On Difference Cordial Labeling of Networks

D. Florence Isido, V.M.Chitra

Abstract: Let \( G \) be a \((p,q) \) graph. Let \( f:V(G)\to \{1,2,3,\ldots,p\} \) be a function. For each edge \( uv \) assign the label \( |f(u)−f(v)| \), \( f \) is called a difference cordial labeling if \( f \) is one to one and map \( |e_f(0)−e_f(1)| \leq 1 \), where \( e_f(1) \) and \( e_f(0) \) denote the number of edges labeled with 1 and labeled not with 1 respectively. A graph with difference cordial labeling is called difference cordial graph. In this paper we investigate the difference cordial labeling of the butterfly network, benes network, path union of cycles and corona product of cycles with \( K_1, K_2 \) and \( K_3 \).

Keywords: Butterfly graph, benes graph, path union of cycles, corona product of cycles with \( K_1, K_2 \) and \( K_3 \) and difference cordial labeling.

I. INTRODUCTION

Rosa introduced the concept of labeling in 1967. Graph labeling is the assignment of integers to vertices or edges or both of a graph subject to certain conditions.

Graph labeling has applications in Coding Theory, Radar location codes, Missile guidance codes, optimal circuit design, X-Ray Crystallographic analysis, cloud computing, data mining and conflict resolution.

Cahit introduced cordial labeling in 1987 and R.Ponraj, S.Sathish Narayanan and R.Kala introduced difference cordial labeling in 2013 and investigated difference cordial labeling of paths, cycles, bipartite graph, complete graph etc. In 2015 Seoud and Salman studied difference cordial labeling in 2013 and investigated difference cordial labeling of paths, cycles, bipartite graph, complete graph etc.

II. PRELIMINARIES

Definition 1: The \( n \) dimensional butterfly network, denoted by \( BF(n) \), has a vertex set \( V = \{x;i\}; x \in V(Q_n) \), \( 0 \leq i \leq n \). Two vertices \( (x;i) \) and \((y;j)\) are linked by an edge in \( BF(n) \) if and only if \( j = i + 1 \) and either (i) \( x = y \) or (ii) \( x \) differs from \( y \) in precisely the \( j \)th bit.

For \( y \), the edge is said to be a straight edge, otherwise the edge is a cross edge. For fixed \( i \) the vertex \( (x;i) \) is a vertex on level \( i \).

Definition 2: Then dimensional benes network consists of back to back butterfly network and it has \( 2n+1 \) levels, \((2n+1)2^n \) vertices and \( n2^{n+2} \) edges.

Definition 3: A path union of graphs is obtained by from \( n \) copies \( G_1, G_2, G_3 \ldots G_n \) of a graph \( G \) by adding an edge from \( G_i \) to \( G_{i+1} \), \( i = 1,2 \ldots n-1 \).

Definition 4: Corona product of graphs of two graphs \( G \) and \( H \) is obtained by taking one copy of \( G \) and \( |V(G)| \) copies of \( H \) and joining each vertex of the \( i \)th copy of \( H \) to the \( i \)th vertex of \( G \), where \( 1 \leq i \leq |V(G)| \).

III. RESULTS AND DISCUSSIONS

Theorem 1: Any butterfly network \( BF(n) \) is difference cordial graph.

Proof: The butterfly network has \( (n+1)2^n \) vertices and \( n2^{n+1} \) edges. In level 0 and level \( n \), vertices are of degree 2 and in other levels, vertices are of degree 4.

Define \( f:V(G)\to \{1,2,3,\ldots,(n+1)2^n\} \)

\[ f(v_i) = (n+1)(i-1) + (i+1), 0 \leq i \leq n - 1 \leq j \leq 2^n \]

By the above labeling the straight edges are labeled with 1 and cross edges are labeled not with 1. The \( n \) dimensional butterfly network has \( n2^n \) edges with label 1. The total number of edge is \( n2^{n+1} \). Therefore the edges with label not with 1 is \( n2^{n+1} - n2^n = n2^n \)

Fig. 1 Butterfly network BF (3)

<table>
<thead>
<tr>
<th>( e_f(0) )</th>
<th>( e_f(1) )</th>
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<tbody>
<tr>
<td>( n2^n )</td>
<td>( n2^n )</td>
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</table>

\[ |e_f(0)−e_f(1)| = 0 \]

Hence any butterfly network is difference cordial graph.

Theorem 2: Any benes network is a difference cordial graph.
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Proof:
The n-dimensional network has \((2n + 1)2^n\) vertices and \(2n + 2\) edges. In level 0 and level \(n-1\), degree of vertices is 2 and in other levels, degree of vertices is 4.

Define \(f: V(G) \rightarrow \{1, 2, 3, \ldots, (2n + 1)2^n\}\)
\(f(v_i) = (2n + 1)(j - 1) + (i + 1), 0 \leq i \leq 2n, 1 \leq j \leq 2^n.\)
The induced edge labeling,
\[f^*(v_{ij}, v_{(i+1)j}) = 1, 0 \leq i \leq 2n, 1 \leq j \leq 2^n\]

The benes network consists of \(n(2^{n+2})\) edges, in that there are \(2n(2^n) = n(2^{n+1})\) straight edges labeled with 1. Therefore the edges labeled not with 1 are equal to \(n(2^{n+2}) - n(2^{n+1}) = n(2^{n+1})\).
Here \(|e_f(0) - e_f(1)| = 0\) Hence benes network is difference cordial graph.

\[
\begin{array}{c|c|c}
\text{edges} & e_f(0) & e_f(1) \\
\hline
\text{n(2^{n+1})} & n(2^{n+1}) & n(2^{n+1}) \\
\end{array}
\]

Fig. 2 Benes network BB \((3)\)

Theorem 3: Any path union of cycle \(C_m\) is a difference cordial graph.

The path union of cycles has \(mn\) vertices and \(mn + n - 1\) edges.

Define \(f: V(G) \rightarrow \{1, 2, 3, \ldots, mn\}\)

Case 1: When \(m\) is even:

When \(i\) is odd,
\[f(v_{2i}) = 2i, f(v_{2i+1}) = 2i - 1, 1 \leq i \leq n\]

When \(i\) is even,
\[f(v_{2i}) = 2i - 1, f(v_{2i+1}) = 2i, 1 \leq i \leq n\]

\[f(v_{2i-1}) = (2n + 1)(2i) + (2j - 1)(i - 1)(m - 2), 1 \leq i \leq n, 2 \leq j \leq \frac{m}{2} - 1\]

\[f(v_{2i}) = (2n + 1)(2i - 1) + (2j)(i - 1)(m - 2), 1 \leq i \leq n, 2 \leq j \leq \frac{m}{2} - 1\]

\[|f(v_{2i-1}) - f(v_{2i})| = |2i - (2i - 1)| = 1, 1 \leq i \leq n\]
The induced edge labeling,

\[ f^*(v_{1}, v_{2}) = 1, 1 \leq i \leq n \]
\[ f^*(v_{(2i-1)}, v_{(2i)}) = 1, 1 \leq i \leq n/2, 2 \leq j \leq \frac{m}{2} \]
\[ f^*(v_{i+1}, v_{i}) = 0, 1 \leq i \leq n, 1 \leq j \leq \frac{m}{2} \]
\[ f^*(v_{(2i-1)}, v_{(2i)}) = 1, 1 \leq i \leq \frac{n}{2} \]
\[ f^*(v_{(2i)}, v_{(2i+1)}) = 0, 1 \leq i \leq \frac{n}{2} \]

Case 1 a: When \( n \) is even

\[ e_f(1) = \frac{(m-1)}{2} + \frac{n}{2} = n \left( \frac{m-2}{2} + \frac{n}{2} \right) - n \left( \frac{m+1}{2} \right) \]
\[ e_f(0) = (m+1)n - 1 - \frac{n(m+1)}{2} = \frac{(m+1)n-1}{2} \]
\[ |e_f(0) - e_f(1)| = 1 \]

Fig. 3 Path union of cycles (m is even)

<table>
<thead>
<tr>
<th>( e_f(1) )</th>
<th>( e_f(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (m+1) \frac{n}{2} )</td>
<td>( (m+1) \frac{n}{2} - 1 )</td>
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</table>

Case 1 b: When \( n \) is odd

\[ e_f(1) = \frac{n(m-1)}{2} + \frac{n-1}{2} = n \left( \frac{m-2}{2} + \frac{n}{2} - \frac{1}{2} - n \left( \frac{m+1}{2} \right) \right) \]
\[ e_f(0) = (m+1)n - 1 - \frac{n(m+1)}{2} = \frac{n(m+1)-1}{2} \]
\[ |e_f(0) - e_f(1)| = 0 \]

Case 2: When \( m \) is odd: \( 1 \leq i \leq n, 2 \leq j \leq \frac{m}{2} \)

\[ f^*(v_{1}) = 2 + n(i-1) \]
\[ f^*(v_{2}) = 1 + n(i-1) \]
\[ f^*(v_{n}) = 3 + n(i-1) \]
\[ f^*(v_{(2i-1)}) = n(i-1) + 2j + 1 \]
\[ f^*(v_{(2i)}) = n(i-1) + 2j \]
\[ f^*(v_{1}) = 2 + n(i-1) \]
\[ f^*(v_{2}) = 1 + n(i-1) + 3n(i+1) \]
\[ f^*(v_{n}) = n(i-1) + 2j + 1 - n(i-1) + 2j \]
\[ 1 \leq i \leq n, 2 \leq j \leq \frac{m}{2} \]
\[ f^*(v_{(2i)}) - f^*(v_{(2i-1)}) = 3, \]

The induced edge labeling is,

\[ f^*(v_{1}, v_{2}) = 1, 1 \leq i \leq n \]
\[ f^*(v_{(2i-1)}, v_{(2i)}) = 1, 1 \leq i \leq n/2 \]
\[ f^*(v_{i+1}, v_{i}) = 0, 1 \leq i \leq n, 1 \leq j \leq \frac{m}{2} \]
\[ f^*(v_{(2i-1)}, v_{(2i)}) = 1, 1 \leq i \leq n, 2 \leq j \leq \frac{m-1}{2} \]
\[ f^*(v_{(2i-1)}, v_{(2i)}) = 1, 2 \leq j \leq \frac{m-1}{2} \]
\[ f^*(v_{(2i)}, v_{(2i+1)}) = 0, 1 \leq j \leq \frac{m-1}{2} \]

Number of edges labeled with 1 is

\[ n \left( \frac{m-1}{2} - 1 \right) + 2n = n \left( \frac{m-1}{2} - 1 + 2 \right) = n \left( \frac{m+1}{2} \right) \]

and total number of edges are \( mn + n - 1 \).

So the number of edges labeled not with 1 is

\[ mn + n - 1 = (n(m+1) - 1) - n \left( \frac{m+1}{2} \right) = n \left( \frac{m+1}{2} \right) - 1 \]

Fig. 4 Path union of cycle (m is odd)

<table>
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<tr>
<th>( e_f(0) )</th>
<th>( e_f(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \left( \frac{m+1}{2} \right) - 1 )</td>
<td>( n \left( \frac{m+1}{2} \right) )</td>
</tr>
</tbody>
</table>

| \( e_f(0) - e_f(1) | = 1 . Hence path union of cycle is a difference cordial graph.

**Theorem 4:** \( C_n \uplus K_1 \) is a difference cordial graph.

Define \( f : V(G) \rightarrow \{ 1, 2, 3, \ldots, 2n \} \)

\[ f(v_{1}) = (2j - 1), 1 \leq j \leq n \]
\[ f(v_{2}) = 2j, 1 \leq j \leq n \]
\[ f(v_{1}) - f(v_{2}) = 1 \leq j \leq n \]
\[ f(v_{1}) - f(v_{2}) = 1 \leq j \leq n \]
\[ f(v_{1}) - f(v_{2}) = 1 \leq j \leq n \]

It induces the edge labeling

\[ f^*(v_{1}, v_{2}) = 1, 1 \leq j \leq n \]
\[ f^*(v_{1}, v_{1}) = 0, 1 \leq j \leq n \]

Therefore \( C_n \uplus K_1 \) is a difference cordial graph.
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Theorem 5: \( C_n \odot K_1 \) is a difference cordial graph. It has \( 3n \) vertices and \( 4n \) edges.

Define \( f: V(G) \rightarrow \{1, 2, 3, \ldots, 3n\} \)
\[
f(v_{1j}) = 3j - 1, 1 \leq j \leq n.
\]
\[
f(v_{2(j-1)}) = 3j - 2, 1 \leq j \leq n.
\]
\[
f(v_{2(2j-1)}) = 3j - 1, 1 \leq j \leq n.
\]
\[
|f(v_{1j}) - f(v_{2(l)})| = |(3j - 1) - (3j - 1)| = 1, 1 \leq j \leq n.
\]
\[
|f(v_{1j}) - f(v_{2(2j-1)})| = |(3j - 1) - (3j - 3j)| = 1, 1 \leq j \leq n.
\]
\[
|f(v_{1j}) - f(v_{2(2j-1)})| = |(3j - 1) - (3j + 1)| = 3, 1 \leq j \leq n.
\]
It induces the edge labeling
\[
f''(v_{1j}, v_{2(j-1)}) = 1, 1 \leq j \leq n
\]
\[
f''(v_{1j}, v_{2(2j-1)}) = 1, 1 \leq j \leq n
\]
\[
f''(v_{1j}, v_{2(2j-1)}) = 0, 1 \leq j \leq n
\]
\[
e_f(1) = n + n = 2n
\]
\[
e_f(0) = n + n = 2n
\]

Illustration:

Define \( f: V(G) \rightarrow \{1, 2, 3, \ldots, 4n\} \)
\[
f(v_{1j}) = 4j - 2, 1 \leq j \leq n
\]
\[
f(v_{2(2j-1)}) = 4j - 3, 1 \leq j \leq n
\]
\[
f(v_{2(2j)}) = 4j - 1, 1 \leq j \leq n
\]
\[
f(v_{3j}) = 4j, 1 \leq j \leq n
\]
\[
|f(v_{1j}) - f(v_{2(l)})| = |(4j - 2) - (4j - 1)| = 1, 1 \leq j \leq n
\]
\[
|f(v_{1j}) - f(v_{2(2j-1)})| = |(4j - 2) - (4j - 3)| = 1, 1 \leq j \leq n
\]
\[
|f(v_{3j}) - f(v_{2(2j)})| = |(4j - (4j - 1)| = 1, 1 \leq j \leq n
\]
\[
e_f(0) = 7n - 3n = 4n
\]

Fig. 6 Corona product of \( C_5 \) with \( K_2 \)

Theorem 6: \( C_n \odot K_2 \) is a difference cordial graph. It has \( 4n \) vertices and \( 7n \) edges

Case 1: Label the vertices in \( C_n \) as well as the vertices in \( K_2 \) consecutively

Number of edges with label 1 is \( (n - 1) + n(2) = 3n - 1 \) and the number of edges not with label 1 is \( 7n - (3n - 1) = 4n + 1 \)

Case 2: When each vertex of \( i^{th} \) copy of \( K_2 \) is connected to the \( i^{th} \) vertex of \( C_n \), \( K_4 \) is formed. Label the vertices of \( K_4 \) consecutively. \( K_4 \) can have at most three edges with label 1, hence

Number of edges with label 1 is \( 3n \) and the number of edges not with label 1 is \( 7n - 3n = 4n \).

In both the cases the labeling is not difference cordial labeling.

IV. CONCLUSION

The difference cordial labeling of butterfly network, benes network, path union of cycles and corona product of cycles with \( K_1 \), \( K_2 \) and \( K_3 \) are investigated.

REFERENCES

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