Oscillation and Nonoscillation Criteria for Generalized Second Order Quasilinear α-Difference Equations

V.Srimanju, SK.Khadar Babu

Abstract: ***New oscillation and nonoscillation theorems are obtained for the generalized second order quasilinear*** α***-difference equation***

$$\Delta_{\alpha(\ell)}\left(|\Delta_{\alpha(\ell)}u(k-\ell)|^{\gamma-1}\Delta_{\alpha(\ell)}u(k-\ell)\right)+p(k)|u(k)|^{\gamma-1}u(k)=0,$$

where $\gamma>0$ ***is a constant and*** $p(k)$ ***is a real valued function with*** $p(k)\geq 0$.

Keywords: Generalized difference equation, Quasilinear, Oscillation.

I. INTRODUCTION

Difference equations represent a fascinating mathematical area on its own as well as a rich field of the applications in such diverse disciplines as population dynamics, operations research, ecology, economics, biology etc. For general background as difference equations with many examples from diverse fields, one can refer to [1]. The theory of difference equations is based on the operator Δ defined as $\Delta u(k)=u(k+1)-u(k), k\in[0,\infty)$.

Even though many authors [1], [9] have suggested the definition of Δ as

$$\Delta u(k)=u(k+\ell)-u(k),\quad \ell\in(0,\infty), \tag{1}$$

no significant progress took place on this line. But recently, when we took up the definition of Δ as given in (1), the theory of difference equations are developed in a different direction. For convenience, we labelled the operator Δ defined by (1) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results in number theory were obtained. By extending theory of Δ_ℓ to complex function, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were established for the solutions of difference equations involving Δ_ℓ. The results obtained can be found in [3-5].

Jerzy Popenda, et.al., [8], while discussing the behavior of solutions of a particular type of difference equation, defined Δ_α as $\Delta_\alpha u(k)=u(k+1)-\alpha u(k)$. This definition of Δ_α is being ignored for a long time.

Recently, in [6], they generalized the definition of Δ_α by $\Delta_{\alpha(\ell)}$ defined as $\Delta_{\alpha(\ell)}u(k)=u(k+\ell)-\alpha u(k)$ for the real valued function $u(k)$ and $\ell\in(0,\infty)$ and also obtained the solutions of certain types of generalized α-difference equations, in particular, the generalized Clairauts α-difference equation, generalized Euler α-difference equation and the generalized α-Bernoulli polynomial $B_{\alpha(n)}(k,\ell)$, which is a solution of the α-difference equation $u(k+\ell)-\alpha u(k)=nk^{n-1},\ for\ n\in\mathbb{N}(1)$.

In this paper we consider the generalized second order quasilinear α-difference equation

$$\Delta_{\alpha(\ell)}\left(|\Delta_{\alpha(\ell)}u(k-\ell)|^{\gamma-1}\Delta_{\alpha(\ell)}u(k-\ell)\right)+p(k)|u(k)|^{\gamma-1}u(k)=0,\ k\geq k^0 \tag{2}$$

where the $\Delta_{\alpha(\ell)}$ is defined as $\Delta_{\alpha(\ell)}u(k)=u(k+\ell)-\alpha u(k)$ is a constant, and $p(k)$ is a real valued function with $p(k)\geq 0$ for $k\geq k^0$.

Oscillation and nonoscillation of equation (2) have investigated intensively, see, for example [1, 2]. In this paper, we obtain new oscillation and nonoscillation criteria for equation (2) which generalize the known results on second order linear difference equations.

A solution $u(k)$ of (2) is said to be oscillatory if the terms $u(k)$ of the real valued function $u(k)$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

II. SOME LEMMAS

Throughout, we shall use the following notations: $\mathbb{N}_\ell=\{0,\ell,2\ell,\ldots\}$, $\mathbb{N}_\ell(a)=\{a,a+\ell,a+2\ell\ldots\}$, where $a\in\mathbb{N}_\ell$, and $\mathbb{N}_\ell(a,b)=\{a,a+\ell,\ldots,b\}$, where $b\in\mathbb{N}_\ell(a)$, $j=k-\left[\frac{k}{\ell}\right]\ell$.

Lemma 1. *(Discrete mean value theorem [1]). Suppose that* $u(k)$ *is defined on* $\mathbb{N}_\ell(a,b)$*.Then, there exists a* $c\in\mathbb{N}_\ell(a+\ell,b-\ell)$ *such that*

$$\Delta_{\alpha(\ell)}x(c)\leq\frac{x(b)-x(a)}{b-a}\leq\Delta_{\alpha(\ell)}x(c-\ell) \tag{3}$$

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or

$$\Delta_{\alpha(l)}x(c-1)\le \frac{x(b)-x(a)}{b-a}\le \Delta_{\alpha(l)}x(c). \qquad (4)$$

Lemma 2 *Let $u(k)$ is a nontrivial solution of equation (2). Assume that $u(k)>0$ for $k\in\mathbb{N}_l(a+1,b)$ and that*

$$\Delta_{\alpha(l)}u(a-1)\ge 0,\ \Delta_{\alpha(l)}u(a)<0, u(b-1)>0, u(b)\le 0, \quad (5)$$

or

$$u(a-1)\le 0,\ u(a)>0, \Delta_{\alpha(l)}u(b-2l)>0, \Delta_{\alpha(l)}u(b-1)\le 0, \qquad (6)$$

where $a,b\in\mathbb{N}_l(k^0)$ and $a<b$. Then the relation

$$(b-a)^{\gamma}\sum_{i=a}^{b-1}p(il+j)>1, \qquad (7)$$

holds.

Proof. We assume that (5) holds. If (6) holds, then the proof is similar. Let

$$r=\max\{s\mid \Delta_{\alpha(l)}u(s-1)\ge 0, \Delta_{\alpha(l)}u(s)<0, s\in\mathbb{N}_l(a,b-1)\}.$$

Since $\Delta_{\alpha(l)}u(b-1)<0$, then $\Delta_{\alpha(l)}u(k-1)<0$ for $k\in\mathbb{N}_l(r+1,b)$. Since $u(k)>0$ for $k\in\mathbb{N}_l(a+1,b)$, it follows that, by (2), $\Delta_{\alpha(l)}(|\Delta_{\alpha(l)}u(k-1)|^{\gamma-1}\Delta_{\alpha(l)}u(k-1))\le 0$. Hence, $\Delta_{\alpha(l)}(-\Delta_{\alpha(l)}u(k-1))^{\gamma}\ge 0$, for $k\in\mathbb{N}_l(r+1,b)$, which implies $\Delta_{\alpha(l)}u(k-1)$ is nonincreasing and $\Delta_{\alpha(l)}u(k-1)<0$ for $k\in\mathbb{N}_l(r+1,b)$. Hence, $u(k)$ is decreasing. By (4) of Lemma 1 and (5), we get

$$\frac{u(r)}{b-r}<\frac{u(r)-u(b)}{b-r}\le -\Delta_{\alpha(l)}u(b-1). \qquad (8)$$

On the other hand, since $\Delta_{\alpha(l)}u(r-1)\ge 0$, sum of equation (2) between $k=r$ and $k=b-1$ shows that

$$\left(-\Delta_{\alpha(l)}u(b-1)\right)^{\gamma}\le \alpha\,|\Delta_{\alpha(l)}u(r-1)|^{\gamma-1}\Delta_{\alpha(l)}u(r-1)$$

$$-(\alpha-1)\sum_{i=r}^{b-2}|\Delta_{\alpha(l)}u(il+j)|^{\gamma-1}\Delta_{\alpha(l)}u(il+j)$$

$$-|\Delta_{\alpha(l)}u(b-1)|^{\gamma-1}\Delta_{\alpha(l)}u(b-1)=\sum_{i=r}^{b-1}p(il+j)u^{\gamma}(il+j). \qquad (9)$$

Since $u(i)$ is decreasing, equation (9) implies

$$\left(-\Delta_{\alpha(l)}u(b-1)\right)^{\gamma}<u^{\gamma}(r)\sum_{i=r}^{b-1}p(il+j). \qquad (10)$$

From equation (8) and equation (10), we obtain

$$(b-a)^{\gamma}\sum_{i=a}^{b-1}p(il+j)\ge (b-r)^{\gamma}\sum_{i=a}^{b-1}p(il+j)>1.$$

This completes the proof of Lemma 2.

Lemma 3 *[7] Assume that $a\ge 0, b\ge 0$. If $\gamma\ge 1$, then $a^{\gamma}+b^{\gamma}\le (a+b)^{\gamma}$.*

Lemma 4 *[7] Assume that $a\ge 0, b\ge 0$. If $0<\gamma\le 1$, then $a^{\gamma}+b^{\gamma}\ge (a+b)^{\gamma}$.*

III. MAIN RESULTS

Let the real valued function k_m be given such that

$$k^0\le k_0<k_1<\ldots<k_m<\ldots, k_m\to\infty \text{ as } m\to\infty, \qquad (11)$$

and introduce the notation

$$\beta_m=\frac{k_{m+1}-k_m}{k_1-k_0} \text{ for } m\in\mathbb{N}_1. \qquad (12)$$

Clearly, we have $\beta_0=1, \beta_m>0$, and $\sum_{m=0}^{\infty}\beta_m=\infty$. Our first result (on the nonoscillation) is the following.

Theorem 5 *Assume that $0<\gamma\le 1$. Further suppose the coefficient $p(k)$ in equation (2) has the property*

$$(k_{m+1}-k_m)^{\gamma}\sum_{i=k_m}^{k_{m+1}-1}p(il+j)\le \eta_m, 0\le\eta_m<1 \text{ for } m\in\mathbb{N}(1), \qquad (13)$$

and there exists the infinite real valued function w_m satisfying the recurrence relation

$$w_{m+1}=\frac{w_m-\eta_m}{\theta_m+w_m-\eta_m}, m\in\mathbb{N}(1), w_0=1, \qquad (14)$$

such that $0<w_m<1$ for $m=1,2,\ldots$, where $\theta_m=\left(\frac{\beta_m}{\beta_m+1}\right)_1^{\gamma}$. Then equation (2) is nonoscillatory.

Proof. We are going to show that the solution $u(k)$ of (2) subject to the conditions $u(k_0-1)\le 0, u(k_0)>0$ remains positive for all $k\in\mathbb{N}_1(k_0)$.

By equation (13), $(k_1-k_0)^{\gamma}\sum_{i=k_0}^{k_1-1}p(il+j)<1$. Therefore by Lemma 2 we have $\Delta_{\alpha(l)}u(k-1)>0$ on $\mathbb{N}_1(k_0+1,k_1)$. In fact, if not, assume that there exists a $k^*\in\mathbb{N}_1(k_0+1,k_1)$ such that $\Delta_{\alpha(l)}u(k^*-2l)>0, \Delta_{\alpha(l)}u(k^*-1)\le 0$, then by Lemma 2, we have $(k_1-k_0)^{\gamma}\sum_{i=k_0}^{k_1-1}p(il+j)>1$ which contradicts

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$(k_1-k_0)^{\gamma}\sum_{i=k_0}^{k_1-1}p(il+j)<1$. Therefore $\Delta_{\alpha(1)}u(k-1)>0$ on $\mathbb{N}_1(k_0+1,k_1)$. Hence $u(k)>0$, $k\in\mathbb{N}_1(k_0,k_1)$ and that $\Delta_{\alpha(1)}\left(\Delta_{\alpha(1)}u(k-1)\right)^{\gamma}\le 0$ for $k\in\mathbb{N}_1(k_0,k_1)$ which means that $\Delta_{\alpha(1)}u(k-1)$ is nonincreasing on $\mathbb{N}_1(k_0+1,k_1)$. Summing equation (2) from k_0 to k_1-1, we obtain

$$(\Delta_{\alpha(1)}u(k_0-1))^{\gamma}-(\Delta_{\alpha(1)}u(k_1-1))^{\gamma}$$
$$+(\alpha-1)\sum_{i=k_0}^{k_1-2}(\Delta_{\alpha(1)}u(il+j))^{\gamma}$$
$$=\sum_{i=k_0}^{k_1-1}p(il+j)u^{\gamma}(il+j). \quad (15)$$

By Lemma 1, we get

$$u(i)\le u(i)-u(k_0-1)$$
$$\le (i+1-k_0)\Delta_{\alpha(1)}u(k_0-1)\text{ for } i\in(k_0,k_1-1). \quad (16)$$

From (15) and (16), we have

$$\alpha(\Delta_{\alpha(1)}u(k_0-1))^{\gamma}-(\Delta_{\alpha(1)}u(k_1-1))^{\gamma}$$
$$-(\alpha-1)\sum_{i=k_0}^{k_1-2}(\Delta_{\alpha(1)}u(il+j))^{\gamma}$$
$$\le(\Delta_{\alpha(1)}u(k_0-1))^{\gamma}(k_1-k_0)^{\gamma}\sum_{i=k_0}^{k_1-1}p(il+j)$$
$$\le \rho_0(\Delta_{\alpha(1)}u(k_0-1))^{\gamma}$$

or

$$(\Delta_{\alpha(1)}u(k_1-1))^{\gamma}\ge\alpha(\Delta_{\alpha(1)}u(k_0-1))^{\gamma}-\rho_0(\Delta_{\alpha(1)}u(k_0-1))^{\gamma}$$
$$=(\alpha-\rho_0)(\Delta_{\alpha(1)}u(k_0-1))^{\gamma}. \quad (17)$$

Now we claim that $u(k)$ is fixed signed on $\mathbb{N}_1(k_1,k_2)$. If not, then there exists a $s_1\in\mathbb{N}_1(k_1-1,k_2)$ such that

$$u(s_1-1)>0,\ u(s_1)\le 0.$$

In view of $\Delta_{\alpha(1)}u(k-1)>0$ on $\mathbb{N}_1(k_0+1,k_1)$, by Discrete Rolleâ€™s Theorem (see [1]), there exists a $a_1\in\mathbb{N}_1(k_0,s_1-1)$ such that

$$\Delta_{\alpha(1)}u(a_1-1)\ge 0\text{and }\Delta_{\alpha(1)}u(a_1)<0.$$

Clearly, $a_1\in\mathbb{N}_1(k_1,k_2-1)$. Using Lemma 3, we obtain

$$\eta_1\ge(k_2-k_1)^{\gamma}\sum_{i=k_1}^{k_2-1}p(il+j)\ge(s_i-a_1)^{\gamma}\sum_{i=a_1}^{s_1-1}p(il+j)>1.$$

Contradicting the assumption $\eta_1<1$. Thus we have shown that

$$u(k)>0\text{ for }k\in\mathbb{N}_1(k_1,k_2). \quad (18)$$

Now we are going to show by mathematical induction the validity of the relations

$$(\Delta_{\alpha(1)}u(k_m-1))^{\gamma}\ge\frac{w_m}{\beta_m^{\gamma}}\left(\sum_{s=k_s}^{m}\beta_s\Delta_{\alpha(1)}u(il+j-1)\right)^{\gamma}, \quad (19)$$

$$(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}\ge$$
$$(\Delta_{\alpha(1)}u(k_m-1))^{\gamma}-\frac{\eta_m}{\beta_m^{\rho}}\left(\sum_{s=0}^{m}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}, \quad (20)$$

$$u(k)>0\text{ for }k\in\mathbb{N}_1(k_m+1,k_m+2l), \quad (21)$$

where w_m is defined by the recurrence relation (14).

We can easily check that the case $m=0$ is covered by (17) and (18). Hence we have to show the formulas (19, 20, 21) for $m+1$ instead of m, assuming their validity for $0,1,\ldots,m$. Since $w_m+1>0$, therefore by (14), $w_m>\eta_m$, and by (19) and (20) we have

$$(\Delta_{\alpha(1)}(u(k_{m+1}-1))^{\gamma}\ge\frac{w_m-\eta_m}{\beta_m^{\gamma}}\left(\sum_{s=0}^{m}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}>0$$

or equivalently

$$\frac{\beta_m^{\gamma}}{w_m-\eta_m}(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}\ge\left(\sum_{s=0}^{m}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}.$$

Hence,

$$\left(\beta_{m+1}^{\gamma}+\frac{\beta_m^{\gamma}}{w_m-\eta_m}\right)(\Delta_{\alpha(1)}(u(k_{m+1}-1))^{\gamma}$$
$$\ge\left(\sum_{s=0}^{m}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}+(\beta_{m+1}\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}.$$

By Lemma 4, for $0<\gamma\le 1$, we get

$$\left(\beta_{m+1}^{\gamma}+\frac{\beta_m^{\gamma}}{w_m-\eta_m}\right)(\Delta_{\alpha(1)}(u(k_{m+1}-1))^{\gamma}$$
$$\ge\left(\sum_{s=0}^{m+1}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma},$$

which is equivalent to

$$(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}\ge\frac{w_{m+1}}{\beta_{m+1}^{\gamma}}\left(\sum_{s=0}^{m+1}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}. \quad (22)$$

By (2) and (21), we get that $\Delta_{\alpha(1)}(|\Delta_{\alpha(1)}u(k-1)|^{\gamma-1}\Delta_{\alpha(1)}u(k-1))\le 0$ which implies

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$|\Delta_{\alpha(1)}u(k-1)|^{\gamma-1}\Delta_{\alpha(1)}u(k-1)$ is decreasing. Hence, for any $n,l\in\mathbb{N}_1(k_{m+1},k_{m+2})$ and $n>l$, we have that $|\Delta_{\alpha(1)}u(n-1)|^{\gamma-1}\Delta_{\alpha(1)}u(n-1)$ $<|\Delta_{\alpha(1)}u(l-1)|^{\gamma-1}\Delta_{\alpha(1)}u(l-1).$

If $\Delta_{\alpha(1)}u(n-1)>0$ and $\Delta_{\alpha(1)}u(l-1)>0$, then $\Delta_{\alpha(1)}u(n-1)<\Delta_{\alpha(1)}u(l-1)$. If $\Delta_{\alpha(1)}u(n-1)<0$ and $\Delta_{\alpha(1)}u(l-1)<0$ then $(-\Delta_{\alpha(1)}u(n-1))^{\gamma}>(-\Delta_{\alpha(1)}u(l-1))^{\gamma}$, which implies $\Delta_{\alpha(1)}u(k-1)<\Delta_{\alpha(1)}u(l-1)$. If $\Delta_{\alpha(1)}u(k-1)<0$ and $\Delta_{\alpha(1)}u(l-1)>0$, clearly, $\Delta_{\alpha(1)}u(k-1)<\Delta_{\alpha(1)}u(l-1)$. Therefore, $\Delta_{\alpha(1)}u(k-1)$ is decreasing on $\mathbb{N}_1(k_{m+1},k_{m+2})$. By Lemma 1, for $i\in\mathbb{N}_1(k_{m+1},k_{m+2}-1)$, we have

$$\frac{u(i)-u(k_{m+1}-1)}{i-k_{m+1}+1}\le\Delta_{\alpha(1)}u(c-1)$$
$$\le\Delta_{\alpha(1)}u(k_{m+1}-1),c\in\mathbb{N}_1(k_{m+1},i-1).$$

Summing (2) from k_{m+1} to $k_{m+2}-1$, we get

$$\alpha(\Delta_{\alpha(1)}u((k_{m+1}-1))^{\gamma}-|\Delta_{\alpha(1)}u(k_{m+2}-1))|^{\gamma-1}\Delta_{\alpha(1)}u(k_{m+2}-1)$$
$$-(\alpha-1)\left[\sum_{i=k_{m+1}}^{k_{m+2}}|\Delta_{\alpha(1)}u(il+j)|^{\gamma-1}\Delta_{\alpha(1)}u(il+j)\right]$$
$$=\sum_{i=k_{m+1}}^{k_{m+2}-1}p(il+j)u^{\gamma}(il+j)$$
$$\le\sum_{i=k_{m+1}}^{k_{m+2}-1}p(il+j)$$
$$\left[u(il+j-1)+\Delta_{\alpha(1)}u(il+j-1)(il+j-k_{m+1}+1)\right]^{\gamma}$$
$$\le\left[u(k_{m+1}-1)+\Delta_{\alpha(1)}u(k_{m+1}-1)(k_{m+2}-k_{m+1})\right]^{\gamma}$$
$$\sum_{i=k_{m+1}}^{k_{m+2}-1}p(il+j).$$

By Lemma 2,

$$u(k_{m+1}-1)\le\sum_{s=0}^{m}\left[u(k_{s+1}-1)-u(k_s-1)\right]$$
$$\le\sum_{s=0}^{m}(k_{s+1}-k_s)\Delta_{\alpha(1)}u(c-1)$$
$$\le\sum_{s=0}^{m}(k_{s+1}-k_s)\Delta_{\alpha(1)}u(k_s-1)=(k_1-k_0)\sum_{s=0}^{m}\beta_s\Delta_{\alpha(1)}u(k_s-1),$$

where $n_s\le c\le n_{s+1}-2l$. Hence

$$u(k_{m+1}-1)+\Delta_{\alpha(1)}u(k_{m+1}-1)\left(k_{m+2}-k_{m+1}\right)$$
$$\le(k_1-k_0)\sum_{s=0}^{m}\beta_s\Delta_{\alpha(1)}u(k_s-1)+(k_1-k_0)\beta_{m+1}\Delta_{\alpha(1)}u(k_{m+1}-1)$$
$$=(k_1-k_0)\sum_{s=0}^{m+1}\beta_s\Delta_{\alpha(1)}u(k_s-1)$$

Therefore

$$(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}-|\Delta_{\alpha(1)}u(k_{m+2}-1)|^{\gamma-1}\Delta_{\alpha(1)}u(k_{m+2}-1)$$
$$\le(k_1-k_0)^{\gamma}\left(\sum_{s=0}^{m+1}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}\sum_{i=k_{m+1}}^{k_{m+2}-1}p(il+j)$$
$$=\left(\frac{k_{m+2}-k_{m+1}}{\beta_{m+1}^{\gamma}}\right)^{\gamma}\left(\sum_{s=0}^{m+1}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}\sum_{i=k_m+1}^{k_{m+2}-1}p(il+j)$$
$$\le\frac{\eta_{m+1}}{\beta_{m+1}^{\gamma}}\left(\sum_{s=0}^{m+1}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}$$

or

$$|\Delta_{\alpha(1)}u((k_{m+2}-1)|^{\gamma-1}\Delta_{\alpha(1)}u(k_{m+2}-1)$$
$$\ge\left(\Delta_{\alpha(1)}u(k_{m+1}-1)\right)^{\gamma}-\frac{\eta_{m+1}}{\beta_{m+1}^{\gamma}}\left(\sum_{s=0}^{m+1}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}. \quad (23)$$

Here $w_{m+2}>0$ implies $w_{m+1}>\eta_{m+1}$. By (22) and (23), we get

$$|\Delta_{\alpha(1)}u(k_{m+2}-1)|^{\gamma-1}\Delta_{\alpha(1)}u(k_{m+2}-1)$$
$$\ge\frac{w_{m+1}-\eta_{m+1}}{\beta_{m+1}^{\gamma}}\left(\sum_{s=0}^{m+1}\beta_s\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}>0,$$

which implies $\Delta_{\alpha(1)}u(k_{m+2}-1)>0$. Now we claim that $u(k)>0$ for $k\in\mathbb{N}_1(k_{m+2},k_{m+3})$. If not, then there exists a $t_{m+2}\in\mathbb{N}_1(k_{m+2}+1,k_{m+3})$ such that

$$u(t_{m+2}-1)>0,\ u(t_{m+2})\le 0.$$

In view of $\Delta_{\alpha(1)}u(k_{m+2}-1)>0$, by Discrete Rolles Theorem (see [1]), there exists a $a_{m+2}\in\mathbb{N}_1(k_{m+2},t_{m+2}-1)$ such that

$$\Delta_{\alpha(1)}u(a_{m+2}-1)\ge 0,\ \Delta_{\alpha(1)}u(a_{m+2})<0.$$

By Lemma 2, we obtain

$$\eta_{m+2}\ge\left(k_{m+3}-k_{m+2}\right)^{\gamma}\sum_{i=k_{m+2}}^{k_{m+3}-1}p(il+j)$$

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$$\geq \left(s_{m+2} - a_{m+2}\right)^{\gamma} \sum_{i=a_{m+2}}^{s_{m+2}-1} p(il+j) > 1.$$

Contradicting the assumption $\eta_{m+2} < 1$. Thus, $u(k) > 0$ for $k \in \mathbb{N}_1(k_{m+2}, k_{m+3})$, which together with (22) and (23), completes the induction step. In this way, we have shown that in (21) the relation $u(k) > 0$ holds for all $k \in \mathbb{N}_1(k_0)$, which complete the proof of Theorem 5.

Theorem 6 *Assume that* $1 \leq \gamma < \infty$. *Further suppose the coefficient* $p(k)$ *in (2) has the property*

$$\left(k_{m+1} - k_m\right)^{\gamma} \sum_{r=k_m}^{k_{m+1}-1} p(il+j) \geq \eta_m, \eta_m > 0 \text{ for } m \in \mathbb{N}, \quad (24)$$

and the recurrence relation

$$y(m+1) = \frac{\eta_{m+1}}{\eta_m} \theta_m \left(\frac{y(m)}{1-y(m)} + \eta_m \right), m \in \mathbb{N}_1(1), y_0 = 1, \quad (25)$$

has no solution such that $0 < y(m) < 1$ for all $m = 1, 2, \cdots$, where $\theta_m = (\beta_m / \beta_{m+1})^{\gamma}$. Then (2) is oscillatory.

Proof. Without loss of generality, assume that (2) has an eventually positive solution $u(k)$. We may suppose that this holds already for $k \in \mathbb{N}_1(k_0)$. Then, by (2), $|\Delta_{\alpha(1)} u(k-1)|^{\gamma-1} \Delta_{\alpha(1)} u(k-1) \leq 0$, hence, $|\Delta_{\alpha(1)} u(k-1)|^{\gamma-1} \Delta_{\alpha(1)} u(k-1)$ is decreasing. We see that either

(i). $\Delta_{\alpha(1)} u(k-1) > 0$ for $k \in \mathbb{N}_1(k^1)$ or

(ii). there exists $k^2 > k^1$ such that $\Delta_{\alpha(1)} u(k-1) < 0$ on $\mathbb{N}_1(k^2)$.

If (ii) holds, then it follows $\Delta_{\alpha(1)}(-\Delta_{\alpha(1)} u(k-1))^{\gamma} \geq 0$ which implies that $\Delta_{\alpha(1)} u(k-1)$ decreases. This and $\Delta_{\alpha(1)} u(k-1) < 0$ on $\mathbb{N}_1(k^2)$ imply that there exists $k^3 > k^2$ such that $u(k) \leq 0$ for $k \in \mathbb{N}_1(k^3)$. This contradicts $u(k) > 0$. Thus, (i) holds and $\Delta_{\alpha(1)}(\Delta_{\alpha(1)} u(k-1)^{\gamma} \leq 0$. Hence, $\Delta_{\alpha(1)} u(k-1)$ decreases for $k \in \mathbb{N}_1(k^1)$.

Summing (2) from k_m to $k_{m+1}-1$, we get

$$\alpha(\Delta_{\alpha(1)} u(k_m-1))^{\gamma} - (\Delta_{\alpha(1)} u(k_{m+1}-1))^{\gamma}$$

$$= (\alpha-1) \sum_{i=k_m}^{k_{m+1}-2} (\Delta_{\alpha(1)} u(il+j))^{\gamma} + \sum_{i=k_m}^{k_{m+1}-1} p(il+j) u^{\gamma}(il+j)$$

$$\geq u^{\gamma}(k_m) \sum_{i=k_m}^{k_{m+1}-1} p(il+j) > 0,$$

hence

$$\Delta_{\alpha(1)} u(k_0-1) > \Delta_{\alpha(1)} u(k_1-1) > \cdots$$

$$> \Delta_{\alpha(1)} u(k_m-1) > \Delta_{\alpha(1)} u(k_{m+1}-1) > \cdots > 0, \quad (26)$$

moreover, by Lemma 1 and (12),

$$u(k_m) = u(k_0) + \sum_{s=1}^{m} [u(k_s) - u(k_s-1)]$$

$$> \sum_{s=1}^{m} (k_s - k_{s-1}) \Delta_{\alpha(1)} u(k_s^*)$$

$$\geq (k_1 - k_0) \sum_{s=1}^{m} (\beta_{s-1}) \Delta_{\alpha(1)} u(k_s-1), \text{ where } k_{s-1}+1 \leq k_s^* \leq k_s - 1.$$

Consequently by (24),

$$\alpha(\Delta_{\alpha(1)} u(k_m-1)^{\gamma} - (\Delta_{\alpha(1)} u(k_{m+1}-1))^{\gamma}$$

$$-(\alpha-1) \sum_{s=k_{m+1}}^{k_m+2} (\Delta_{\alpha(1)} u(il+j))^{\gamma}$$

$$> (k_1 - k_0)^{\gamma} \sum_{i=k_m}^{k_{m+1}-1} p(il+j) \left(\sum_{s=1}^{m} \beta_{s-1} \Delta_{\alpha(1)} u(k_s-1) \right)^{\gamma}$$

$$\geq \frac{\eta_m}{\beta_m^{\gamma}} \left(\sum_{s=1}^{m} \beta_{s-1} \Delta_{\alpha(1)} u(k_s-1) \right)^{\gamma}.$$

This implies the following two relations:

$$\frac{\eta_m}{\beta_m^{\gamma}} \left(\sum_{s=1}^{m} \beta_{s-1} \Delta_{\alpha(1)} u(k_s-1) \right)^{\gamma} < (\Delta_{\alpha(1)} u(k_s-1))^{\gamma}, \quad (27)$$

$$(\Delta_{\alpha(1)} u(k_{m+1}-1))^{\gamma}$$

$$< (\Delta_{\alpha(1)} u(k_m-1))^{\gamma} - \frac{\eta_m}{\beta_m^{\gamma}} \left(\sum_{s=0}^{m} \beta_{s-1} \Delta_{\alpha(1)} u(k_s-1) \right)^{\gamma}. \quad (28)$$

Let the real valued function $y(0), y(1), \cdots$ be defined by the recurrence relation (25) as long as it is possible. Then we claim that

$$y(m) \Delta_{\alpha(1)}(k_m-1)^{\gamma}$$

$$\leq \frac{\rho_m}{\beta_m^{\gamma}} \left(\sum_{s=1}^{m} \beta_{s-1} \Delta_{\alpha(1)} u(k_s-1) \right)^{\gamma} \text{ for } m = 0, 1, \cdots,$$

$$0 < y(m) < 1 \text{ for } m = 1, 2, \cdots, \quad (29)$$

The proof of this relation proceeds on mathematical induction.

It is easy to show that (29) holds for $m = 0$ and $m = 1$. Suppose that (29) is proved already for $0, 1, \cdots, m$. Then we are going to show it for $m+1$.

By (27) and (29), we have

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$$\Delta_{\alpha(1)}u(k_m-1)^{\gamma}-\frac{\eta_m}{\beta_m^{\gamma}}\left(\sum_{s=1}^{m}\beta_{s-1}\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}$$

$$\le(1-y(m))(\Delta_{\alpha(1)}u(k_m-1))^{\gamma}.$$

and by (28) and (29),

$$(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}<(1-y(m))(\Delta_{\alpha(1)}u(k_m-1))^{\gamma}$$

$$\le\frac{(1-y(m))}{y(m)}y(m)(\Delta_{\alpha(1)}u(k_m-1))^{\gamma}$$

$$\le\frac{1-y(m)}{y(m)}\frac{\eta_m}{\beta_m^{\gamma}}\left(\sum_{s=1}^{m}\beta_{s-1}\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}$$

or

$$\frac{y(m)}{1-y(m)}\frac{\beta_m^{\gamma}}{\eta_m}(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}<\left(\sum_{s=1}^{m}\beta_{s-1}\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}.$$

By adding $\beta_m^{\gamma}(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}$ to both sides and using Lemma 3, we obtain

$$\frac{\beta_m^{\gamma}}{\eta_m}(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}\frac{y(m)}{1-y(m)}+\eta_m$$

$$<\left(\sum_{s=1}^{m}\beta_{s-1}\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}+\beta_m^{\gamma}(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}$$

$$\le\left(\sum_{s=1}^{m+1}\beta_{s-1}\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}$$

By multiplying the resulting inequality by $\eta_{m+1}/\beta_{m+1}^{\gamma}$, we get

$$y(m+1)(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}<\frac{\eta_{m+1}}{\beta_{m+1}^{\gamma}}\left(\sum_{s=1}^{m+1}\beta_{s-1}\Delta_{\alpha(1)}u(k_s-1)\right)^{\gamma}$$

$$<(\Delta_{\alpha(1)}u(k_{m+1}-1))^{\gamma}. \qquad (30)$$

Where $y(m+1)$ is given by (25). This proves (29) for $m+1$ because $y(m+1)>0$ follows from (25), and $y_{m+1}<1$ follows from (30) and from (27) applied to $m+1$ instead of m. Hence the induction step is completed and (29) holds for all $m\in\mathbb{N}_1$. All this was made under the indirect assumption that there exists a solution $u(k)$ with $u(k)>0$ for all sufficiently large k. According to the assumptions of Theorem 6, the recurrence relation (25) has no solution for all m, and this contradiction proves our theorem.

Remark 7. *For the case where $\gamma=1$, we observe that in [10], Theorems 5 and 6] the real valued function k_m is $2^m k_0$, therefore we have $\beta_m=2^m$ and $\theta_m=1/2$, i.e., θ_m is fixed for all $m\in\mathbb{N}_1$.*

IV. CONCLUSION

New oscillation and non oscillation theorems are obtained for the generalized second order quasilinear α-difference equation.

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