponov function. The graphical illustrations are presented to compare the results with the system not including delay term by suitable example. Also, the numerical simulations are presented to identify the weighted functions with different kernels values of $\alpha$ and $\beta$ in which the system exhibits rich dynamics. For different kernels of $\alpha$, $\beta$ in the system (4.1) we observed the following behavior in all three species:

i) There is considerable growth rate for all three species as the values of $\alpha$ and $\beta$ varies from 0.01 to 0.2, and moreover the second mutual species is existent first and remains converge to the equilibrium point later. So one can observe that the dynamical system is unstable for these delay terms.

ii) As the range of $\alpha$ and $\beta$ varies from [0.6, 0.95], all three species exists and the growth rates of them are gradually increasing and then later reaching their asymptotic values. Further the kernels’ values in the above interval don’t more effect on mutual species and so the system is stable.

iii) When the kernel values are 1.01, all three species having oscillatory growth in the random environment with high intensity of first mutual species comparing to other two species.

iv) As soon as the kernels cross the value 1.01, the dynamical system lost its stability. So hopf bifurcation can observe at that point to the system.

REFERENCES


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