Oscillation and Nonoscillation Criteria for Generalized Second Order Quasilinear \( \alpha \)-Difference Equations

V.Srimanju, SK.Khadar Babu

Abstract: New oscillation and nonoscillation theorems are obtained for the generalized second order quasilinear \( \alpha \)-difference equation

\[
\Delta_{\alpha(1)} \left( \left| \Delta_{\alpha(1)} \mu(k-1) \right|^{\frac{1}{\alpha}} \Delta_{\alpha(1)} \mu(k) \right) + p(k) \left| u(k) \right|^{\frac{1}{\alpha}} u(k) = 0, \quad \gamma > 0
\]

where \( \gamma \) is a constant and \( p(k) \) is a real valued function with \( p(k) \geq 0 \).

Keywords: Generalized difference equation, Quasilinear, Oscillation.

I. INTRODUCTION

Difference equations represent a fascinating mathematical area on its own as well as a rich field of the applications in such diverse disciplines as population dynamics, operations research, ecology, economics, biology etc. For general background as difference equations with many examples from diverse fields, one can refer to [1]. The theory of difference equations is based on the operator \( \Delta \) defined as

\[
\Delta u(k) = u(k+1) - u(k), \quad k \in [0, \infty).
\]

Even though many authors [1], [9] have suggested the definition of \( \Delta \) as

\[
\Delta u(k) = u(k+1) - u(k), \quad 1 \in (0, \infty),
\]

no significant progress took place on this line. But recently, when we took up the definition of \( \Delta \) as given in (1), the theory of difference equations are developed in a different direction. For convenience, we labelled the operator \( \Delta \) defined by (1) as \( \Delta_1 \) and by defining its inverse \( \Delta^{-1}_1 \), many interesting results in number theory were obtained. By extending theory of \( \Delta_1 \) to complex function, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were established for the solutions of difference equations involving \( \Delta_1 \). The results obtained can be found in [3-5].

Jerzy Popenda, et.al., [8], while discussing the behavior of solutions of a particular type of difference equation, defined \( \Delta_\alpha \) as \( \Delta_\alpha u(k) = u(k+1) - \alpha u(k) \). This definition of \( \Delta_\alpha \) is being ignored for a long time.

Recently, in [6], they generalized the definition of \( \Delta_\alpha \) by \( \Delta_{\alpha(1)} \) defined as \( \Delta_{\alpha(1)} \mu(k) = u(k+1) - \alpha u(k) \) for the real valued function \( u(k) \) and \( 1 \in (0, \infty) \) and also obtained the solutions of certain types of generalized \( \alpha \)-difference equations, in particular, the generalized Clairauts \( \alpha \)-difference equation, generalized Euler \( \alpha \)-difference equation and the generalized \( \alpha \)-Bernoulli polynomial \( B_{\alpha(1)}(k,1) \), which is a solution of the \( \alpha \)-difference equation \( u(k+1) - \alpha u(k) = n \alpha^{n-1} \), for \( n \in \mathbb{N}(1) \).

In this paper we consider the generalized second order quasilinear \( \alpha \)-difference equation

\[
\Delta_{\alpha(1)} \left( \left| \Delta_{\alpha(1)} \mu(k-1) \right|^{\frac{1}{\alpha}} \Delta_{\alpha(1)} \mu(k) \right) + p(k) \left| u(k) \right|^{\frac{1}{\alpha}} u(k) = 0, \quad k \geq k_0
\]

where the \( \Delta_{\alpha(1)} \) is defined as \( \Delta_{\alpha(1)} \mu(k) = u(k+1) - \alpha u(k) \) is a constant, and \( p(k) \) is a real valued function with \( p(k) \geq 0 \) for \( k \geq k_0 \).

Oscillation and nonoscillation of equation (2) have investigated intensively, see, for example [1, 2]. In this paper, we obtain new oscillation and nonoscillation criteria for equation (2) which generalize the known results on second order linear difference equations.

A solution \( u(k) \) of (2) is said to be oscillatory if the terms \( u(k) \) of the real valued function \( u(k) \) are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

II. SOME LEMMAS

Throughout, we shall use the following notations:

\[ N_1 = \{0, 1, 2, 1L \}, \quad N_1(a) = \{a, a+1, a+2 1L \}, \quad \text{where} \ a \in N_1, \ \text{and} \ N_1(a, b) = \{a, a+1, 1L, b\}, \ \text{where} \ b \in N_1(a), \ j = k - \left[ \frac{k}{1} \right]. \]

Lemma 1. (Discrete mean value theorem [1]). Suppose that \( u(k) \) is defined on \( N_1(a, b) \). Then, there exists a \( c \in N_1(a+1, b-1) \) such that

\[
\Delta_{\alpha(1)}u(c) \leq \frac{x(b) - x(a)}{b - a} \leq \Delta_{\alpha(1)}u(c-1)
\]

Revised Manuscript Received on May 07, 2019.

V.Srimanju, Corresponding Author, Department of Mathematics, VIT University, Vellore, Tamilnadu, India

SK.Khadar Babu, Assistant Professor(Sr.), Department of Mathematics, VIT University, Vellore, Tamilnadu, India

International Journal of Innovative Technology and Exploring Engineering (IJITEE)
ISSN: 2278-3075, Volume-8, Issue-7, May 2019

Published By:
Blue Eyes Intelligence Engineering
& Sciences Publication

1136
Oscillation and Nonoscillation Criteria for Generalized Second Order Quasilinear \( \alpha \)-Difference Equations

or

\[
\Delta_{\alpha(k)}x(c-1) \leq \frac{x(b)-x(a)}{b-a} \leq \Delta_{\alpha(k)}x(c). \tag{4}
\]

**Lemma 2** Let \( u(k) \) is a nontrivial solution of equation (2). Assume that \( u(k) > 0 \) for \( k \in \mathbb{N}_1(a+1, b) \) and that

\[
\Delta_{\alpha(k)}u(a-1) \geq 0, \Delta_{\alpha(k)}u(a) < 0, a(b-1) > 0, u(b) \leq 0, \tag{5}
\]

or

\[
u(a-1) \leq 0, u(a) > 0, \Delta_{\alpha(k)}u(b-1) > 0, \Delta_{\alpha(k)}u(b-1) \leq 0, \tag{6}
\]

where \( a, b \in \mathbb{N}_1(k^0) \) and \( a < b \). Then the relation

\[
(b-a)^\gamma \sum_{i=a}^{b-1} p(il + j) > 1,
\]

holds.

**Proof.** We assume that (5) holds. If (6) holds, then the proof is similar. Let

\[
r = \max \{s | \Delta_{\alpha(k)}u(s-1) \geq 0, \Delta_{\alpha(k)}u(s) < 0, s \in \mathbb{N}_1(a, b-1)\}.
\]

Since \( \Delta_{\alpha(k)}u(b-1) < 0 \), then \( \Delta_{\alpha(k)}u(k-1) < 0 \) for \( k \in \mathbb{N}_1(r+1, b) \). Since \( u(k) > 0 \) for \( k \in \mathbb{N}_1(a+1, b) \), it follows that, by (2),

\[
\Delta_{\alpha(k)}(\Delta_{\alpha(k)}u(k-1)) \geq 0, \quad \text{for} \quad k \in \mathbb{N}_1(r+1, b).
\]

Hence, \( \Delta_{\alpha(k)}u(k-1) \) is nonincreasing and \( \Delta_{\alpha(k)}u(k-1) < 0 \) for \( k \in \mathbb{N}_1(r+1, b) \). Hence, \( u(k) \) is decreasing. By (4) of Lemma 1 and (5), we get

\[
u(r) < \frac{u(r)-u(b)}{b-r} \leq -\Delta_{\alpha(k)}u(b-1).
\]

On the other hand, since \( \Delta_{\alpha(k)}u(r-1) \geq 0 \), sum of equation (2) between \( k = r \) and \( k = b-1 \) shows that

\[
\left(\Delta_{\alpha(k)}u(b-1)\right)^\gamma \leq \alpha \left| \Delta_{\alpha(k)}u(r-1) \right|^{-\gamma} \Delta_{\alpha(k)}u(r-1)
\]

\[
-(\alpha-1) \sum_{i=r}^{b-2} \left| \Delta_{\alpha(k)}u(il + j) \right|^{-\gamma} \Delta_{\alpha(k)}u(il + j)
\]

\[
-\left| \Delta_{\alpha(k)}u(b-1) \right|^{-\gamma} \Delta_{\alpha(k)}u(b-1) = \sum_{i=r}^{b-1} p(il + j)w^\gamma(il + j).
\]

Since \( u(i) \) is decreasing, equation (9) implies

\[
\left(\Delta_{\alpha(k)}u(b-1)\right)^\gamma < u^\gamma(r) \sum_{i=r}^{b-1} p(il + j).
\]

From equation (8) and equation (10), we obtain

\[
(b-a)^\gamma \sum_{i=a}^{b-1} p(il + j) \geq (b-r)^\gamma \sum_{i=a}^{b-1} p(il + j) > 1.
\]

This completes the proof of Lemma 2.

**Lemma 3** [7] Assume that \( a \geq 0, b \geq 0 \). If \( \gamma \geq 1 \), then \( a^\gamma + b^\gamma \geq (a+b)^\gamma \).

**Lemma 4** [7] Assume that \( a \geq 0, b \geq 0 \). If \( 0 < \gamma \leq 1 \), then \( a^\gamma + b^\gamma \geq (a+b)^\gamma \).

**III. MAIN RESULTS**

Let the real valued function \( k_m \) be given such that

\[
k^0 \leq k_0 < k_1 < L \leq k_m < L \to \infty \text{ as } m \to \infty,
\]

and introduce the notation

\[
\beta_m = \frac{k_{m+1} - k_m}{k_1 - k_0} \text{ for } m \in \mathbb{N}_1.
\]

Clearly, we have \( \beta_0 = 1, \beta_m > 0 \), and \( \sum_{m=0}^\infty \beta_m = \infty \). Our first result (on the nonoscillation) is the following.

**Theorem 5** Assume that \( 0 < \gamma \leq 1 \). Further suppose the coefficient \( p(k) \) in equation (2) has the property

\[
k_{m+1} - k_m \sum_{i=m}^{k_{m+1}-1} p(il + j) \leq \eta_m \to 0 \leq \eta_m < 1 \text{ for } m \in \mathbb{N}(1),
\]

and there exists the infinite real valued function \( w_m \) satisfying the recurrence relation

\[
w_{m+1} = \frac{w_m - \eta_m}{\theta_m + w_m - \eta_m}, \quad m \in \mathbb{N}(1), \quad w_0 = 1,
\]

such that \( 0 < w_m \) for \( m = 1, 2, L \), where

\[
\theta_m = \left( \frac{\beta_m}{\beta_{m+1}} \right)^\gamma.
\]

Then equation (2) is nonoscillatory.

**Proof.** We are going to show that the solution \( u(k) \) of (2) subject to the conditions \( u(k_0 - 1) \leq 0, u(k_0) > 0 \) remains positive for all \( k \in \mathbb{N}_1(k_0) \).

By equation (13), \( (\kappa - k_0)^\gamma \sum_{i=k_0}^{k_{m+1}-1} p(il + j) < 1 \). Therefore by Lemma 2 we have \( \Delta_{\alpha(k)}u(k-1) > 0 \) on \( \mathbb{N}_1(k_0 + 1, k_1) \).

In fact, if not, assume that there exists a \( k^* \in \mathbb{N}_1(k_0 + 1, k_1) \) such that \( \Delta_{\alpha(k)}u(k^* - 2) > 0, \Delta_{\alpha(k)}u(k^* - 1) \leq 0 \), then by Lemma 2, we have \( (k_1 - k_0)^\gamma \sum_{i=k_0}^{k_{m+1}-1} p(il + j) > 1 \) which contradicts...
\((k_1 - k_0)^{y} \sum_{i=k_0}^{k_1-1} p(i + j) < 1\). Therefore \(\Delta_{u(k)}(k-1) > 0\) on \(N_1(k_0+1,k_1)\). Hence \(u(k) > 0, k \in N_1(k_0,k_1)\) and that \(\Delta_{u(k)}(k-1)\) is nonincreasing on \(N_1(k_0+1,k_1)\).

Summing equation (2) from \(k_0\) to \(k_1 - 1\), we obtain
\[
\begin{align*}
\Delta_{u(k_1)}(k_1-1) - \Delta_{u(k_1)}(k_1-1) & = (\alpha - 1) \sum_{i=k_0}^{k_1-2} \Delta_{u(i)}(i + j))^{y} \\
& = \sum_{k_0}^{k_1-1} p(i + j)u^{y} (i + j).
\end{align*}
\]

By Lemma 1, we get
\[
\begin{align*}
u(i) \leq & (\alpha - 1)u(k_0 - 1) \\
& \leq (i + 1 - k_0)\Delta_{u(k_0)}(k_0 - 1) \text{ for } i \in (k_0,k_1 - 1).
\end{align*}
\]

From (15) and (16), we have
\[
\begin{align*}
\Delta_{u(k_1)}(k_1-1) - \Delta_{u(k_1)}(k_1-1) & = (\alpha - 1) \sum_{i=k_0}^{k_1-2} \Delta_{u(i)}(i + j))^{y} \\
& \leq \rho_0(\Delta_{u(k_0)}(k_0 - 1))^{y} \text{ or }
\end{align*}
\]
\[
\begin{align*}
(\Delta_{u(k)}(k_1 - 1))^{y} & \geq (\alpha - 1)\Delta_{u(k_0)}(k_0 - 1) - \rho_0(\Delta_{u(k_0)}(k_0 - 1))^{y} \\
& \geq (\alpha - \rho_0)\Delta_{u(k_0)}(k_0 - 1))^{y}.
\end{align*}
\]

Now we claim that \(u(k)\) is fixed signed on \(N_1(k_1,k_2)\). If not, then there exists a \(s_1 \in N_1(k_1,k_2)\) such that \(u(s_1 - 1) > 0, u(s_1) \leq 0\).

In view of \(\Delta_{u(k_1)}(k_1 - 1) > 0\) on \(N_1(k_0+1,k_1)\), by Discrete Rolle\textsuperscript{TM}S\textsuperscript{TM} Theorem (see [1]), there exists a \(a_1 \in N_1(k_0,s_1 - 1)\) such that \(\Delta_{u(a_1)}(a_1 - 1) > 0\) and \(\Delta_{u(a_1)}(a_1) = 0\).

Clearly, \(a_1 \in N_1(k_1,k_2 - 1)\). Using Lemma 3, we obtain
\[
\begin{align*}
\eta_i & \geq (k_2 - k_1)^{y} \sum_{i=1}^{k_2-1} p(i + j) \geq (s_1 - a_1)^{y} \sum_{i=1}^{s_1-1} p(i + j) > 1.
\end{align*}
\]

Contradicting the assumption \(\eta_i < 1\). Thus we have shown that \(u(k) > 0\) for \(k \in N_1(k_1,k_2)\).

Now we are going to show by mathematical induction the validity of the relations
\[
\begin{align*}
(\Delta_{u(k)}(k_m - 1))^{y} & \geq u(k) > 0 \text{ for } k \in N_1(k_m + 1,k_m + 21), \quad \text{(21)}
\end{align*}
\]
where \(w_m\) is defined by the recurrence relation (14).

We can easily check that the case \(m = 0\) is covered by (17) and (18). Hence we have to show the formulas (19, 20, 21) for \(m+1\) instead of \(m\), assuming their validity for \(0,1,L,m\). Since \(w_m + 1 > 0\), therefore by (14), \(w_m > \eta_m\), and by (19) and (20) we have
\[
\begin{align*}
(\Delta_{u(k)}(k_m + 1))^{y} & \geq u(k) > 0 \text{ for } k \in N_1(k_m + 1,k_m + 21), \quad \text{(21)}
\end{align*}
\]

or equivalently
\[
\begin{align*}
\frac{\beta_m^{y}}{w_m - \eta_m} (\Delta_{u(k)}(k_m + 1))^{y} & \geq \left( \sum_{i=0}^{m+1} \beta_i \Delta_{u(k)}(k_i - 1) \right)^{y}.
\end{align*}
\]

Hence,
\[
\begin{align*}
\left( \beta_{m+1}^{y} + \frac{\beta_m^{y}}{w_m - \eta_m} \right) (\Delta_{u(k)}(k_m + 1))^{y} & \geq \left( \sum_{i=0}^{m+1} \beta_i \Delta_{u(k)}(k_i - 1) \right)^{y} + \left( \beta_{m+1} \Delta_{u(k)}(k_m + 1) \right)^{y}.
\end{align*}
\]

By Lemma 4, for \(0 < \gamma \leq 1\), we get
\[
\begin{align*}
\left( \beta_{m+1}^{y} + \frac{\beta_m^{y}}{w_m - \eta_m} \right) (\Delta_{u(k)}(k_m + 1))^{y} & \geq \left( \sum_{i=0}^{m+1} \beta_i \Delta_{u(k)}(k_i - 1) \right)^{y} + \left( \beta_{m+1} \Delta_{u(k)}(k_m + 1) \right)^{y}.
\end{align*}
\]
Oscillation and Nonoscillation Criteria for Generalized Second Order Quasilinear $\alpha$-Difference Equations

$|\Delta_{\alpha(t)} u(k - 1)|^{\alpha - 1} \Delta_{\alpha(t)} u(k - 1)$ is decreasing. Hence, for any $n, l \in \mathbb{N}_1 (k_{m+1}, k_{m+2})$ and $n > 1$, we have that $|\Delta_{\alpha(t)} u(n - 1)|^{\alpha - 1} \Delta_{\alpha(t)} u(n - 1)$ $< |\Delta_{\alpha(t)} u(l - 1)|^{\alpha - 1} \Delta_{\alpha(t)} u(l - 1)$.

If $\Delta_{\alpha(t)} u(n - 1) > 0$ and $\Delta_{\alpha(t)} u(l - 1) > 0$, then

$\Delta_{\alpha(t)} u(n - 1) < \Delta_{\alpha(t)} u(l - 1)$. If $\Delta_{\alpha(t)} u(n - 1) < 0$ and

$\Delta_{\alpha(t)} u(l - 1) < 0$ then $(-\Delta_{\alpha(t)} u(n - 1))^{\alpha - 1} > (-\Delta_{\alpha(t)} u(l - 1))^{\alpha - 1}$,

which implies $\Delta_{\alpha(t)} u(k - 1) < \Delta_{\alpha(t)} u(l - 1)$. If $\Delta_{\alpha(t)} u(n - 1) < 0$ and $\Delta_{\alpha(t)} u(l - 1) > 0$, clearly,

$\Delta_{\alpha(t)} u(k - 1) < \Delta_{\alpha(t)} u(l - 1)$. Therefore, $\Delta_{\alpha(t)} u(k - 1)$ is decreasing on $\mathbb{N}_1 (k_{m+1}, k_{m+2})$. By Lemma 1, for $i \in \mathbb{N}_1 (k_{m+1}, k_{m+2})$, we have

$u(i) - u(k_{m+1} - 1) \leq \Delta_{\alpha(t)} u(c - 1)$

$\leq \Delta_{\alpha(t)} u(k_{m+1} - 1), c \in \mathbb{N}_1 (k_{m+1}, i - 1)$.

Summing (2) from $k_{m+1}$ to $k_{m+2} - 1$, we get

$\alpha(\Delta_{\alpha(t)} u(k_{m+1} - 1))^{\alpha - 1} - |\Delta_{\alpha(t)} u(k_{m+2} - 1))^{\alpha - 1} \Delta_{\alpha(t)} u(k_{m+2} - 1)$

$- (\alpha - 1) \sum_{i = k_{m+1}}^{k_{m+2} - 1} |\Delta_{\alpha(t)} u(i + j)|^{\alpha - 1} \Delta_{\alpha(t)} u(i + j)$

$\leq \sum_{i = k_{m+1}}^{k_{m+2} - 1} p(i + j) u(i + j)$

$\leq \sum_{i = k_{m+1}}^{k_{m+2} - 1} p(i + j)$

$\left[u(i + j - 1) + \Delta_{\alpha(t)} u(i + j - 1)(i + j - k_{m+1} + 1)\right]^{\alpha - 1}$

$\leq \left[u(k_{m+1} - 1) + \Delta_{\alpha(t)} u(k_{m+1} - 1)(k_{m+2} - k_{m+1})\right]^{\alpha - 1}$

$\sum_{i = k_{m+1}}^{k_{m+2} - 1} p(i + j)$.

By Lemma 2,

$u(k_{m+1} - 1) \leq \sum_{j = 0}^{m} u(k_{j+1} - 1) - u(k_{j+1} - 1)$

$\leq u(k_{m+1} - 1) + \Delta_{\alpha(t)} u(k_{m+1} - 1)(k_{m+2} - k_{m+1})$

where $n_j \leq c \leq n_{j+1} - 2l$. Hence

$u(k_{m+1} - 1) + \Delta_{\alpha(t)} u(k_{m+1} - 1)(k_{m+2} - k_{m+1})$
\[ (s_{m+2} - a_{m+2})^\gamma \sum_{i=0}^{s_{m+1}-1} p(\bar{i} + j) > 1. \]

Contradicting the assumption \( \eta_{m+2} < 1 \). Thus, \( u(k) > 0 \) for \( k \in \mathbb{N}_1 (k_{m+2}, k_{m+3}) \), which together with (22) and (23), completes the induction step. In this way, we have shown that in (21) the relation \( u(k) > 0 \) holds for all \( k \in \mathbb{N}_1 (k_0) \), which complete the proof of Theorem 5.

**Theorem 6** Assume that \( 1 \leq \gamma < \infty \). Further suppose the coefficient \( p(k) \) in (2) has the property

\[ (k_{m+1} - k_m)^\gamma \sum_{i=k_m}^{k_{m+1}-1} p(\bar{i} + j) \geq \eta_m, \eta_m > 0 \text{ for } m \in \mathbb{N}, \]  

and the recurrence relation

\[ y(m+1) = \frac{\eta_{m+1}}{\eta_m} \theta_m \left( \frac{y(m)}{1 - y(m)} + \eta_m \right), m \in \mathbb{N} \setminus \{1\}, y_0 = 1, \]  

has no solution such that \( 0 < y(m) < 1 \) for all \( m = 1, 2, \ldots \), where \( \theta_m = (\beta_m / \beta_{m+1})^\gamma \). Then (2) is oscillatory.

**Proof.** Without loss of generality, assume that (2) has an eventually positive solution \( u(k) \). We may suppose that this holds already for \( k \in \mathbb{N}_1 (k_0) \). Then, by (2),

\[ |\Delta_{a(i)}(u(k_1) - 1)\]  

\[ \Delta_{a(i)}(u(k_1) - 1) \leq 0, \]  

hence

\[ |\Delta_{a(i)}(u(k_1) - 1)|^{-1} \Delta_{a(i)}(u(k_1) - 1) \]  

\[ \Delta_{a(i)}(u(k_1) - 1) \]  

is decreasing. We see that either

(i). \( \Delta_{a(i)}(u(k_1) - 1) > 0 \) for \( k \in \mathbb{N}_1 (k_1) \) or

(ii). there exists \( k^2 > k^1 \) such that \( \Delta_{a(i)}(u(k_1) - 1) < 0 \) on \( \mathbb{N}_1 (k_2) \).

If (ii) holds, then it follows \( \Delta_{a(i)}(u(k_1) - 1) \geq 0 \) which implies that \( \Delta_{a(i)}(u(k_1) - 1) \) decreases. This and \( \Delta_{a(i)}(u(k_1) - 1) \) on \( \mathbb{N}_1 (k_2) \) imply that there exists \( k^3 > k^2 \) such that \( u(k_1) \leq 0 \) for \( k \in \mathbb{N}_1 (k_3) \). This contradicts \( u(k_1) > 0 \). Thus, (i) holds and \( \Delta_{a(i)}(u(k_1) - 1) \) decreases for \( k \in \mathbb{N}_1 (k_1) \).

Summing (2) from \( k_m \) to \( k_{m+1} - 1 \), we get

\[ a(\Delta_{a(i)}(u(k_m) - 1)) = (\alpha - 1) \sum_{i=k_m}^{k_{m+2}} (\Delta_{a(i)}(u(\bar{i} + j)) + \sum_{i=k_m}^{k_{m+1}} p(\bar{i} + j)u(\bar{i} + j) \]

\[ \geq u(\bar{k}) \sum_{i=k_m}^{k_{m+1}} p(\bar{i} + j) > 0, \]

hence

\[ \Delta_{a(i)}(u(k_0) - 1) \Delta_{a(i)}(u(k_1) - 1) > 0, \]

\[ \Delta_{a(i)}(u(k_m) - 1) > 0 \]

and

\[ \Delta_{a(i)}(u(k_{m+1}) - 1) > 0, \]

\[ \Delta_{a(i)}(u(k_{m+1}) - 1) > 0, \]

moreover, by Lemma 1 and (12),

\[ u(k_m) = u(k_0) + \sum_{s=1}^{m} [u(k_s) - u(k_{s-1})] \]

\[ \geq (k_1 - k_0) \sum_{i=1}^{m} (\beta_i - 1) \Delta_{a(i)}(u(k_{s-1}). \]

Consequently by (24),

\[ \alpha(\Delta_{a(i)}(u(k_m) - 1)) - (\Delta_{a(i)}(u(k_{m+1}) - 1))^{-1} \]

\[ -(\alpha - 1) \sum_{s=1}^{m} (\Delta_{a(i)}(u(\bar{i} + j)) \]

\[ \geq (k_1 - k_0) \sum_{i=k_m}^{k_{m+1}} p(\bar{i} + j) \left( \sum_{s=1}^{m} \beta_i - 1 \Delta_{a(i)}(u(k_{s-1}) \right) \]

\[ \geq \eta_m \left( \sum_{s=1}^{m} \beta_i - 1 \Delta_{a(i)}(u(k_{s-1}) \right) \]

This implies the following two relations:

\[ \eta_m \left( \sum_{s=1}^{m} \beta_i - 1 \Delta_{a(i)}(u(k_{s-1}) \right) < (\Delta_{a(i)}(u(k_{s-1}) \right) \]

\[ \Delta_{a(i)}(u(k_{m+1}) - 1) \]

\[ \Delta_{a(i)}(u(k_{m+1}) - 1) \]

Let the real valued function \( y(0), y(1), \ldots \) be defined by the recurrence relation (25) as long as it is possible. Then we claim that

\[ y(m)(\Delta_{a(i)}(k_m) - 1) \]

\[ \leq \frac{\eta_m}{\beta_1} \left( \sum_{s=1}^{m} \beta_i - 1 \Delta_{a(i)}(u(k_{s-1}) \right) \]

\[ 0 < y(m) < 1 \text{ for } m = 1, 2, \ldots \]

The proof of this relation proceeds on mathematical induction.

It is easy to show that (29) holds for \( m = 0 \) and \( m = 1 \). Suppose that (29) is proved already for \( 0, 1, \ldots, m \). Then we are going to show it for \( m + 1 \).

By (27) and (29), we have
Oscillation and Nonoscillation Criteria for Generalized Second Order Quasilinear $\alpha$ -Difference Equations

$$\Delta_{\alpha(t)}u(k_m-1)^\gamma - \frac{\eta_m}{\beta_m^\gamma} \left( \sum_{s=1}^{m} \beta_{s-1} \Delta_{\alpha(t)}u(k_s-1) \right)^\gamma \leq (1 - y(m))\left( \Delta_{\alpha(t)}u(k_m-1) \right)^\gamma.$$ 

and by (28) and (29),

$$\left( \Delta_{\alpha(t)}u(k_{m+1}-1) \right)^\gamma < (1 - y(m))\left( \Delta_{\alpha(t)}u(k_m-1) \right)^\gamma$$

$$\leq \frac{1 - y(m)}{y(m)} y(m) \left( \Delta_{\alpha(t)}u(k_m-1) \right)^\gamma$$

or

$$\frac{y(m)}{1 - y(m)} \eta_m^\gamma \left( \Delta_{\alpha(t)}u(k_m-1) \right)^\gamma < \left( \sum_{s=1}^{m} \beta_{s-1} \Delta_{\alpha(t)}u(k_s-1) \right)^\gamma.$$ 

By adding $\beta_m^\gamma \left( \Delta_{\alpha(t)}u(k_{m+1}-1) \right)^\gamma$ to both sides and using Lemma 3, we obtain

$$\frac{\beta_m^\gamma}{\eta_m} \left( \Delta_{\alpha(t)}u(k_{m+1}-1) \right)^\gamma \leq \left( \sum_{s=1}^{m} \beta_{s-1} \Delta_{\alpha(t)}u(k_s-1) \right)^\gamma + \beta_m^\gamma \left( \Delta_{\alpha(t)}u(k_m-1) \right)^\gamma.$$ 

By multiplying the resulting inequality by $\eta_{m+1}/\beta_{m+1}^\gamma$, we get

$$y(m+1)\left( \Delta_{\alpha(t)}u(k_{m+1}-1) \right)^\gamma < \eta_{m+1}^\gamma \left( \sum_{s=1}^{m+1} \beta_{s-1} \Delta_{\alpha(t)}u(k_s-1) \right)^\gamma$$

$$< (\Delta_{\alpha(t)}u(k_{m+1}-1))^{\gamma}.$$ 

(30)

Where $y(m+1)$ is given by (25). This proves (29) for $m + 1$ because $y(m+1) > 0$ follows from (25), and $y_{m+1} < 1$ follows from (30) and from (27) applied to $m + 1$ instead of $m$. Hence the induction step is completed and (29) holds for all $m \in \mathbb{N}_1$. All this was made under the indirect assumption that there exists a solution $u(k)$ with $u(k) > 0$ for all sufficiently large $k$. According to the assumptions of Theorem 6, the recurrence relation (25) has no solution for all $m$, and this contradiction proves our theorem.

Remark 7. For the case where $\gamma = 1$, we observe that in [10, Theorems 5 and 6] the real valued function $k_m$ is $2^m k_0$, therefore we have $\beta_m = 2^m$ and $\theta_m = 1/2$, i.e., $\theta_m$ is fixed for all $m \in \mathbb{N}_1$.

IV. CONCLUSION

New oscillation and non oscillation theorems are obtained for the generalized second order quasilinear $\alpha$ -difference equation.

REFERENCES