Coefficient Inequalities of a Subclass of Analytic Functions Defined Using $q$-Differential Operator

K. R. Karthikeyan, M. Musthafa Ibrahim, K. Srinivasan

Abstract: By making use of a $q$-analogue of the Sălăgean differential operator, we define a new subclass of analytic functions. We obtain Fekete-Szegö inequality and coefficient inequality of certain class satisfying bi-convex criteria. Fekete-Szegö inequality of several well-known classes are obtained as special cases from our results. Applications of the result are also obtained on the class defined by convolution.

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I. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $A$ denote the class of functions of the form
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)

analytic in the open unit disk $\mathcal{U}$. Also we let $S$ denote the class of all function in $A$ which are univalent in $\mathcal{U}$. The well known example in this class is the Koebe function $k(z)$, defined by
$$k(z) = \frac{z}{(z-1)^2} = z + \sum_{n=2}^{\infty} \frac{n}{n-1} z^n.$$

The Bieberbach conjecture about the coefficient of the univalent functions in the unit disk was formulated by Bieberbach [2] in the year 1916. The conjecture states that for every function $f \in S$ given by (1.1), we have $|a_n| \leq n$ for every $n$. Strictly inequality holds for all $n$ unless $f$ is the Koebe function or one of its rotation. For many years, this conjecture remained as a challenge to mathematicians. After the proof of $|a_3| \leq 3$ by Lowner in 1923, Fekete-Szegö [7] surprised the mathematicians with the complicated inequality
$$|a_3 - \mu a_2^2| \leq 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right),$$

which holds good for all values $0 \leq \mu \leq 1$. Note that this inequality region was thoroughly investigated by Schaefer and Spencer [16].

For a class functions in $A$ and a real (or more generally complex) number $\mu$, the Fekete-Szegö problem is all about finding the best possible constant $C(\mu)$ so that $|a_3 - \mu a_2^2| \leq C(\mu)$ for every function in $A$. Many papers have been devoted to this problem see [3, 4, 5, 8, 11, 10]. In this paper, we obtain the estimates of $a_2\cdot a_3$ and also the Fekete-Szegö inequality for a subclass of functions of complex order defined using subordination.

It is well known that every function $f \in S$ has a function $f^{-1}$, defined by
$$f^{-1}[f(z)] = z; (z \in \mathcal{U})$$
and
$$f[f^{-1}(w)] = w; \quad (|w| < r_0(f) ; r_0 f \geq \frac{1}{4}).$$

In fact, the inverse function $f^{-1}$ is given by
$$f^{-1}[w] = w - a_2 w^2 + (2 a_2 w^2 - a_3) w^3 - (5 a_2^3 - 5 a_2 a_3 + a_4) w^4 + \ldots.$$  \hspace{1cm} (1.2)

A function $f \in A$ is said to be biunivalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$. Let $f$ and $g$ be analytic in the open unit disk $\mathcal{U}$. The function $f$ is subordinate to $g$ written as $f < g$ in $\mathcal{U}$, if there exist a function $w$ analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1; (z \in \mathcal{U})$ such that $f(z) = g(w(z))$. ($z \in \mathcal{U}$).

Let $S^*(\alpha)$ and $C(\alpha)$ denote the well known subclasses of the univalent function class $S$ which are respectively defined as follows:
$$S^*(\alpha) = \left\{ f \in A : \Re \left( \frac{zf''(z)}{f'(z)} \right) > \alpha ; \quad 0 \leq \alpha < 1 \right\}$$
and
$$C(\alpha) = \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha ; \quad 0 \leq \alpha < 1 \right\}$$

Using Alexander transform, it follows that $f(z) \in C(\alpha)$ if and only if $zf''(z) \in S^*(\alpha)$.

$q$-calculus has been studied by various authors due to the fact that applications of basic Gaussian hypergeometric function to the diverse subjects of combinatorics, quantum theory, number theory, statistical mechanics, are constantly being uncovered.

The $q$-difference operator denoted as $D_q f(z)$ is defined by

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K. R. Karthikeyan, Department of Mathematics and Statistics, Caledonian College of Engineering, Muscat, Sultanate of Oman. kr_karthikeyan1979@yahoo.com
M. Musthafa Ibrahim, College of Engineering, University of Buraimi, Al Buraimi, Sultanate of Oman
K. Srinivasan, Department of Mathematics, Presidency College (Autonomous), Chennai-600005, Tamilnadu, India.
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D_qf(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad (f \in \mathcal{A}, z \in \mathcal{U} - \{0\})

and \( D_qf(0) = f'(0) \), where \( q \in (0,1) \). It can be easily seen that \( D_qf(z) \to f'(z) \) as \( q \to 1^- \). If \( f(z) \) is of the form (1.1), a simple computation yields

\[
D_qf(z) = 1 + \sum_{n=1}^{\infty} \frac{1-q^n}{1-q} a_n z^n, \quad (z \in \mathcal{U}).
\]

The inverse function of (1.3) is given by

\[
D_qg(w) = 1 - (1 + q) a_2 w - (1 + q + q^2) a_3 w^2 - \ldots
\]

The q-analogue of Sălăgean differential operator (see [15])

\[
R_q^n f(z) : \mathcal{A} \to \mathcal{A} \quad \text{for } m \in \mathbb{N},
\]

is formed as follows.

\[
R_q^n f(z) = f(z)
\]

\[
R_q^1 f(z) = z(D_q f(z))
\]

\[
R_q^{1-n} f(z) = R_q^n \left( R_q^{1-n} f(z) \right).
\]

Motivated by [6], we define the following.

**Definition 1.1** Let \( \phi(z) \) be analytic in \( \mathcal{U} \) with \( \phi(0) = 1 \). A function \( f(z) \in \mathcal{A} \) is said to be in the class \( \mathcal{C}_q^m(\lambda; \phi) \) satisfies the differential inequality if and only if

\[
\left( (1-\lambda) z (R_q^n f(z))^{\lambda + \lambda} (R_q^{n+1} f(z))^{\lambda} \right)^{1/\lambda} < \phi(z)
\]

(1.4)

\( (z \in \mathcal{U}; m \in \mathbb{N}; k \geq 0) \)

where \( (q) \) denotes the q-derivative of \( f \) as defined in (1.3).

**Remark 1.1** We note that by specializing \( m, \lambda, q, k \) in the function class \( \mathcal{C}_q^m(\lambda; \phi) \), we obtain several well-known and new subclasses of analytic functions. Here we list a few of them:

1. If we let \( q = 1 \) then the \( \mathcal{C}_q^m(\lambda; \phi) \) reduces to the class \( \mathcal{C}_n^m(\lambda; \phi) \), studied by Darwish, Lashin and Alnayyef [6].
2. If we let \( q = 1 \) and \( k = 1 \) then the \( \mathcal{C}_q^m(\lambda; \phi) \) reduces to the class \( \mathcal{M}_{n,m}(\phi) \) studied by Orhan and Gunes [14].

**Lemma 1.1** [12] Let the function \( \phi(z) \) given by

\[
\phi(z) = \sum_{n=1}^{\infty} B_n z^n
\]

be convex on \( \mathcal{U} \). If \( h(z) \) is \( \phi(z) \)-like in \( \mathcal{U} \), then \( |h_n| \leq |B_n|, n \in \mathbb{N} = \{1,2,3,\ldots\} \).

**Lemma 1.2** [10] If \( p(z) = 1 + c_2 z + c_2 z^2 + \cdots \) is a function with positive real part on \( \mathcal{U} \) and \( A \) is a complex number, then

\[
|c_2 - \mu c_1^2| \leq 2 \max\{1; 2\mu - 1\}.
\]

The result is sharp for the functions given by

\[
p(z) = \frac{1+c^2}{1-z^2} \quad \text{and} \quad \frac{1+z^2}{1-2z}
\]

**II. FEKETE-SZEGÖ INEQUALITY**

We begin with the following result.

**Theorem 2.1**

Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 \neq 0) \).

If \( f(z) \in \mathcal{C}_q^n(\lambda; \phi) \), then

\[
|a_3 - \mu a_2^2| \leq \frac{|B_3|}{1-\mu} \frac{1}{2q(q+1)(1+q+q^2)^{\lambda}(1-\lambda+\lambda(1+q+q^2)^{\lambda})}
\]

\[
\max \left\{ 1, \frac{|B_3|}{1-\mu} \frac{1}{2q(q+1)(1+q+q^2)^{\lambda}(1-\lambda+\lambda(1+q+q^2)^{\lambda})} \right\}
\]

\[
B_3(1+q+q^2)^{\lambda}(1-\lambda+\lambda(1+q+q^2)^{\lambda}) - \mu
\]

\[
\frac{2q(1+q+q^2)^{\lambda}(1-\lambda+\lambda(1+q+q^2)^{\lambda})}{(1+q+q^2)^{\lambda}(1-\lambda+\lambda(1+q+q^2)^{\lambda})}
\]

**Proof.** Let \( f(z) \in \mathcal{C}_q^n(\gamma; \phi) \), then there exists a Schwarz function \( w(z) \) in \( \mathcal{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( \mathcal{U} \) such that

\[
\left( \frac{(1-\lambda) z (R_q^n f(z))^{\lambda + \lambda} (R_q^{n+1} f(z))^{\lambda}}{(1-\lambda) R_q^n f(z) + \lambda R_q^{n+1} f(z)} \right)^{1/\lambda} = |w(z)|
\]

(2.1)

Define the function \( f(z) \) by

\[
p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad z \in \mathcal{U}.
\]

(2.2)

Since \( w(z) \) is Schwarz function, it can be easily seen that

\[
R e\left( \frac{1+w(z)}{1-w(z)} \right) > 0.
\]

Then \( R e(p(z)) > 0 \) and \( p(0) = 1 \). Therefore

\[
\phi(w(z)) = \frac{p(z)}{p(z)+1} = \phi \left( \frac{1}{c_2 - c_1^2} \right) \frac{z^2 + \left( c_1^2 - c_2^2 \right) z^3 + \cdots }{1 + \frac{1}{2} B_1 c_2 + \frac{1}{2} B_1 \left( c_2 - c_1^2 \right) + \frac{1}{4} B_2 c_2^2 z^2 + \cdots }.
\]

(2.3)

Now by substituting (2.1) in (2.3),

\[
\left( \frac{(1-\lambda) z (R_q^n f(z))^{\lambda + \lambda} (R_q^{n+1} f(z))^{\lambda}}{(1-\lambda) R_q^n f(z) + \lambda R_q^{n+1} f(z)} \right)^{1/\lambda} = 1 + \frac{1}{2} B_1 c_2 + \frac{1}{2} B_1 \left( c_2 - c_1^2 \right) + \frac{1}{4} B_2 c_2^2 z^2 + \cdots .
\]

From this equation, we get

\[
q(1+q)^{\lambda}(1-\lambda + \lambda(1+q)^{\lambda}) a_2 = \frac{1}{2} B_1 c_1
\]

and

\[
q(1+q)^{\lambda}(1+q + q^2)^{\lambda}(1-\lambda + \lambda(1+q + q^2)^{\lambda}) a_3 - q(1+q)^{\lambda}(1+q + q^2)^{\lambda}(1-\lambda + \lambda(1+q + q^2)^{\lambda}) a_2^2 = \frac{B_3}{2} c_2 - \frac{B_2}{4} c_2^2 + \frac{B_2}{4} c_2^2.
\]

(2.4)

Or, equivalently

\[
a_2 = \frac{B_3}{2} c_2 - \frac{B_2}{4} c_2^2 + \frac{B_2}{4} c_2^2
\]

\[
a_3 = \frac{B_3}{2} c_2 - \frac{B_2}{4} c_2^2 + \frac{B_2}{4} c_2^2
\]

\[
on q(1+q)^{\lambda}(1+q + q^2)^{\lambda}(1-\lambda + \lambda(1+q + q^2)^{\lambda}) a_2^2
\]

On computation, we have

\[
|a_3 - \mu a_2^2| \leq \frac{1}{4} A_2 (1+q+q^2)^{\lambda}(1-\lambda + \lambda(1+q+q^2)^{\lambda})
\]

(2.5)

where
On rearranging the terms and taking modulus on both sides, the result follows on the application of the Lemma 1.2. The result is sharp for the functions and . This completes the proof of the theorem.

If we let and in Theorem 2.1, we have

**Corollar y 2.2** (see [13]) Let satisfy the inequality

\[
(2.7)
\]

then

\[
(3.1)
\]

From the equivalent subordination condition proved by Kuroki and Owa in [9], the inequality (2.7) can be rewritten in the form

\[
(3.2)
\]

(3.3)\]

where and have the forms respectively. It follows from (3.2) and (3.3), we deduce

\[
(3.4)
\]

(3.5)

(3.6)

From (3.4) and (3.6) we obtain

\[
(3.7)
\]

III. COEFFICIENT INEQUALITIES OF BIUNIVALENT FUNCTIONS

In this section, we obtain coefficient estimates of a biconvex functions.

**Theorem 3.1** Let \( f(z) \) be of the form (1.1) and let \( g \) denote the inverse function of \( f \) given by (1.4). Suppose that \( f \) and \( g \) satisfies the following analytic condition

\[
(3.1)
\]

where \( B_n = \frac{\beta - \alpha}{n\pi} \) and \( \left(1 - e^{2\pi n\alpha}(1 - \alpha)/(\beta - \alpha)\right) \). From the equivalent subordination condition proved by Kuroki and Owa in [9], the inequality (2.7) can be rewritten in the form

\[
(3.2)
\]

(3.3)\]

where \( p(z) \) and \( q(w) \) have the forms

\[
(3.3)
\]

\[
(3.4)
\]

(3.5)

and

\[
(3.6)
\]

respectively. It follows from (3.2) and (3.3), we deduce

\[
(3.4)
\]

\[
(3.5)
\]

\[
(3.6)
\]

and

\[
(3.7)
\]

From (3.4) and (3.6) we obtain
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By adding (3.5) and (3.7), we get

$$2(1 + q)^2m[q(1 + q^2)(1 - \lambda + \lambda(1 + q + q^2))]a_3^2.$$ (3.8)

Since $p, q \in \mathbb{H}(U)$, applying Lemma 1.1, we have

$$|p_m| = \left|\frac{m^m}{m!}\right| = |E_m|, m \in \mathbb{N}$$ (3.9)

and

$$|q_m| = \left|\frac{m^m}{m!}\right| = |E_m|, m \in \mathbb{N}.$$ (3.10)

Applying (3.9), (3.10) and Lemma 1.1 for the coefficients $p_1, p_2, q_1$ and $q_2$, we readily get

$$\left|\frac{m^m}{m!}\right| \leq |E_m|, m \in \mathbb{N}.$$ (3.11)

Subtracting (3.5) from (3.7) we have

$$p_2 - q_2 = 2q(1 + q)(1 + q + q^2)^m(1 - \lambda + \lambda(1 + q + q^2)^2)a_3.$$ (3.11)

Applying (3.9), (3.10) and Lemma 1.1 once again for the coefficients $p_1, p_2, q_1$ and $q_2$, we readily get

$$\left|\frac{m^m}{m!}\right| \leq |E_m|, m \in \mathbb{N}.$$ (3.11)

This completes the proof of Theorem 3.1.

Remark 3.1 We note that Theorem 3.1 can be reduced to various other well-known and new results (see Altinkaya and Yalçın[1]).

IV. CONCLUSION

This paper define a new subclass of analytic functions to obtain Fekete-Szego Inequality.

REFERENCES