Some Bounds for the Sum of Domination Number and Chromatic Number of Network Graphs in Terms of Number of Edges

K. Thiagarajan, P. Mansoor

Abstract: ***In the advanced world of technology, graph theory has become an essential tool in studying problems in social networks, computer based networks and other complex network systems. To identify and solve those problems, the best method is to have a graph theoretical framework on it. In graph theory, several inequalities involving number of edges, domination number and chromatic number of a graph G have been established. In this paper we shall discuss some results on the domination number of connected expanded network graphs and introduce some inequalities connecting number of edges, domination number and chromatic number for some special types of graphs which will be useful in studying various network related issues.***

Keywords: ***Chromatic Number, C_n+ Graph, C_n+C_n Graph, Domination Number, Expanded Network Graph, Semi Node, Wheel Graph.***

I. INTRODUCTION AND PRELIMINARIES

A graph G consists of a non-empty set Vof points, called nodes, and a set Eof two point subsets of V, called edges connecting pairs of nodes. A graph G is said to be connected if there exists a path between every pairs of nodes of G. A graph G is said to be complete if each node in G is adjacent to the other nodes of G. A complete graph on n nodes is denoted by K_n which has $\frac{n(n-1)}{2}$ edges. A graph G is said to be k- regular if all node of Ghas of degree k.

A cycle is a graph having equal number of nodes and edges whose nodes can be arranged around a circle so that two nodes are adjacent if and only if they appear consecutively along the circle.

A dominating set for a graph G = (V, E) is a subset S of V such that every node not in S is adjacent to at least one node of S. The domination number $\gamma(G)$ of a graph G is the number of nodes in a minimal dominating set for G. A nodev in a graph G dominates itself and each of its adjacent nodes.

A proper colouring of a graph G is a labelling of nodes of G such that no two adjacent nodes receive the same colour. A colouring of a graph G using at most k colours is called a k- colouring. The chromatic number $\chi(G)$ of a graph G is the minimum number of colours required for colouring G properly.

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The C_n+ graph is a graph obtained from C_n, a cycle on nnodes$v_1, v_2, v_3, \ldots, v_n$, by placing new nodes$u_i$ for allv_i and joining by the edges (v_i, u_i) for i= 1, 2, 3, …, n. Obviously, the number of vertices and the number of edges for C_n+ graphs are the same and both are equal to 2n.

The C_n+C_n graph is a graph obtained from C_n by connecting a new node for each node of C_n and joining the new nodes consecutively along a circle.

The wheel graph $C_{w,n}, (n \geq 3)$is a graph obtained from a cycle $C_n, (n \geq 3)$ by connecting a new node to all of its n-vertices. The wheel graph $C_{w,n}, (n \geq 3)$ has $n+1$nodes and 2n edges.

A semi graph G is a pair (V, E) where V is a non-empty set whose elements are called nodes of G and E is the set of edges whose elements are the n-tuples of distinct nodes for variousn ≥ 2, satisfying the following two conditions:

(i) Any two edges have at most one node in common.

(ii) Two edges $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_k)$ are considered to be equal if and only if k = n and either $u_i = v_i$ or $u_i = v_{n-i}+1$ for i = 1, 2, 3, …, n.

Thus, the edges $(u_1, u_2, \ldots, u_n)$ are the same as the edges $(u_n, u_{n-1}, \ldots, u_1)$.

When splicing a graph G, the new nodes obtained are called semi nodes and the new edges formed by the decomposition of edges are called semi edges.

II. THE DOMINATION NUMBER OF A CONNECTED EXPANDED NETWORK GRAPH

Theorem 1: The domination number of a connected semi node introduced expanded network graph of C_n+ is the same as that of a connected C_n+ graph. In other words, there is no change in the domination number after introducing semi nodes in the C_n+ graphs.

Proof: Let *S* be a minimal dominating set of a connectedC_n+ graph with$|S| = \gamma$. Then each node of C_n +is a neighbour of some nodes in *S*. If we introduce a semi node between any two pairs of adjacent nodes of C_n+, then obviously it will become the neighbour of some node in *S*, because *S* is a dominating set. Hence there is no change in the elements of the minimal dominating set *S* after introducing a semi node. That is the domination number after introducing semi nodes in the C_n+ graphs is same as the domination number of the C_n+ graphs.

Example 1: ConsiderC_3+ graph

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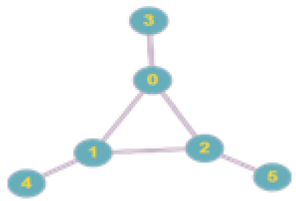
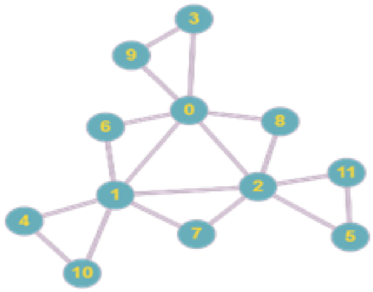
Here the domination number$\gamma = 3$.

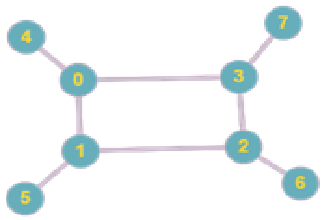


The semi node introduced $C_3 +$ graph is given by

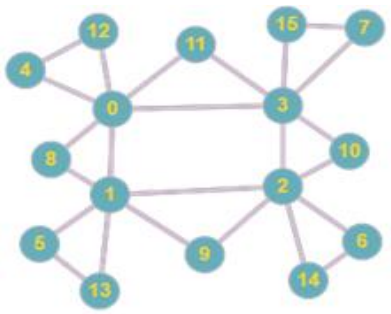


Here the domination number γ=3.

Example 2: Consider $C_4 +$ graph



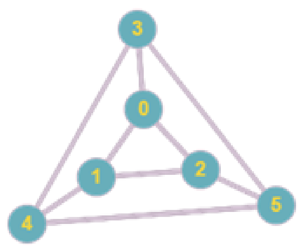
The domination number $\gamma = 4$.
The semi node introduced $C_4 +$ graph is given by



Here the domination number $\gamma = 4$.

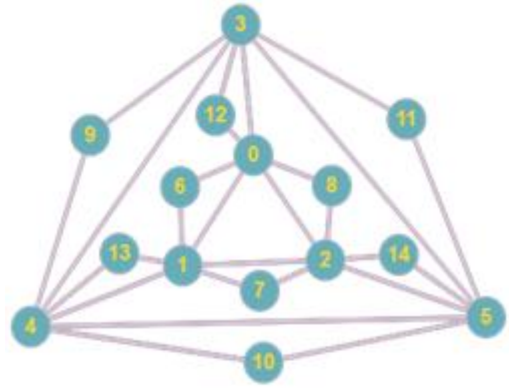
Remark 1: The above theorem 1need not be true for regular connected graph $C_n + C_n$ having $2n$ nodes. That is the domination number of a regular connected graph $C_n + C_n$need not be same as that of the seminode introduced expanded network graph $C_n + C_n$.

Example3: Consider the 3- regular graph $C_3 + C_3$having 6 vertices given by



The domination number $\gamma = 2$

The semi node introduced expanded network graph $C_3 + C_3$ is as follows:



The domination number $\gamma = 3$

III. INEQUALITIES INVOLVING NUMBER OF EDGES, DOMINATION NUMBER AND CHROMATIC NUMBER

Theorem 2: Let *G* be a complete graph having $n \geq 4$ nodes. Then $\gamma + \chi < m$, where γ denotes the domination number, χ denotes the chromatic number and m denotes the number of edges.

Proof: The proof is by induction on n, the number of nodes. For $n = 4$, the result is true, since the graph is K_4 for which $\gamma = 1$, $\chi = 4$ and $m = 6$ so that $\gamma + \chi = 1 + 4 = 5 < 6 = m$. For $n = 5$, the result is also true, since the graph is K_5 for which $\gamma = 1$, $\chi = 5$ and $m = 10$ so that $\gamma + \chi = 1 + 5 = 6 < 10 = m$.

Assume that the result is true for every complete graph having $n = p$, where $p \geq 4$ nodes. Then is $\gamma + \chi = m$ implies$1 + p < \frac{p(p-1)}{2}$. ……………..(1)

Consider a complete graph having $n = p + 1$ number of nodes. Then $\gamma = 1$, $\chi = p + 1$ and $= \frac{(p+1)p}{2}$. In order to complete the proof, it suffices only to prove that$\gamma + \chi < \frac{(p+1)p}{2}$.

Now, $\gamma + \chi = 1 + (p + 1) = (1 + p) + 1 < \frac{p(p-1)}{2} + 1$, from (1)

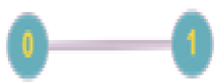
$$= \frac{p^2 - p + 2}{2} = \frac{p^2 - p + p - p + 2}{2}$$

$$= \frac{p^2 + p - 2p + 2}{2} = \frac{(p + 1)p - 2(p - 1)}{2}$$

$= \frac{(p+1)p}{2} - (p - 1) < \frac{(p+1)p}{2}$, as $p - 1 \geq 3$ for $p \geq 4$, by assumption.

Hence by mathematical induction, the proof is complete.

Remark 2: For K_2 and K_3, $\gamma + \chi > m$



Here $\gamma = 1$, $\chi = 2$ and $m = 1$, therefore $\gamma + \chi = 1 + 2 = 3 > 1 = m$.

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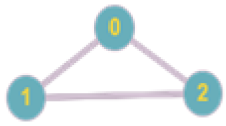
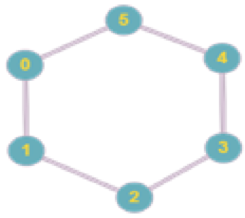


Here$\gamma = 1$, $\chi = 3$ and$m = 3$, therefore $\gamma + \chi = 1 + 3 = 4 > 3 = m$.

Theorem3: Let G be a connected 2-regular graph having $n \geq 4$ nodes. Then$\gamma + \chi \leq m$, where γ denotes the domination number, χ denotes the chromatic number and m denotes the number of edges.

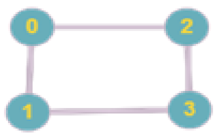
Example 4: Consider C_6, the 2-regular graph having 6 nodes.

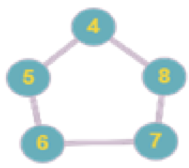


Here $\gamma = 2$, $\chi = 2$and $m = 6$. Therefore $\gamma + \chi = 4 < 6 = m$.

Remark 3: The equality condition in the above theorem, that is,$\gamma + \chi = m$ holds only when $n = 4$ and 5.

Consider $C_4$and C_5;



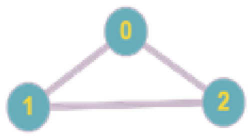


Here, for C_4, $\gamma = 2$,$\chi = 2$ and $m = 4$, therefore $\gamma + \chi = 2 + 2 = 4 = m$.

For C_5,$\gamma = 2$, $\chi = 3$ and $m = 5$, therefore $\gamma + \chi = 2 + 3 = 5 = m$.

Remark 4: For $n = 3$, $\gamma + \chi > m$.

Consider the graphC_3,



Here $\gamma = 1$, $\chi = 3$and$m = 3$, therefore $\gamma + \chi = 1 + 3 = 4 > 3 = m$.

Remark 5: For 1-regular graph consisting two nodes, $\gamma + \chi > m$.

$\gamma = 1$, $\chi = 2$ and $m = 1$, therefore$\gamma + \chi = 1 + 2 = 3 > 1 = m$.

Theorem 4: For every wheel graph$C_{w,n}$, where $n \geq 3$, $\gamma(C_{w,n}) + \chi(C_{w,n}) < m(C_{w,n})$.

Proof: Let $C_{w,n}$, $(n \geq 3)$ be a wheel graph obtained from C_n, $(n \geq 3)$ by linking a new vertex v with all of the n nodes, that is v is adjacent to all of the n nodes. Then the number of nodes is $n + 1$,the number of edges is $m(C_{w,n}) = 2n \geq 6$ andthe domination number is $\gamma(C_{w,n}) = 1$.

For the graph C_n, $(n \geq 3)$,the chromatic number is given by

$$\chi(C_n) = \begin{cases} 2 \; if \; n \; is \; even \\ 3 \; if \; n \; is \; odd \end{cases}.$$

Since v is adjacent to all of the n nodes, in order to colour $C_{w,n}$, $(n \geq 3)$ properly, a new colour is required and hence the chromatic number of $C_{w,n}$, $(n \geq 3)$is

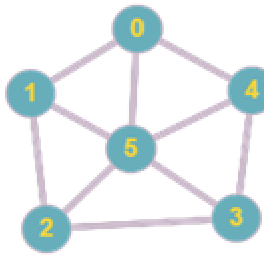
$$\chi(C_{w,n}) = \begin{cases} 3 \; if \; n \; is \; even \\ 4 \; if \; n \; is \; odd \end{cases}.$$

Now $\gamma(C_{w,n}) + \chi(C_{w,n}) = 1 + \begin{cases} 3 \; if \; n \; is \; even \\ 4 \; if \; n \; is \; odd \end{cases}$

$$= \begin{cases} 4 \; if \; n \; is \; even \\ 5 \; if \; n \; is \; odd \end{cases}.$$

In both cases, $\gamma(C_{w,n}) + \chi(C_{w,n}) < m(C_{w,n})$.

Example 5: Consider the wheel graph $C_{w,5}$



Here $\gamma = 1$, $\chi = 4$, $m = 10$. Therefore $\gamma + \chi = 1 + 4 = 5 < 10 = m$.

IV. CONCLUSION

In this paper we proved that the domination number of a semi node introduced expanded network graph of C_n +is same as that of theC_n + graph. We discussed some inequalities involving number of edges, domination number and chromatic number for complete graphs, 2-regular connected graphs and wheel graphs that are useful in network related problems.

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