Finitely Generated L-Slice for a locale L

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***Abstract:** The notion of an action of a locale L on a join semilattice J with bottom element 0_J is developed and is utilized to form the entity (σ, J), which we call L-slice, that has properties which could be studied algebraically as well as topologically. We investigate the properties of L-slice (σ, J) of a locale L. We have proved that the product of two L-slices of a locale is an L-slice. The notion of finitely generated L-slice of a locale L is introduced and we have shown that every finitely generated L-slice (σ, J), of a locale L with n generators is isomorphic to the quotient slice of the L-slice (Π, L^n)*

***Keywords:** Single phase multilevel inverter, Multi carriers, Sixty Degree PWM, Twenty seven levels, Cascaded multilevel inverter, and Distortion Factor.*

I. INTRODUCTION

Among many introductions to topology, a particular view that has arisen in Theo-retical Computer Science starts with the theory of domains as defined by Scott and Strachey [10] to provide a mathematical foundation for semantics of programming languages, establishing that domains could be put into a topological setting. Duality between Frames and topological spaces have been utilized to make a connection between syntactical and semantical approach to logic. But the application of Stone duality in modal logic require a duality for Boolean algebras or distributive lattices endowed with additional operations. This has inspired the concept of action of a locale on a join semilattice introduced in this paper.

In this paper we have taken up the following study which is relevant in the above context. Given a locale L and a join semilattice J with bottom element 0_J , we have introduced a new concept called L-slice of a locale L denoted by (σ, J), to be an action of the locale L on the join semilattice J together with a set of conditions. The L-slice of a locale L though algebraic in nature adopts topological properties such as compactness of L through the action . The notion of finitely generated L-slice of a locale L is introduced and we have shown that every finitely generated L-slice (σ, J) of a locale L with n generators is isomorphic to the quotient slice of the L-slice (Π, L^n).

The content of this paper has been divided into three sections. Section 1 includes some preliminary concepts of locale theory required for next sections.

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Section 2 deals with the concept of L-slice of a locale L and its various properties needed for next section. Section 3 explains finitely generated L-slice of a locale L and its properties

II. PRELIMINARIES

Definition 1.1. [9] A frame is a complete lattice L satisfying the infinite distributivity law ⋁ A ∧ b = ⋁ { a ∧ b | a ∈ A }, for any subset A ⊆ L and any b ∈ L.
Definition 1.2. [9] A map f : L→ M between frames L, M preserving all finite meets (including the top 1) and all joins (including the bottom 0) is called a frame homomorphism. A bijective frame homomorphism is called a frame isomorphism.
Remark. The category of frames is denoted by **Frm**. The opposite of category **Frm** is the category **Loc** of locales. We can represent the morphism in **Loc** as the infima -preserving f : L →M such that the corresponding left adjoint f : M→ L preserves finite meet. If we do not refer to the morphisms in the category **Loc** of locales and the category **Frm** of frames, then the objects frames and locales are same.
Examples 1.3. [9] i. The lattice of open subsets of topologicalspace.
ii.The Boolean algebra B of all regularly open subsets of Real-line,R.
Definition 1.4. [6] A subset I of a locale L is said to be an ideal,if
i. I is a sub-join-semilattice of L; that is 0_L∈ I and a∈I; b∈I impliesa⊔b∈ I;and
ii. I is a lower set; that is a ∈ I and b ≤ a imply b ∈ I. ∈ I
If a ∈ L, the set $\downarrow(a) = \{x \in L: x \leq a\}$ is an ideal of L. $\downarrow(a)$ is the smallest ideal containing a and is called the principal ideal generated by a. A proper ideal I is prime if x Π $y \in I$ implies that either $x \in I$ or $y \in I$[6].

III. L-SLICE AND ITS PROPERTIES

Definition 2.1. [10] Let L be a locale with bottom element 0_L, top element 1_L and (σ, J)be a L - slice with bottom element 0_J. By the"action of L on J"we mean a function $\sigma: L \times J \to J$ such that the following conditions are satisfied
1.$\sigma(a, x_1 \vee x_2) = \sigma(a, x_1) \vee \sigma(a, x_2)$ for all $a \epsilon L$ and for all $x_1, x_2 \epsilon J$.
2. .$\sigma(a, 0_J) = 0_J$ for all $a \epsilon L$.
3. $\sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x))$ for all , $b \epsilon L$, $x \epsilon J$.
4. $\sigma(1_L, x) = x$ and $\sigma(0_L, x) = 0_J$ for all $x \epsilon J$.

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5. $\sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x)$, for , $b \in L$, $x \in J$.

If σ is an action of the locale L on a join semilattice J, then we call (σ, J) as L-slice.

Next proposition gives sufficient conditon for a subset S ⊆O(L) , the collection of all order preserving maps on L, to be an L-slice.

Proposition 2.2. [10] Let L be a locale, and let S be a set of order preserving maps on L such that: i.

The constant map **0** ∈ S (**0** takes everything to 0).

Ii. If f; g ∈ S, then f ∨ g ∈ S.

iii. For all a ∈ L and for all f ∈ S, the meet of the constant map a and f is in S (i.e. f ⊓a ∈ S). Then the map σ : L × S →S defined by σ (a,f)(x)= f(x) ⊓ a is an action of L on S.

Examples 2.3[10]

1.Let L be a locale and I be any ideal of L .Consider each $x \in I$ and define $\sigma: L \times I \to I$ as $\sigma(a, x) = a \wedge x, \ a \in L$. It can be easily seen that (σ, I) is a L –slice.

2.Let L be a chain with top and bottom elements and J be any join semilattice with bottom element 0_J.Define $\sigma: L \times J \to J$ by $\sigma(a, j) = j$, for every $a \neq 0_L$ and $\sigma(0_L, j) = 0_J$. This is called a trivial L-slice.

3.Any locale L can be viewed as the meet L –slice$(\sqcap, L)$ where the action σ is defined as $\sigma(a, x) = a \sqcap x$.

Proposition 2.4. The product of two L-slices of a locale L is an L-slice.

Proof. Let (σ_1, J_1),(σ_2, J_2) be two L-slices of a locale L. Since J_1, J_2 are join semilattices with bottom elements, $J_1 \times J_2$ is a join semilattice with bottom $(0_{J1}, 0_{J2})$. Define σ: $L \times (J_1 \times J_2) \to J_1 \times J_2$ by $\sigma(a,(x,y)) = (\sigma_1(a,x), \sigma_2(a,y))$.Then

1. $\sigma(a, (x_1, y_1) \vee (x_2, y_2)) = \sigma(a, (x_1 \vee x_2, y_1 \vee y_2)) = (\sigma_1(a, x_1 \vee x_2), \sigma_2(a, y_1 \vee y_2)) = (\sigma_1(a, x_1) \vee \sigma_1(a, x_2), \sigma_2(a, y_1) \vee \sigma_2(a, y_2)) = (\sigma_1(a, x_1), \sigma_2(a, y_1)) \vee (\sigma_1(a, x_2), \sigma_2(a, y_2)) = \sigma(a, (x_1, y_1)) \vee \sigma(a, (x_2, y_2))$

2. $\sigma(a, (0_{J1}, 0_{J2})) = (\sigma_1(a, 0_{J1}), \sigma_2(a, 0_{J2})) = (0_{J1}, 0_{J2})$

3. $\sigma(a \sqcap b, (x,y)) = (\sigma_1(a \sqcap b, x), \sigma_2(a \sqcap b, y)) = (\sigma_1(a, \sigma_1(b,x)), \sigma_2(a, \sigma_2(b,y))) = \sigma(a, (\sigma_1(b,x), \sigma_2(b,y)) = \sigma(a, \sigma(b,(x,y))$

4. $\sigma(1_L, (x,y)) = (\sigma_1(1_L, x), \sigma_2(1_L, y)) = (x,y)$
$\sigma(0_L, (x,y)) = (\sigma_1(0_L, x), \sigma_2(0_L, y)) = (0_{J1}, 0_{J2})$

5. $\sigma(a \sqcup b, (x,y)) = (\sigma_1(a \sqcup b, x), \sigma_2(a \sqcup b, y)) = (\sigma_1(a,x) \vee \sigma_1(a,y), \sigma_2(a,x) \vee \sigma_2(a,y)) = (\sigma_1(a,x), \sigma_2(a,x)) \vee (\sigma_1(a,y), \sigma_2(a,y) = \sigma(a,(x,y)) \vee \sigma(b,(x,y))$

Thus σ is an action on $J_1 \times J_2$ and $(\sigma, J_1 \times J_2)$ is a L-slice of locale L.

Definition 2.5. [10] Let (σ,J) be an L-slice of a locale L. A subjoin semilattice J' of J is said to be L-subslice of J if J' is closed under action by elements of L.

Examples 2.6. [10] 1. Let L be a locale and O(L) denotes the collection of all order preserving maps on L. Then (σ, O(L)) is an L-slice, where σ : L× O(L) → O(L) is defined by σ(a, **f**)= f_a, where $\mathbf{f}_a$: L → L is defined by $\mathbf{f}_a(x) = \mathbf{f}(x) \sqcap a$. Let K={**f** ∈ O(L) : **f**(x)≤x, ∀x ∈L}. Then (σ, K) is an L-subslice of the L-slice (σ, O(L)).

2.Let (σ, J) be an L-slice and let x ∈ (σ, J).. Define <x>={σ(a, x); a ∈ L}. Then (σ, <x>) is an L-subslice of (σ, J) and it is the smallest L-subslice of (σ, J) containing x.

Proposition 2.7. [10] The intersection of any family of L-subslices of an L-slice (σ, J) is again an L-subslice of (σ, J).

Deftnition 2.8. Let (σ, J) be an L-slice of a locale L. An equivalence relation R on (σ, J) is called an L-slice congruence if

i. xRy implies $x \vee zRy \vee z$ for any $x, y, z \in (\sigma, J)$

ii. xRy implies $\sigma(a, x) R \sigma(a, y)$ for all $a \in L, x, y \in (\sigma, J)$.

Definition 2.9[10] Let (σ, J) and (μ, K) be L-slices. A map $f: (\sigma, J) \to (\mu, K)$ is said to be L-slice homomorphism if

i) $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$, for all $x_1, x_2 \in (\sigma, J)$.

ii) $f(\sigma(a, x)) = \mu(a, f(x))$ for all $a \in L$ and all $x \in (\sigma, J)$.

Examples 2.10. [10] i. Let (σ, J) be an L-slice and (σ, J') be an L-subslice of (σ, J). Then the inclusion map i : (σ, J') →(σ, J) is an L-slice homomorphism. ii. Let I =↓ (a), J=↓(b) be principal ideals of the locale L. Then (σ, I), (σ, J) are L-slices. Then the map f : (σ, I) →(σ, J) defined by f (x)= x⊓ b is an L-slice homomorphism.

Proposition 2.11. Let (σ, J), (μ, K) be two L-slices of a locale L and let f: (σ, J) → (μ, K) be an L-slice homomorphism. Then the relation R on (σ, J) defined by xRy if and only if f(x)= f (y) is a congruence on (σ, J)

Deftnition 2.12. The L-slice congruence R discussed in proposition 2.11 is called natural congrurence associated with the L-slice homomorphism **f** : (σ, J) → (μ, K).

Let R be a congruence on (σ, J) and let J/R denotes the collection of all equivalence classes with respect to the relation R. Then J/R is a join semilattice with bottom element $[0_J]$ where the partial order ≤ on J/R is defined by $[x] \leq [y]$ if and only if $x \leq y$ in (σ, J). In the next proposition, we will show that (γ, J/R) is an L-slice where the action γ : L×J/R→ J/R is defined by $\gamma(a, [x]) = [\sigma(a, x)]$.

Definition2.13.[10] Let(σ, J), (μ, K)be two L-slices.A map $f: (\sigma, J) \to (\mu, K)$is said to be an L-slice isomorphism if

i)f is one-one

ii)f is onto

iii)f is a L-slice homomorphism.

Proposition 2.14. If R is a congruence relation on (σ, J), then (γ, J/R) is an L-slice.

Deftnition 2.15. Let (σ, J) be an L-slice of a locale L and R be a congruence on (σ, J). Then the L-slice $(\gamma, J/R)$ described in proposition 2.14 is called quotient slice of L-slice (σ, J) with respect to the congruence R.

Proposition 2.16. L-slice Isomorphism theorem Let (σ, J), (μ, K) be two L-slices of a locale L and let f : (σ,J) → (μ, K) be an L-slice homomorphism. Let R be the natural congruence associated with the L-slice homomorphism f.

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Then the quotient slice (γ, J/R) of (σ, J) is isomorphic to the subslice (μ, imf) of the L-slice (μ, K).

IV. FINITELY GENERATED L-SLICE

The notion of finitely generated L-slice of a locale *L* is introduced and we have shown that every finitely generated L-slice (σ, J) of a locale L with n generators is isomorphic to the quotient slice of the L-slice (⊓, L^n).

Deftnition 3.1. Let (σ, J) be an L-slice of a locale *L*. A subset *S* of *(σ, J)* is said to be span of the set $\{x_1, x_2, ...x_n\} \subseteq (\sigma, J)$ if each $x \in S$ can be written as $x=\bigvee_{i=1}^{n} \sigma(a_i, x_i)$ where $a_i \in L$.

Proposition 3.2. Let (σ, J) be an L-slice of a locale L and $\{x_1, x_2, ...x_n\} \subseteq (\sigma, J)$. Let S=Spanp({$x_1$, x_2, x_{n}}). Then (σ, S) is a subslice of (σ, J).

Proof: Let $x, y \in S$. Then there is $a_1, a_2, ..., a_n, b_1, b_2, b_n \in L$ such that $x=\bigvee_{i=1}^{n} \sigma(a_i, x_i)$, $y = \bigvee_{i=1}^{n} \sigma(b_i, x_i)$.

$$x \vee y = \bigvee_{i=1}^{n} \sigma(a_i, x_i) \vee \bigvee_{i=1}^{n} \sigma(b_i, x_i)$$

$$= \bigvee_{i=1}^{n} \sigma(a_i, x_i) \vee \sigma(b_i, x_i)$$

$$= \bigvee_{i=1}^{n} \sigma(a_i \sqcup b_i, x_i)$$

∈ S. Therefore *S* is a subjoin semilattice of (*σ*, *J*).

Let $a \in L$. Then

$$\sigma(a, x) = \sigma\left(a, \bigvee_{i=1}^{n} \sigma(a_i, x_i\right) = \bigvee_{i=1}^{n} \sigma(a, \sigma(a_i, x_i)) = \bigvee_{i=1}^{n} \sigma(a \sqcap a_i, x_i) \in S.$$

Hence (*σ, S*) is a subslice of (*σ, J*)

Deftnition 3.3. An L-slice (*σ, J*) of a locale *L* is said to be finitely generated if there is a finite subset *S*⊆ (*σ, J*) such that (*σ, J*)= *Span*(*S*). Elements of *S* are called generators of the L-slice (*σ, J*).

An L-slice (*σ, J*) of a locale *L* is said to be generated by *n* elements if there is a finite subset *S⊆(σ, J)* having *n* elements such that (*σ,J*)=*Span*(*S*) and there is no subset *T⊆(σ, J)*having less than *n* elements which spans the L-slice (*σ, J*).

Example 3.4. If L is a locale, then (⊓, L) is a finitely generated L-slice.

Deftnition 3.5. An L-slice (*σ, J*) with a single generator *x* is called cyclic L-slice. (*σ, J*) is a cyclic L-slice if (*σ*, *<x>)=(σ, J*).

Proposition 3.6. Let (σ, J) be an L-slice of a locale L and let S be a finite subset of (σ, J) such that Span(S)=(σ, J). Then Span(T)=(σ, J) for all subset T of (σ, J) such that S⊆T.

Proof: Let $S=\{x_1, x_2, ...x_n\}$ be such that *Span(S)*=(*σ, J*). Then for any x ∈ (σ, J), $x = \bigvee_{i=1}^{n} \sigma(a_i, x_i)$. If $z_i \in T$, then $x = \bigvee_{i=1}^{n} \sigma(b_i, x_i)$,where $b_i = a_i$ if $z_i \in S$ and $b_i = 0_L$ if $z_i \in T - S$. Hence Span(T)=(σ,J).

Proposition 3.7. Let (σ, J) and(μ, K)be L-slices of a locale L, and let (σ, J) be finitely generated with generators $\{x_1, x_2, ...x_n\}$. If f : (σ, J) → (μ,K) is an onto L-slice homomorphism, then (μ,K) is finitely generated.

Proof: Let y ∈(μ, K). There exist $x \in (\sigma, J)$ such that y=f(x). Since (*σ, J*) is finitely generated, there is $a_1, a_2, ...a_n \in L$ such that $x = \bigvee_{i=1}^{n} \sigma(a_i, x_i)$.

$y=f(\bigvee_{i=1}^{n} \sigma(a_i, x_i))=\bigvee_{i=1}^{n} f(\sigma(a_i, x_i))=$
$\bigvee_{i=1}^{n} \mu(a_i, f(x_i))$.

Therefore $\{f(x_1), f(x_2), ...f(x_n)\}$ generates (*μ, K*).

Proposition 3.8. Let (σ, J) be a finitely generated L-slice of a locale L with generators $\{x_1, x_2, ...x_n\}$. Then φ : (⊓, L^n) → (σ, J) defined by $\varphi(a_1, a_2, ...a_n)=\bigvee_{j=1}^{n} \sigma(a_j, x_j)$ is an onto L-slice homomorphism.

Proof. By Proposition 2.4, (⊓, L^n) is an L-slice of a locale L.

$$\varphi\left(\bigvee_{i=1}^{n}(a_{1i}, a_{2i}, ... a_{ni}\right) = \varphi(\bigvee_{i=1}^{n} a_{1i}, \bigvee_{i=1}^{n} a_{2i}, ... \bigvee_{i=1}^{n} a_{ni})$$

$$= \bigvee_{j=1}^{n} \sigma(\bigvee_{i=1}^{n} a_{ji}, x_j) = \bigvee_{j=1}^{n}\bigvee_{i=1}^{n} \sigma(a_{ji}, x_j)$$

$$= \left(\bigvee_{i=1}^{n} \varphi(a_{1i}, a_{2i}, ... a_{ni}\right)$$

Thus *φ* preserves join.

$$\varphi(a \sqcap (a_1, a_2, ... a_n)) = \varphi(a \sqcap a_1, a \sqcap a_2, ... a \sqcap a_n)$$

$$= \bigvee_{i=1}^{n} \sigma(a \sqcap a_i, x_i) = \bigvee_{i=1}^{n} \sigma(a, \sigma(a_i, x_i)$$

$$= \sigma(a, \bigvee_{i=1}^{n} \sigma(a_i, x_i) = \sigma(a, \varphi(a_1, a_2, ... a_n))$$

Hence *φ* is an L-slice homomorphism.

Let *y* ∈(*σ, J*). Then $y=\bigvee_{i=1}^{n} \sigma(a_i, x_i)$.So $(a_1, a_2, ... a_n) \in$ (⊓, L^n) such that φ $(a_1, a_2, ... a_n)$=y. Hence φ is onto.

Corollary 3.9. Let (σ , J) be a finitely generated L-slice of a locale L with generators $\{x_1, x_2, ...x_n\}$. Then (σ, J) is isomorphic to the quotient L-slice (⊓, L^n/R) of the product L-slice (⊓, L^n).

Proof. By proposition 3.8, φ : (⊓, L^n) → (σ,J) defined by $\varphi(a_1, a_2, ... a_n)=\bigvee_{j=1}^{n} \sigma(a_j, x_j)$ is an onto L-slice homomorphism. Let R be the congruence xRy if and only if φ(x) = φ(y). Then by isomorphism theorem for L-slices imφ = (σ,J) is isomorphic to the quotient L-slice (⊓, L^n/R).

V. CONCLUSION

In this paper we have discussed various topological properties of Lattice.

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