On The Dynamical Behaviour of Three Species Food Chain Model with Time Delay

Y. Suresh Kumar, N. Seshagiri Rao, B. V. Appa Rao

Abstract**:** ***A three species model in an ecosystem involving a mutualist interaction among two species and a predator is considered across an autonomous system of intrigro-ordinary delay differential equations. Due to the gestation of the predator, the delay term is proposed to the predator functional response in the model equations. The delay length is estimated how far the stability of the interior equilibrium continues to hold, if the interior equilibrium point is asymptotically stable with no delay term is under consideration. Further the local and global stabilities are discussed by perturbed method and a suitable Lyapunov technique. Also observed the increasing delay can cause a bifurcation of the stable equilibrium into periodic solutions. Finally, the numerical solutions are compared with theoretical results of the model at the end.***

Index Terms**:** ***Delay, Equilibrium points, Local and global stability, Lyapunov technique, Mutual species, Numerical simulations , Predator.***

I. INTRODUCTION

The discussion on local and global stability of ecological models on different types of interactions on species growth is very engrossing and demanding mathematically and biologically. The introduction to mathematical modeling in life sciences can be found in [19]. Ever since the stability of the complex ecological models were studied by [22, 23, 24]. The comprehensive report on theoretical ecology can be seen in [26]. [4, 15] have studied some applications on life and social sciences, which are connected to ordinary differential equations. A detailed discussion on more than one species population models can be observed in [3, 8, 16]. The numerical study of commensalism interaction between two species had been investigated by [25]. In particular, the more detailed discussion on models with mutualism interactions can be had from [1, 7, 9, 13, 28, 29, 31].

In biologically, the evaluation process of one species interaction with other species doesn't respond instantaneously. For this the time delay has been implemented to the species in model equations to see how long the species will sustain in nature. The introduction of population growth models with time delay have been motivated by the influence of the past history as well as the present situations of the species in models. The impact of delay can change the stable equilibrium point to unstable and vice versa in the biological models. Particularly, the time delay plays a key role to marks out the stability of systems, which involves the growth response functions to the species. The effect of the time delay further stabilizes or destabilizes the system. Many authors have been included the time delays to the different species into their biological models. The general discussions on delayed biological systems can be had from the monographs of [2, 5, 6, 20] and also the detailed investigation on delayed predator-prey models can be found in [10, 11, 12, 14, 17, 18, 21, 27, 30].

The aim of this paper is to establish the local and global stabilities of a system involving three species of a predator, two species mutual interaction with continuous time delay. The mathematical model is formulated in section 2. We identified the possible equilibrium points of the system in section 3. The local stability is discussed through perturbed method in section 4. The global stability at the positive equilibrium point is investigated with a suitable Lypunov function in section 5. We investigated the numerical results to support the theoretical results in section 6. At the end, we concluded with a short discussion in section 7.

II. MODEL FORMULATION

In this sub section, a mathematical model for a three species predator and the mutual interaction between two species with time delay is given here under in terms of intigro- differential equations:

$$\frac{dx_1}{dt} = a_1x_1 - \alpha_{11}x_1^2 + \alpha_{12}x_1x_2 - \alpha_{13}x_1y$$
$$\frac{dx_2}{dt} = a_2x_2 - \alpha_{22}x_2^2 + \alpha_{21}x_1x_2 - \alpha_{23}x_2\int_{-\infty}^{t} K_1(t-u)y(u)du$$
$$\frac{dy}{dt} = a_3y - \alpha_{33}y^2 + \alpha_{31}x_1y + \alpha_{32}y\int_{-\infty}^{t} K_2(t-u)x_2(u)du$$

(2.1)

Nomenclature:

x_1, x_2, y	:	The populations' densities of mutual species and a predator.
a_1, a_2, a_3	:	The natural growth rates of all three species.

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$\alpha_{11},\alpha_{22},\alpha_{33}$	:	The decreasing rates of three species because of their limited resources available in nature.
α_{12},α_{21}	:	The increasing rate for the mutual species due to their support.
α_{13},α_{23}	:	The rate decrease due to predator feed on mutual species.
α_{31},α_{32}	:	The rates of increase for the predator form the mutual species.
$K_1(t-u), K_2(t-u)$	:	The weight factors for the second mutual species and a predator.

Normalizing the kernals K_1 and K_2 in (2.1) with the following conditions

$$\int_0^\infty K_1(z)dz = 1, \int_0^\infty K_2(z)dz = 1, \int_0^\infty zK_1(z)dz < \infty \text{ and}$$

$$\int_0^\infty zK_2(z)dz < \infty \qquad (2.\ 2)$$

By employing delay kernals conditions in (2.2), it becomes

$$\frac{dx_1}{dt} = a_1x_1 - \alpha_{11}x_1^2 + \alpha_{12}x_1x_2 - \alpha_{13}x_1y$$

$$\frac{dx_2}{dt} = a_2x_2 - \alpha_{22}x_2^2 + \alpha_{21}x_1x_2 - \alpha_{23}x_2\int_0^\infty K_1(z)y(t-z)dz$$

$$\frac{dy}{dt} = a_3y - \alpha_{33}y^2 + \alpha_{31}x_1y + \alpha_{32}y\int_0^\infty K_2(z)x_2(t-z)dz$$

(2.3)

with conditions: $x_1(0) = 0, x_2(0) = 0, y(0) = 0$.

III. Equilibrium States

In this section, all positive equilibrium points of the system (2.3) are indentified by setting $\frac{dx_i}{dt} = 0\ (i = 1, 2)$ and $\frac{dy}{dt} = 0$.

(2.3) E_1 : All species extinct equilibrium state

$$\bar{x}_1 = 0\ ;\ \bar{x}_2 = 0\ ;\ \bar{y} = 0$$

E_2 : Only the first mutual survival equilibrium state

$$\bar{x}_1 = \frac{a_1}{\alpha_{11}}\ ;\ \bar{x}_2 = 0\ ;\ \bar{y} = 0$$

E_3 : Only the second mutual species survival equilibrium state

$$\bar{x}_1 = 0\ ;\ \bar{x}_2 = \frac{a_2}{\alpha_{22}}\ ;\ \bar{y} = 0$$

E_4 : Only predator survival equilibrium state

$$\bar{x}_1 = 0\ ;\ \bar{x}_2 = 0\ ;\ \bar{y} = \frac{a_3}{\alpha_{33}}$$

E_5 : The predator free equilibrium

$$E_5 : \bar{x}_1 = \frac{a_2\alpha_{12} + a_1\alpha_{22}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\ ;\ \bar{x}_2 = \frac{a_2\alpha_{11} + a_1\alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}\ ;\ \bar{y} = 0$$

This is positive under the condition $\alpha_{11}\alpha_{22} > \alpha_{12}\alpha_{21}$.

E_6 : Second mutual species washed out equilibrium

$$E_6 : \bar{x}_1 = \frac{a_1\alpha_{33} - a_3\alpha_{13}}{\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31}}\ ;\ \bar{x}_2 = 0\ ;\ \bar{y} = \frac{a_1\alpha_{31} + a_3\alpha_{11}}{\alpha_{11}\alpha_{33} + \alpha_{13}\alpha_{31}}$$

This is positive under the condition $a_1\alpha_{33} > a_3\alpha_{13}$.

E_7 : First mutual species washed out equilibrium

$$E_7 : \bar{x}_1 = 0\ ;\ \bar{x}_2 = \frac{a_2\alpha_{33} - a_3\alpha_{23}}{\alpha_{22}\alpha_{33} + \alpha_{23}\alpha_{32}}\ ;\ \bar{y} = \frac{a_2\alpha_{32} + a_3\alpha_{22}}{\alpha_{22}\alpha_{33} + \alpha_{23}\alpha_{32}}$$

This case exists only when $a_2\alpha_{33} > a_3\alpha_{23}$.

E_8 : The coexistence state

$$\bar{x}_1 = \frac{-\alpha_{12}\alpha_{23}a_3 - a_3\alpha_{13}\alpha_{22} + a_2\alpha_{12}\alpha_{33} - a_2\alpha_{13}\alpha_{32} + a_1\alpha_{33}\alpha_{22} + a_1\alpha_{23}\alpha_{32}}{-\alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{12}\alpha_{31}\alpha_{23} + \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{31}\alpha_{22}\alpha_{13} + \alpha_{32}\alpha_{21}\alpha_{13} + \alpha_{23}\alpha_{11}\alpha_{32}}\ ;$$

$$\bar{x}_2 = \frac{-\alpha_{13}\alpha_{21}a_3 + a_2\alpha_{13}\alpha_{31} - a_3\alpha_{11}\alpha_{23} - a_1\alpha_{31}\alpha_{23} + a_1\alpha_{33}\alpha_{21} + a_2\alpha_{11}\alpha_{33}}{-\alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{12}\alpha_{31}\alpha_{23} + \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{31}\alpha_{22}\alpha_{13} + \alpha_{32}\alpha_{21}\alpha_{13} + \alpha_{23}\alpha_{11}\alpha_{32}}\ ;$$

$$\bar{y} = \frac{-\alpha_{12}\alpha_{21}a_3 + a_3\alpha_{11}\alpha_{22} + a_2\alpha_{12}\alpha_{31} + a_1\alpha_{31}\alpha_{22} + a_1\alpha_{32}\alpha_{21} + a_2\alpha_{11}\alpha_{32}}{-\alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{12}\alpha_{31}\alpha_{23} + \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{31}\alpha_{22}\alpha_{13} + \alpha_{32}\alpha_{21}\alpha_{13} + \alpha_{23}\alpha_{11}\alpha_{32}}$$

This case eists provided the following four conditions hold

(C_1) : $a_2\alpha_{12}\alpha_{33} + a_1\alpha_{33}\alpha_{22} + a_1\alpha_{23}\alpha_{32} > \alpha_{12}\alpha_{23}a_3 + a_3\alpha_{13}\alpha_{22} + a_2\alpha_{13}\alpha_{32}$

(C_2): $a_2\alpha_{13}\alpha_{31} + a_1\alpha_{33}\alpha_{21} + a_2\alpha_{11}\alpha_{33} > a_3\alpha_{11}\alpha_{23} + a_1\alpha_{31}\alpha_{23} + \alpha_{13}\alpha_{21}a_3$

(C_3): $a_3\alpha_{11}\alpha_{22} + a_2\alpha_{12}\alpha_{31} + a_1\alpha_{31}\alpha_{22} + a_1\alpha_{32}\alpha_{21} + a_2\alpha_{11}\alpha_{32} > \alpha_{12}\alpha_{21}a_3$

(C_4): $\alpha_{12}\alpha_{31}\alpha_{23} + \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{31}\alpha_{22}\alpha_{13} + \alpha_{32}\alpha_{21}\alpha_{13} + \alpha_{23}\alpha_{11}\alpha_{32} > \alpha_{12}\alpha_{21}\alpha_{33}$

We note that the equilibrium points of (2.1) and (2.3) are same with those same kernels.

IV. Local Stability Analysis

Local stability around the corresponding equilibrium points of the system can be observed based on the nature of eigen values of a Jacobian matrix for the linearized perturbed equations of the dynamical system. Let us

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take $u_1 = A_1 e^{\lambda t}, u_2 = A_2 e^{\lambda t}, u_3 = A_3 e^{\lambda t}$, the system (2.3) becomes

$$\begin{aligned}
\frac{du_1}{dt} &= (a_1 - 2\alpha_{11}x_1 + \alpha_{12}x_2 - \alpha_{13}y)u_1 + \alpha_{12}x_1u_2 - \alpha_{13}x_1u_3 \\
\frac{du_2}{dt} &= \alpha_{21}x_2u_1 + (a_2 - 2\alpha_{22}x_2 + \alpha_{21}x_1 - \alpha_{23}y)u_2 - \alpha_{23}x_2k_1^*(\lambda)u_3 \\
\frac{du_3}{dt} &= \alpha_{31}yu_1 + \alpha_{32}yk_2^*(\lambda)u_2 - (a_3 - 2\alpha_{33}y + \alpha_{31}x_1 + \alpha_{32}x_2)u_3
\end{aligned} \tag{4.1}$$

where $k_1^*(\lambda) = \int_0^\infty k_1(z)\exp(-\lambda z)dz$ and $k_2^*(\lambda) = \int_0^\infty k_2(z)\exp(-\lambda z)dz$ are the Laplace Transforms of $k_1(z)$ and $k_2(z)$.

The associated Jacobian matrix for the system (4.1) after linearization is as follows

$$\frac{dU}{dt} = J_E U \tag{4.2}$$

where

$$J_E = \begin{bmatrix} a_1 - 2\alpha_{11}x_1 + \alpha_{12}x_2 - \alpha_{13}y & \alpha_{12}x_1 & -\alpha_{13}x_1 \\ \alpha_{21}x_2 & a_2 - 2\alpha_{22}x_2 + \alpha_{21}x_1 - \alpha_{23}y & -\alpha_{23}x_2k_1^*(\lambda) \\ \alpha_{31}y & \alpha_{32}yk_2^*(\lambda) & a_3 - 2\alpha_{33}y + \alpha_{31}x_1 + \alpha_{32}x_2 \end{bmatrix} \tag{4.3}$$

The system is stable when all eigen values of Jacobian matrix J_E are negative reals or negative real parts when the eiven values are complex at that particular equilibrium point.

In the following prepositions, the local stability nature of the dynamical system at existing eight equilibrium points is discussed.

Proposition: 1 The extinct equilibrium state E_1 is always unstable.

Proof: The eigen values for the following Jacobian matrix at this existent state are all positive (i.e. $\lambda_1 = a_1, \lambda_2 = a_2$ and $\lambda_3 = a_3$) and hence the system is unstable at this E_1.

$$J_{E_1} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

Proposition: 2 Only first mutual survival equilibrium state E_2 is a saddle node.

Proof: In this case the following is the Jacobian matrix and whose eigen values are $\lambda_1 = a_1 - \frac{a_2\alpha_{12}}{\alpha_{22}}, \lambda_2 = -a_2 < 0$ and $\lambda_3 = a_3 + \frac{a_2\alpha_{32}}{\alpha_{22}} > 0$, so this equilibrium point is a saddle point for any possible conditions of λ_1.

$$J_{E_2} = \begin{bmatrix} a_1 - \frac{a_2\alpha_{12}}{\alpha_{22}} & 0 & 0 \\ \frac{a_2\alpha_{21}}{\alpha_{22}} & -a_2 & -\frac{a_2\alpha_{23}K_1^*(\lambda)}{\alpha_{22}} \\ 0 & 0 & a_3 + \frac{a_2\alpha_{32}}{\alpha_{22}} \end{bmatrix}$$

Proposition: 3 The second mutual species survival equilibrium state E_3 is unstable.

Proof: Jacobian matrix in this case is

$$J_{E_3} = \begin{bmatrix} -a_1 & \frac{a_1\alpha_{12}}{\alpha_{11}} & -\frac{a_1\alpha_{13}}{\alpha_{11}} \\ 0 & a_2 + \frac{a_1\alpha_{21}}{\alpha_{11}} & 0 \\ 0 & 0 & a_3 + \frac{a_1\alpha_{31}}{\alpha_{11}} \end{bmatrix}$$

The corresponding eigen values of which are

$\lambda_1 = -a_1 < 0, \lambda_2 = a_2 + \frac{a_1\alpha_{21}}{\alpha_{11}} > 0$ and

$\lambda_3 = a_3 + \frac{a_1\alpha_{31}}{\alpha_{11}} > 0$. One can observe that the present steady state is unstable always.

Proposition: 4 If $a_1 < \frac{a_3\alpha_{13}}{\alpha_{33}}$ and $a_2 < \frac{a_3\alpha_{23}}{\alpha_{33}}$ then the predator survival equilibrium state E_4 is stable otherwise is unstable.

Proof: The relevant Jacobian matrix at this state is

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$$J_{E_4}=\begin{bmatrix} a_1-\dfrac{a_3\alpha_{13}}{\alpha_{33}} & 0 & 0 \\ 0 & a_2-\dfrac{a_3\alpha_{23}}{\alpha_{33}} & 0 \\ \dfrac{a_3\alpha_{31}}{\alpha_{33}} & \dfrac{a_3\alpha_{32}K_2^*(\lambda)}{\alpha_{33}} & -a_3 \end{bmatrix}$$

$\lambda_1=a_1-\dfrac{a_3\alpha_{13}}{\alpha_{33}}$, $\lambda_2=a_2-\dfrac{a_3\alpha_{23}}{\alpha_{33}}$, $\lambda_3=-a_3$ are the eigen values of this matrix J_{E_4}. All three eigen values are negative if $a_1<\dfrac{a_3\alpha_{13}}{\alpha_{33}}$ and $a_2<\dfrac{a_3\alpha_{23}}{\alpha_{33}}$ so the system is stable otherwise it is unstable at E_4.

Proposition: 5 Only the predator extinct equilibrium point E_5 is also always unstable.

Proof: The Jacobian matrix of the equilibrium point E_5 is

$$J_{E_5}=\begin{bmatrix} a_1-2\alpha_{11}\bar{x}_1+\alpha_{12}\bar{x}_2 & \alpha_{12}\bar{x}_1 & -\alpha_{13}\bar{x}_1 \\ \alpha_{21}\bar{x}_2 & a_2-2\alpha_{22}\bar{x}_2+\alpha_{21}\bar{x}_1 & -\alpha_{23}\bar{x}_2K_1^*(\lambda) \\ 0 & 0 & a_3+\alpha_{32}\bar{x}_2+\alpha_{31}\bar{x}_1 \end{bmatrix}$$

The characteristic equation of this matrix is

$$[\lambda-(a_3+\alpha_{32}\bar{x}_2+\alpha_{31}\bar{x}_1)]\begin{bmatrix}\lambda^2-\lambda\left(a_1+a_2+\alpha_{12}\bar{x}_2+\alpha_{21}\bar{x}_1-2\alpha_{11}\bar{x}_1-2\alpha_{22}\bar{x}_2\right) \\ +\begin{pmatrix} a_1a_2+a_1\alpha_{21}\bar{x}_1-2a_2\alpha_{11}\bar{x}_1-2a_1\alpha_{22}\bar{x}_2+a_2\alpha_{12}\bar{x}_2 \\ +4\alpha_{11}\alpha_{22}\bar{x}_1\bar{x}_2-2\alpha_{11}\bar{x}_1^{\,2}-2\alpha_{12}\alpha_{22}\bar{x}_2^{\,2}\end{pmatrix}\end{bmatrix}=0$$

From above one can observe the steady state is unstable.

Proposition: 6 Only the second mutual species washed out equilibrium state E_6 is stable for satisfying certain condition (4.4) otherwise is unstable.

Proof: The corresponding Jacobi matrix at this equilibrium point is

$$J_{E_6}=\begin{bmatrix} a_1-2\alpha_{11}\bar{x}_1-\alpha_{13}\bar{y} & \alpha_{12}\bar{x}_1 & -\alpha_{13}\bar{x}_1 \\ 0 & a_2+\alpha_{21}\bar{x}_1-\alpha_{23}\bar{y} & 0 \\ \alpha_{31}\bar{y} & \alpha_{32}\bar{y}K_2^*(\lambda) & a_3-2\alpha_{33}\bar{y}+\alpha_{31}\bar{x}_1 \end{bmatrix}$$

The secular equation of this matrix is

$$[\lambda-(a_2+\alpha_{21}\bar{x}_1-\alpha_{23}\bar{y})]\begin{bmatrix}\lambda^2-\lambda\left(a_1+a_3+\alpha_{31}\bar{x}_1-\alpha_{13}\bar{y}-2\alpha_{11}\bar{x}_1-2\alpha_{33}\bar{y}\right) \\ +\begin{pmatrix} a_1a_3+a_1\alpha_{31}\bar{x}_1-2a_1\alpha_{33}\bar{y}-2a_3\alpha_{11}\bar{x}+4\alpha_{11}\alpha_{33}\bar{x}_1\bar{y} \\ -2\alpha_{11}\alpha_{31}\bar{x}_1^{\,2}-a_3\alpha_{13}\bar{y}+2\alpha_{13}\alpha_{33}\bar{y}^2\end{pmatrix}\end{bmatrix}=0$$

The steady state is stable if satisfies the condition

$$\begin{aligned}&a_2+\alpha_{21}\bar{x}_1-\alpha_{23}\bar{y}<0,\\ &a_1+a_3+\alpha_{31}\bar{x}_1+\alpha_{21}\bar{x}_1<2\alpha_{11}\bar{x}_1+2\alpha_{33}\bar{y}+\alpha_{13}\bar{y}\\ &a_1a_3+a_1\alpha_{31}\bar{x}_1+4\alpha_{11}\alpha_{33}\bar{x}_1\bar{y}+2\alpha_{13}\alpha_{33}\bar{y}^2> \\ &2a_1\alpha_{33}\bar{y}+2a_3\alpha_{11}\bar{x}+2\alpha_{11}\alpha_{31}\bar{x}_1^{\,2}+a_3\alpha_{13}\bar{y}\end{aligned}\qquad(4.4)$$

Otherwise is unstable.

Proposition: 7 The first mutual species washed out equilibrium state E_7 is stable satisfying the condition (4.5), otherwise is unstable.

Proof: The Jacobian matrix for this state is

$$J_{E_7}=\begin{bmatrix} a_1+\alpha_{12}\bar{x}_2-\alpha_{13}\bar{y} & 0 & 0 \\ \alpha_{21}\bar{x}_2 & a_2-2\alpha_{22}\bar{x}_2-\alpha_{33}\bar{y} & -\alpha_{23}\bar{x}_2K_1^*(\lambda) \\ \alpha_{31}\bar{y} & \alpha_{32}\bar{y}K_2^*(\lambda) & a_3-2\alpha_{33}\bar{y}+\alpha_{32}\bar{x}_2 \end{bmatrix}$$ T

he Characterstic equation is

The characteristic equation of the system is

$$[\lambda-(a_1+\alpha_{12}\bar{x}_2-\alpha_{13}\bar{y})]\begin{bmatrix}\lambda^2-\lambda\left(a_2+a_3+\alpha_{32}\bar{x}_2-2\alpha_{22}\bar{x}_2-3\alpha_{33}\bar{y}\right) \\ +\begin{pmatrix} a_2a_3+a_2\alpha_{32}\bar{x}_2-2a_2\alpha_{33}\bar{y}-2a_3\alpha_{22}\bar{x}_2+4\alpha_{22}\alpha_{33}\bar{x}_2\bar{y}-2\alpha_{22}\alpha_{32}\bar{x}_2^{\,2} \\ -a_3\alpha_{33}\bar{y}+2\alpha_{33}\alpha_{33}\bar{y}^2-\alpha_{33}\alpha_{32}\bar{x}_2\bar{y}+\alpha_{32}\alpha_{23}\bar{x}_2\bar{y}K_1^*(\lambda)K_2^*(\lambda)\end{pmatrix}\end{bmatrix}=0$$

The system is stable if holds the condition (4.5) as follows, otherwise is unstable.

$$\begin{aligned}&a_1+\alpha_{12}\bar{x}_2-\alpha_{13}\bar{y}<0,\\ &a_2+a_3+\alpha_{32}\bar{x}_2<2\alpha_{22}\bar{x}_2+3\alpha_{33}\bar{y},\\ &a_2a_3+a_2\alpha_{32}\bar{x}_2+4\alpha_{22}\alpha_{33}\bar{x}_2\bar{y}+2\alpha_{33}\alpha_{33}\bar{y}^2\\ &+\alpha_{32}\alpha_{23}\bar{x}_2\bar{y}K_1^*(\lambda)K_2^*(\lambda)\;>2a_2\alpha_{33}\bar{y}\\ &+2a_3\alpha_{22}\bar{x}_2+2\alpha_{22}\alpha_{32}\bar{x}_2^{\,2}+a_3\alpha_{33}\bar{y}+\alpha_{33}\alpha_{32}\bar{x}_2\bar{y}\end{aligned}$$

(4.5)

Proposition: 8 If $b_1>0$, $b_3>0$ and $b_1b_2-b_3>0$, then the positive equilibrium state E_8 is locally asymptotically stable, otherwise it is unstable.

Proof: For the coexistence point, the corresponding Jacobian matrix is

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$$J_{E_8}=\begin{bmatrix} a_1-2\alpha_{11}\bar{x}_1+\alpha_{12}\bar{x}_2-\alpha_{13}\bar{y} & \alpha_{12}\bar{x}_1 & -\alpha_{13}\bar{x}_1 \\ \alpha_{21}\bar{x}_2 & a_2-2\alpha_{22}\bar{x}_2+\alpha_{21}\bar{x}_1-\alpha_{23}\bar{y} & -\alpha_{23}\bar{x}_2K_1^*(\lambda) \\ \alpha_{31}\bar{y} & \alpha_{32}\bar{y}K_2^*(\lambda) & a_3-2\alpha_{33}\bar{y}+\alpha_{32}\bar{x}_2+\alpha_{31}\bar{x}_1 \end{bmatrix}$$

The characteristic equation of J_{E_8} is

$$\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0.$$

Here

$$\begin{aligned} b_1 &= -(\lambda_1+\lambda_2+\lambda_3) \\ b_2 &= \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1-\alpha_{12}\alpha_{21}\bar{x}_1\bar{x}_2 \\ &\quad +\alpha_{23}\alpha_{32}\bar{x}_2\bar{y}K_1^*(\lambda)K_2^*(\lambda)+\alpha_{13}\alpha_{31}\bar{x}_1\bar{y} \\ b_3 &= \lambda_3\alpha_{12}\alpha_{21}\bar{x}_1\bar{x}_2+\alpha_{12}\alpha_{23}\alpha_{31}\bar{x}_1\bar{x}_2\bar{y}K_1^*(\lambda) \\ &\quad -\alpha_{13}\alpha_{21}\alpha_{32}\bar{x}_1\bar{x}_2\bar{y}K_2^*(\lambda) \\ &\quad -\lambda_1\alpha_{23}\alpha_{32}\bar{x}_2\bar{y}K_1^*(\lambda)K_2^*(\lambda)+\lambda_2\alpha_{13}\alpha_{31}\bar{x}_1\bar{y}-\lambda_1\lambda_2\lambda_3 \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= a_1-2\alpha_{11}\bar{x}_1+\alpha_{12}\bar{x}_2-\alpha_{13}\bar{y}, \\ \lambda_2 &= a_2-2\alpha_{22}\bar{x}_2+\alpha_{21}\bar{x}_1-\alpha_{23}\bar{y}, \\ \lambda_3 &= a_3-2\alpha_{33}\bar{y}+\alpha_{32}\bar{x}_2+\alpha_{31}\bar{x}_1 \end{aligned}$$

Now

$$\begin{aligned} b_1b_2-b_3 &= -\lambda_1^2(\lambda_2+\lambda_3)-\lambda_2^2(\lambda_1+\lambda_3)-\lambda_3^2(\lambda_1+\lambda_2)-2\lambda_1\lambda_2\lambda_3 \\ &\quad -(\lambda_1+\lambda_3+2\lambda_2)\alpha_{13}\alpha_{31}\bar{x}_1\bar{y} \\ &\quad -(\lambda_2+\lambda_3)\alpha_{23}\alpha_{32}\bar{x}_2\bar{y}K_1^*(\lambda)K_2^*(\lambda) \\ &\quad +(\lambda_1+\lambda_2)\alpha_{12}\alpha_{21}\bar{x}_1\bar{x}_2 \\ &\quad +(-\alpha_{12}\alpha_{23}\alpha_{31}K_1^*(\lambda)+\alpha_{13}\alpha_{21}\alpha_{32}K_2^*(\lambda))\bar{x}_1\bar{x}_2\bar{y} \end{aligned}$$

The positive equilibrium point is locally asymptotically, by Routh-Hurwitz criterion, if $b_1>0,\ b_3>0$ and $b_1b_2-b_3>0$.

V. GLOBAL STABILITY

The global stability of the system is discussed at the positive equilibrium point E_8 in this sub section with suitable Lyapunov's function.

Theorem 1: The positive equilibrium state $E_8\left(\bar{x}_1,\bar{x}_2,\bar{y}\right)$ for the system (4.1) is globally asymptotically stable if it holds the condition.

$$\left(\alpha_{11}+\alpha_{22}+\alpha_{33}+\frac{1}{2}\alpha_{13}\right)>\frac{1}{2}\left(\alpha_{12}+\alpha_{21}+\alpha_{23}+\alpha_{31}+\alpha_{32}\right)$$

Proof: The suitable Lipunov function for the system (4.1) is as follows:

$$\begin{aligned} V(x_1,x_2,y) &= x_1-\bar{x}_1\left[1-\log\left(\frac{x_1}{\bar{x}_1}\right)\right] \\ &\quad +x_2-\bar{x}_2\left[1-\log\left(\frac{x_2}{\bar{x}_2}\right)\right] \\ &\quad +y-\bar{y}\left[1-\log\left(\frac{y}{\bar{y}}\right)\right] \\ &\quad +\frac{1}{2}\alpha_{23}\int_0^\infty k_1(z)\int_{t-z}^{t}(y-\bar{y})^2\,dudz \\ &\quad +\frac{1}{2}\alpha_{32}\int_0^\infty k_2(z)\int_{t-z}^{t}(x_2-\bar{x}_2)^2\,dudz \end{aligned} \tag{5.1}$$

For all positive values of x_1,x_2,y, the above function $V(x_1,x_2,y)$ in equation (5.1) is non zero at the positive equilibrium state. Differentiating (5.1) with respect to the time along the solution of (4.1) is

$$\begin{aligned} \frac{dV}{dt} &= \left(1-\frac{\bar{x}_1}{x_1}\right)x_1'+\left(1-\frac{\bar{x}_2}{x_2}\right)x_2' \\ &\quad +\left(1-\frac{\bar{y}}{y}\right)y'+\frac{1}{2}\alpha_{23}\int_0^\infty k_1(z)\left(y(t)-\bar{y}\right)^2dz \\ &\quad -\frac{1}{2}\alpha_{23}\int_0^\infty k_1(z)\left(y(t-z)-\bar{y}\right)^2dz \\ &\quad +\frac{1}{2}\alpha_{32}\int_0^\infty k_2(z)\left(x_2(t)-\bar{x}_2\right)^2dz \\ &\quad -\frac{1}{2}\alpha_{32}\int_0^\infty k_2(z)\left(x_2(t-z)-\bar{x}_2\right)^2dz \end{aligned}$$

(5.2)

$$\begin{aligned} \frac{dV}{dt} &= (x_1-\bar{x}_1)\left(a_1-\alpha_{11}x_1+\alpha_{12}x_2-\alpha_{13}y\right) \\ &\quad +(x_2-\bar{x}_2)\left(a_2-\alpha_{22}x_2+\alpha_{21}x_1-\alpha_{23}\int_0^\infty K_1(z)y(t-z)dz\right) \\ &\quad +(y-\bar{y})\left(a_3-\alpha_{33}y+\alpha_{31}x_1+\alpha_{32}\int_0^\infty K_2(z)x_2(t-z)dz\right) \\ &\quad +\frac{1}{2}\alpha_{23}\int_0^\infty k_1(z)\left(y(t)-\bar{y}\right)^2dz-\frac{1}{2}\alpha_{23}\int_0^\infty k_1(z)\left(y(t-z)-\bar{y}\right)^2dz \\ &\quad +\frac{1}{2}\alpha_{32}\int_0^\infty k_2(z)\left(x_2(t)-\bar{x}_2\right)^2dz-\frac{1}{2}\alpha_{32}\int_0^\infty k_2(z)\left(x_2(t-z)-\bar{x}_2\right)^2dz \end{aligned}$$

(5.3)

The equation (5.3) becomes

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$$\frac{dV}{dt}=-\alpha_{11}(x_1-\bar{x}_1)^2+(x_1-\bar{x}_1)\left[\alpha_{12}(x_2-\bar{x}_2)-\alpha_{13}(y-\bar{y})\right]$$
$$-\alpha_{22}(x_2-\bar{x}_2)\left[(x_2-\bar{x}_2)+\alpha_{21}(x_1-\bar{x}_1)\right]$$
$$-\alpha_{33}(y-\bar{y})\left[(y-\bar{y})+\alpha_{31}(x_1-\bar{x}_1)\right]+\frac{1}{2}\alpha_{23}(y-\bar{y})^2$$
$$-\frac{1}{2}\alpha_{23}\int_0^\infty k_1(z)\left(y(t-z)-\bar{y}\right)^2dz+\frac{1}{2}\alpha_{32}\left(x_2-\bar{x}_2\right)^2$$
$$-\frac{1}{2}\alpha_{32}\int_0^\infty k_2(z)\left(x_2(t-z)-\bar{x}_2\right)^2dz \tag{5.4}$$

by the following parameters values of a_1, a_2, a_3 :

$$a_1=\alpha_{11}\bar{x}_1-\alpha_{12}\bar{x}_2+\alpha_{13}\bar{y}$$

$$;a_2=\alpha_{22}\bar{x}_2-\alpha_{21}\bar{x}_1+\alpha_{23}\int_0^\infty K_1(z)y(t-z)dz \text{ and}$$

$$a_2=\alpha_{22}\bar{x}_2-\alpha_{21}\bar{x}_1+\alpha_{23}\int_0^\infty K_1(z)y(t-z)dz \text{ and}$$

$$a_3=\alpha_{33}\bar{y}-\alpha_{31}\bar{x}_1-\alpha_{32}\int_0^\infty K_2(z)x_2(t-z)dz$$

Using the inequalities

$$ab\le\frac{a^2+b^2}{2},\int_0^\infty k_1(z)[y(t-z)-\bar{y}]^2dz\le\int_0^\infty k_1(z)dz=1$$

and $\int_0^\infty k_2(z)[x_2(t-z)-\bar{x}_2]^2dz\le\int_0^\infty k_2(z)dz=1$

Equation (5.4) becomes

$$\frac{dV}{dt}\le-\alpha_{11}(x_1-\bar{x}_1)^2+\frac{1}{2}\left[\alpha_{12}(x_2-\bar{x}_2)^2+\alpha_{12}(x_1-\bar{x}_1)^2\right]$$
$$-\frac{1}{2}\left[\alpha_{13}(x_1-\bar{x}_1)^2+\alpha_{13}(y-\bar{y})^2\right]-\alpha_{22}(x_2-\bar{x}_2)^2$$
$$+\frac{1}{2}\left[\alpha_{21}(x_1-\bar{x}_1)^2+\alpha_{21}(x_2-\bar{x}_2)^2\right]-\alpha_{33}(y-\bar{y})^2$$
$$+\frac{1}{2}\alpha_{31}(y-\bar{y})^2+\frac{1}{2}\left[\alpha_{31}(x_1-\bar{x}_1)^2+\alpha_{23}(y-\bar{y})^2\right]$$
$$-\frac{1}{2}\left[\alpha_{23}\left(y(t-z)-\bar{y}\right)^2-\alpha_{32}\left(x_2-\bar{x}_2\right)^2\right]$$
$$-\frac{1}{2}\alpha_{32}\left(x_2(t-z)-\bar{x}_2\right)^2 \tag{5.5}$$

$$\Rightarrow\frac{dV}{dt}\le-\left\|\alpha_{11}-\frac{1}{2}\alpha_{12}+\frac{1}{2}\alpha_{13}-\frac{1}{2}\alpha_{21}-\frac{1}{2}\alpha_{31}\right\|(x_1-\bar{x}_1)^2$$
$$-\left\|\alpha_{22}-\frac{1}{2}\alpha_{12}-\frac{1}{2}\alpha_{21}-\frac{1}{2}\alpha_{32}\right\|(x_2-\bar{x}_2)^2$$
$$-\left\|\alpha_{33}-\frac{1}{2}\alpha_{31}-\frac{1}{2}\alpha_{23}+\frac{1}{2}\alpha_{13}\right\|(y-\bar{y})^2-\frac{1}{2}\left\|\alpha_{23}+\alpha_{32}\right\|$$

$$\Rightarrow\frac{dV}{dt}\le-n(x_1-\bar{x}_1)^2-n(x_2-\bar{x}_2)^2-n(y-\bar{y})^2 \tag{5.6}$$

where

$$n=\min\left(\left(\alpha_{11}+\alpha_{22}+\alpha_{33}+\frac{1}{2}\alpha_{13}\right)-\frac{1}{2}(\alpha_{12}+\alpha_{21}+\alpha_{23}+\alpha_{31}+\alpha_{32})\right).$$

This shows that $\frac{dV}{dt}<0$ for all positive values of x_1, x_2 and y . Hence by well known stability theorem the function $V(x_1,x_2,y)$ is asymptotically globally stable at the positive equilibrium state E_8 . Therefore the dynamical model (2.1) is asymptotically stable at the same positive equilibrium state satisfying the above condition.

VI. NUMERICAL SIMULATION

Using transformation method, the system of equations (4.1) is converted the following fifth order of ordinary differential equations:

$$\frac{dx_1}{dt}=a_1x_1-\alpha_{11}x_1^2+\alpha_{12}x_1x_2-\alpha_{13}x_1y$$
$$\frac{dx_2}{dt}=a_2x_2-\alpha_{22}x_2^2+\alpha_{21}x_1x_2-\alpha_{23}x_2w_1$$
$$\frac{dy}{dt}=a_3y-\alpha_{33}y^2+\alpha_{31}x_1y+\alpha_{32}yw_2$$
$$\frac{dw_1}{dt}=x_3-\alpha_{23}w_1$$
$$\frac{dw_2}{dt}=y+\alpha_{32}w_2 \tag{6.1}$$

Defining the kernels as follows

$$w_1=\int_{-\infty}^t k_1(t-u)y(u)du,$$
$$w_2=\int_{-\infty}^t k_2(t-u)x_2(u)du,\ k_1(u)=e^{-\alpha u},$$
$$k_2(u)=e^{-\beta u},\alpha>0,\beta>0$$

Now the system of equations (6.1) is solved numerically to observe its behavior using MATLAB in both the cases of the absence and the presence of time delay in dynamical system as follows:

A. *Simulation in the absence of a time delay*

System of equations without delay is the following form and is solved numerically for suitable parametric values in the system with same package; we get the following results illustrated by the Figures 1 and 2.

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$$\frac{dx_1}{dt} = a_1x_1 - \alpha_{11}x_1^2 + \alpha_{12}x_1x_2 - \alpha_{13}x_1y$$
$$\frac{dx_2}{dt} = a_2x_2 - \alpha_{22}x_2^2 + \alpha_{21}x_1x_2 - \alpha_{23}x_2y \qquad (6.2)$$
$$\frac{dy}{dt} = a_3y - \alpha_{33}y^2 + \alpha_{32}x_2y + \alpha_{31}x_1y$$

Let $a_1 = 12.52,\ \alpha_{11} = 12.48,\ \alpha_{12} = 22.96,\ \alpha_{13} = 16.96,\ a_2 = 18.56,\ \alpha_{21} = 12.48,\ \alpha_{22} = 9.52,\ \alpha_{23} = 21.44,\ a_3 = 2.88,\ \alpha_{31} = 2.64,\ \alpha_{32} = 2.98,\ \alpha_{33} = 14.8,\ x_1(0) = 10,\ x_2(0) = 10,\ y(0) = 5$. The solution curves for system (6.2) are shown below for the above parameters

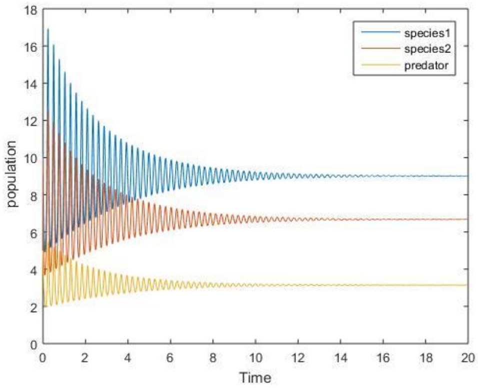

Fig.1

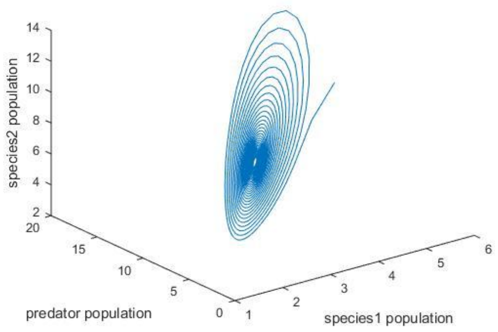

Fig2

The solutions of system (6.2) shown in the above figures are asymptotically stable in the absence of delay arguments. Time series analysis shows periodic monotonic decay oscillations' converging to the equilibrium point $E(9.009, 6.677, 3.147)$ is shown in phase portrait. Hence the system is asymptotically stable.

B. *Simulation in the presence of a time delay*

Delay effect for the system (6.1) for different values of α and β are illustrated with same above parameter values.

S.No.	Figures	Description
1	Odd number figured from 3 to 13.	Shows the variation of x_1, x_2 and y with time to the system (4.1).
2	Even number figured from 4 to 14.	The phase portrait of x_1, x_2 and y for the system (4.1).

The delay kernels for different values α and β for the system of equations are plotted as follows:

S.No.	Parameter values of α and β & Converging Equilibrium point	Nature of the system
1	$\alpha = 0.01; \beta = 0.01$ $E(0.595, 0, 0.301)$	The second mutual species extinct first and both the predator and the first mutual population are increasing from initial population to certain extent and stabilize at fixed population exhibits linear growth dynamics.
2	$\alpha = 0.6; \beta = 0.6$ $E(1.75, 1.35, 0.854)$	The mutual species populations are dominating the predator species due to the less attack rate of the predator over the mutual species.
3	$\alpha = 0.9; \beta = 0.9$ $E(5.567, 4.025, 2.088)$	All three species exists and there will be no considerable growth rates after the time 2 units in the interval [0, 20].

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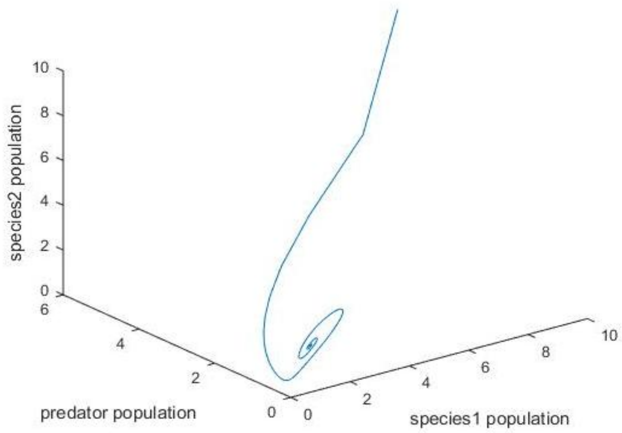
4	$\alpha = 0.95; \beta = 0.95$ E(7, 5.127, 2.529)	With this time delay terms all three species gradually increasing with low growth rate. In this case the mutual species dominate the predator initially and then converging to fixed equilibrium point, and make stable.
5	$\alpha = 1.01; \beta = 1.01$ E(9.754, 7.24, 3.373)	The periodic solutions for all three species in the interval [0, 20] exhibit the instability of the system.
6	$\alpha = 1.02; \beta = 1.02$ E(4.301, 3.201, 2.172)	The oscillatory population growths with high intensity in mutual species comparing the predator on the time interval [0, 20] with cyclic nature in the dynamical system indicate the instability at this equilibrium point.



$\alpha = 0.01; \beta = 0.01$, E(0.595, 0, 0.301)

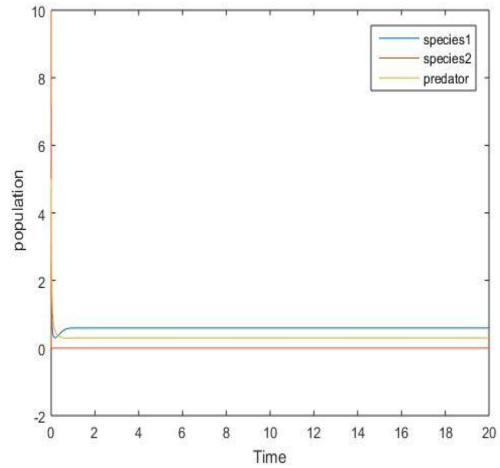


FIG.3

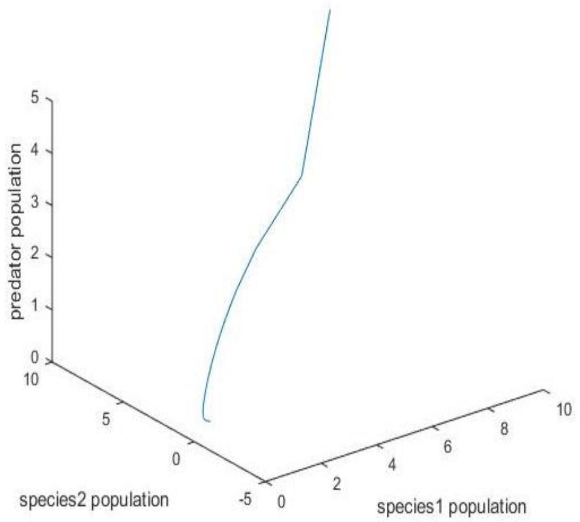


FIG.4

$\alpha = 0.6,\ \beta = 0.6,\ E(1.75, 1.35, 0.854)$

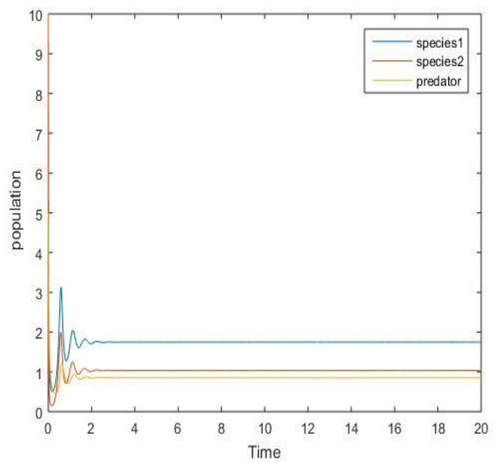


Fig.5

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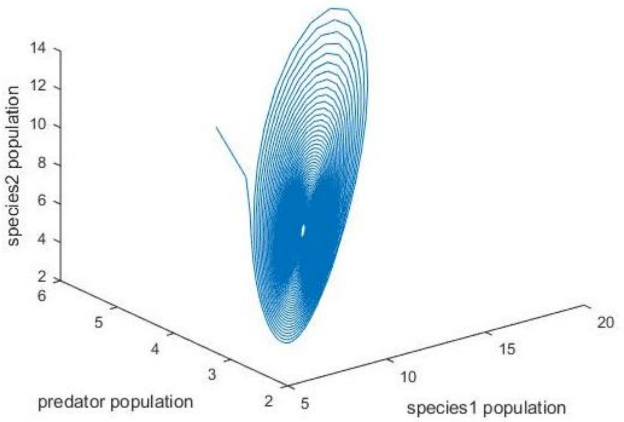

Fig.6

$\alpha = 0.9, \beta = 0.9,$ E(5.567, 4.025, 2.088)

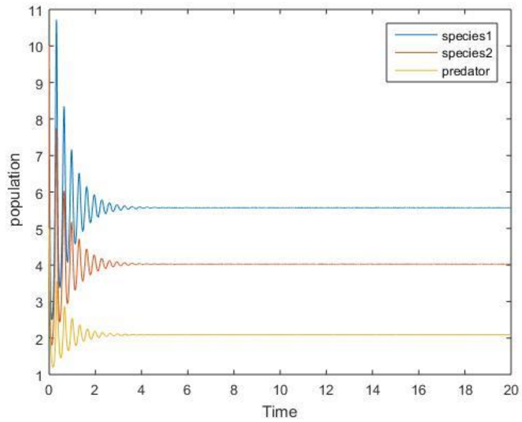

Fig.7

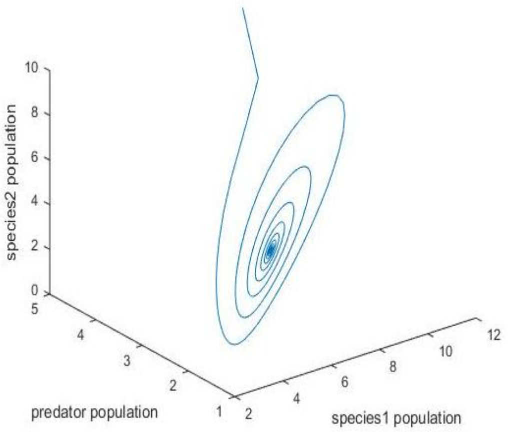

Fig.8

$\alpha = 0.95, \beta = 0.95,$ E(7, 5.127, 2.529)

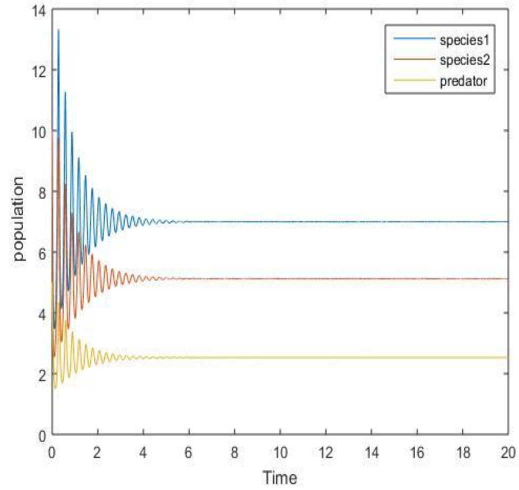

Fig.9

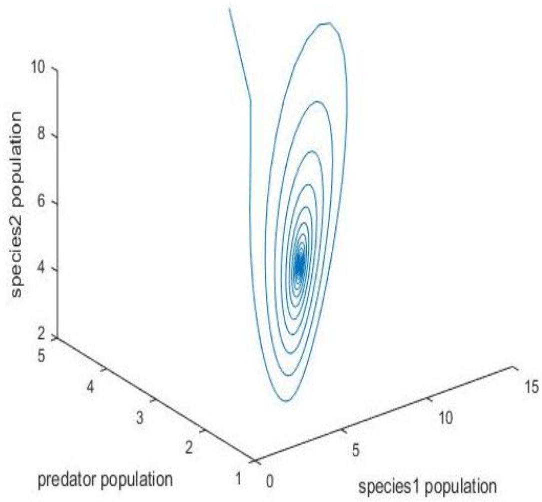

Fig.10

$\alpha = 1.01, \beta = 1.01,$ E(9.754, 7.24, 3.373)

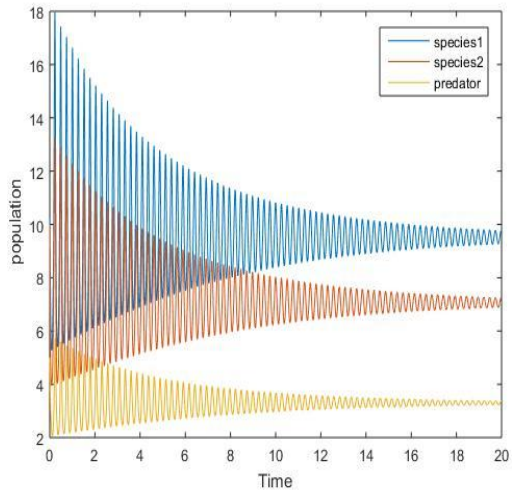

Fig.11

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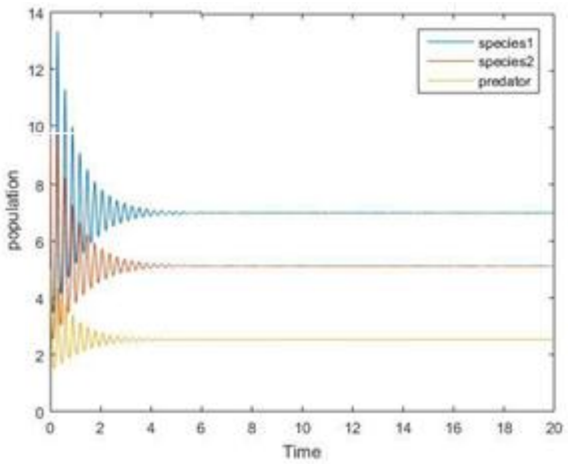

Fig.12

$\alpha = 1.02, \beta = 1.02,$ E(4.301, 3.201, 2.172)

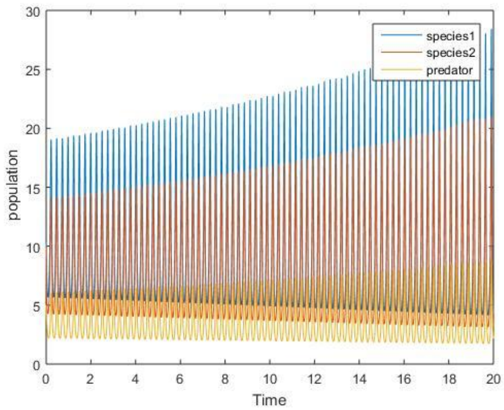

Fig.13

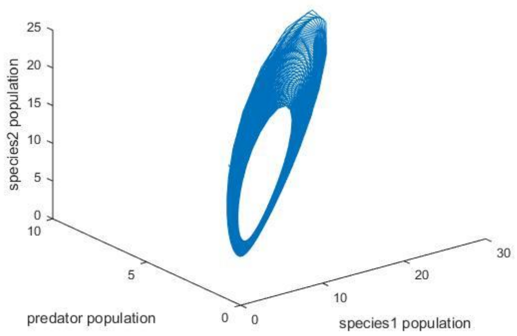

Fig.14

VII. CONCLUSIONS

In the present paper, we explore an ecological model of a predator and two mutual species for investigation. The time delay is imposed on the predator **-** second mutual species and all three species are within in the limited resources. We identified the positive equilibrium points for the model satisfying certain conditions, and the local stability is discussed around those of the dynamical system. Further, we also studied the system global stability at coexistence state using suitable Laponov function. The graphical illustrations are presented to compare the results with the system not including delay term by suitable example. Also, the numerical simulations are presented to identify the weighted functions with different kernels values of α and β in which the system exhibits rich dynamics. For different kernels of α , β in the system (4.1) we observed the following behavior in all three species:

i) There is considerable growth rate for all three species as the values of α and β various from 0.01 to 0.2, and moreover the second mutual species is existent first and remains converge to the equilibrium point later. So one can observe that the dynamical system is unstable for these delay terms.

ii) As the range of α and β varies from [0.6, 0.95], all three species exists and the growth rates of them are gradually increasing and then later reaching their asymptotic values. Further the kernels' values in the above interval don't more effect on mutual species and so the system is stable.

iii) When the kernel values are 1.01, all three species having oscillatory growth in the random environment with high intensity of first mutual species comparing to other two species.

iv) As soon as the kernels cross the value 1.01, the dynamical system lost its stability. So hopf bifurcation can observe at that point to the system.

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