A Note on Weierstrass Transform of Hyperfunctions

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***Abstract:** In this paper the Weierstrass transform of Sato's hyperfunctions and some of its properties are studied using the concept of defining function of hyperfunctions and the Laplace transform of hyperfunctions.*

Keywords: Hyperfunctions, Laplace transform, Weierstrass transform

I. INTRODUCTION

Mikio Sato introduced the idea of hyperfunctions to mention his generalization about the concept of functions. Urs Graf used Sato's idea to generalize the concept of real variable function using the classical complex function theory of analytic function and applied various transforms like Laplace transform, Fourier transform, Hilbert transform, Mellin transforms, Hankel transform to a class of hyperfunctions in his book. [1]

In this paper Weierstrass transform is applied to a class of hyperfunctions having bounded exponential growth. We have established some properties of this background using the relation between Weierstrass and Laplace transforms

II. PRELIMINARIES

We denote the upper half-plane and lower half-plane of the plane C of complex numbers by $C_+ = \{z \in C: Iz>0\}$, $C_- = \{z \in C: Iz<0\}$ respectively.

Definition 2.1[1]: For an open interval I of the real line R, the open subset, $N(I)$ of the set of all complex numbers C is called a *complex neighborhood* of I, if $N(I) \setminus I$ is an open subset of $N(I)$.

We let $N_+(I) = N(I) \cap C_+$ and $N_-(I) = N(I) \cap C_-$

$O(N(I)\setminus I)$ is denoted as the ring of holomophic functions in $N(I)\setminus I$. For an interval I, subset of R, a function $F(z) \in O(N(I) \setminus I)$ can be written as $F(z) = F_+(z)$ for $z \in N_+(I)$ and $F_-(z)$ for $z \in N_-(I)$ where $F_+(z) \in O(N_+(I))$ and $F_-(z) \in O(N_-(I))$, are called upper and lower component of $F(z)$ respectively.Generally the upper component $F_+(z)$ and lower component $F_-(z)$ of $F(z)$ need not be related to each other

Definition 2.2[1]: Two functions $F(z)$ and $H(z)$ in $O(N(I) \setminus I)$ are equivalent if for $z \in N_1(I) \cap N_2(I)$, $H(z) = F(z) + \varphi(z)$, with $\varphi(z) \in O(N(I))$ where $N_1(I)$ and $N_2(I)$ are complex neighborhoods of I of $F(z)$ and $H(z)$ respectively.

Definition 2.3[1]: An equivalence class of functions $F(z) \in O(N(I) \setminus I)$ defined a hyperfunction $f(x)$ on I. It is denoted by $f(x) = [F(z)] = [F_+(z), F_-(z)]$. $F(z)$ is called defining or generating function of the hyperfunction. The set of all hyperfunctions defined on the real interval I is denoted by $B(I)$.

$$B(I) = O(N(I) \setminus I) \setminus O(N(I))$$

A real analytic function $\varphi(x)$ on I is defined by the fact that $\varphi(x)$ can be analytically continued to a full neighborhood U containing I. Then we have $\varphi(z) \in O(U)$. For a complex neighborhood $N(I)$ containing U we can write $B(I) = O(N(I) \setminus I) \setminus A(I)$, where $A(I)$ is the ring of all real analytic functions on I. Thus a hyperfunction $f(x) \in B(I)$ is determined by a defining function $F(z)$ which is holomorphic in an adjacent neighborhood of the real interval I. It is only determined upto a real analytic function on I.

$$f(x) = F(x+i0) - F(x-i0)$$

$$= \lim_{s\to 0^+} \{F_+(x+is) - F_-(x-is)\},$$

provided the limit exists.

Example [1]: *Dirac delta function* at *x=0* is $\delta(x) = [\frac{-1}{2\pi iz}]$, the defining function is $F(z) = \frac{-1}{2\pi iz}$. *F(z)* is defined except at *z=0*. At *z=0*, *F(z)* has an isolated singularity, a pole of order *1*. For every real number $x \neq 0$,

$\lim_{\varepsilon\to 0^+}\{F_+(x+i\varepsilon) - F_-(x-i\varepsilon)\}$ exists and equal to 0

Definition 2.4[1]: A hyperfunction $f(x)$ is called holomorphic at $x = t$, if the lower and upper component of the defining function can be analytically continued to a full(two- dimensional) neighborhood of the real point t i.e. the upper/ lower component can be analytically continued across t into the lower/upper half-plane.

Definition 2.5[1]: Let $f(x) = [F_+(z), F_-(z)]$ be a hyperfunction, holomorphic at both endpoints of the finite interval $[a, b]$, then the definite integral of *f(x)* over *[a,b]* is defined and denoted by

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$$\int_a^b f(x)dx = \int_{\gamma_{a,b}^+} F_+(z)dz - \int_{\gamma_{a,b}^-} F_-(z)dz = -\oint_{(a,b)} F(z)dz$$

where the contour $\gamma_{a,b}^+$runs in N_+ from a to b above the real axis, and the contour $\gamma_{a,b}^-$ is in N_- from a to b below the real axis.

Example: $\int_{-\infty}^{\infty} \delta(x)dx = -\oint \frac{-1}{2\pi iz}dz = 1$

Definition 2.6[1]: Let Σ_0 be the largest open subset of R where the hyperfunction $f(x) = [F(z)]$ is vanishing. Its complement $K_0 = R \setminus \Sigma_0$ is called the support of the hyperfunction $f(x)$. It is denoted by $suppf(x)$.

Let Σ_1 be the largest open subset of the real line where the hyperfunction $f(x) = [F(z)]$ is holomorphic. Its complement $K_1 = R\setminus \Sigma_1$ is called the singular support of the hyperfunction $f(x)$. It is denoted by $singsuppf(x)$.

Consider open sets $J = (a, 0) \cup (0, b)$ with some $a<0$ and some $b>0$ and compact subsets $K = [a', a''] \cup [b', b'']$ with $a<a' \le a'' <0$ and $0<b' \le b'' <b$. Also consider the following open neighborhoods $[-\delta, \infty) + iJ$ and $(-\infty, \delta] + iJ$ of R_+ and R_- respectively for some $\delta>0$

Introduce the subclass $O(R_+)$ of hyperfunctions $f(x) = [F(z)]$ on R having the following properties

(i)The support $suppf(x)$ is contained in $[0,\infty)$

(ii)Either the support $suppf(x)$ is bounded on the right by a finite number $\beta>0$ or among all equivalent defining functions, there exists, $F(z)$ defined in $[-\delta, \infty) + iJ$ such that for any compact $K \subset J$ there exist some real constant $M' >0$ and σ' such that $|F(z)| \le M'e^{\sigma' Rz}$ holds uniformly for all $z \in [0, \infty) + iK$.

Because $suppf(x) \subset R_+$ and since the singular support $singsuppf$ is a subset of the support, we have $singsuppf \subset R_+$. Therefore $f(x)$ is a holomorphic hyperfunction for all $x<0$. Moreover, the fact that $F_+(x + i0) - F_-(x - i0) = 0$ for all $x<0$ shows that $F(z)$ is real analytic on the negative real axis. Hence $f(x) \in O(R_+)$ implies that $\chi_{(-\varepsilon,\infty)}f(x) = f(x)$ for any $\varepsilon>0$.

Definition 2.7[1]: The subclass of hyperfunctions O (R_+) is called the class of *right-sided originals.*

For an unbounded support $supp\, f(x)$, let $\sigma_- = inf\, \sigma'$, where the infimum is taken over all σ' and all equivalent defining functions satisfying (ii). This number $\sigma_- = \sigma_-(f)$ is called the growth index of $f(x) \in O(R_+)$. It has the following properties

(i) $\sigma_- \le \sigma'$

(ii)For every $\varepsilon>0$ there is a σ' with $\sigma_- \le \sigma' \le \sigma_- + \varepsilon$ and an equivalent defining function $F(z)$ such that $|F(z)| \le M'e^{\sigma' Rz}$ uniformly for all $z \in [0, \infty) + iK$.

For bounded support supp $f(x)$, we let $\sigma_-(f) = -\infty$

Definition 2.8[1]: The Laplace transform of a right-sided original $f(x) = [F(z)] \in O(R_+)$ is defined as

$$\hat{f}(s) = L[f(x)](s)$$

$$= -\int_{\infty}^{(0+)} e^{-sz}F(z)dz$$

The image function $\hat{f}(s)$ of $f(x) \in O(R_+)$ is holomorphic in the right half-plane $Rs>\sigma_-(f)$

Similarly, we introduce the subclass $O(R_-)$ of hyperfunctions specified by

(i)The support $suppf(x)$ is contained in $R_- = (-\infty,0]$

(ii)Either the support $suppf(x)$ is bounded on the left by a finite number $\alpha<0$, or we demand that among all equivalent defining functions there is one, denoted by $F(z)$ and defined in $(-\infty, \delta]+iJ$ such that for any compact subset $K \subset J$ there are some real constants $M'' >0$ and σ'' such that $|F(z)| \le M''e^{\sigma'' Rz}$ holds uniformly for $z \in (-\infty, 0] + iK$.

Definition 2.9[1]: The set $O(R_-)$ is said to be the class of left-sided originals.

For an unbounded support let $\sigma_+ = sup\sigma''$, where the supremum is taken over all σ'' and all equivalent defining functions satisfying (ii). The number $\sigma_+ = \sigma_+(f)$ is called the growth index of $f(x) \in O(R_-)$. It has the properties

(i)$\sigma'' \le \sigma_+$.

(ii)For every $\varepsilon >0$ there is a σ'' such that $\sigma_+ - \varepsilon \le \sigma'' \le \sigma_+$ and a defining function $F(z)$ such that $|F(z)| \le M''e^{\sigma'' Rz}$ uniformly for $z \in (-\infty, 0] +iK$.

If the support is bounded, we set $\sigma_+(f) = +\infty$

Definition 2.10[1]: The Laplace transform of a left-sided original $f(x)=[F(z)]\in O(R_-)$ is defined by

$$\hat{f}(s) = L[f(x)](s)$$

$$= -\int_{-\infty}^{(0+)} e^{-sz}F(z)dz$$

The image function $\hat{f}(s)$ of $f(x) \in O(R_-)$ is holomorphic in the left half-plane $Rs<\sigma_+(f)$

With a left-sided original $g(x) \in O(R_-)$ with growth index $\sigma_+(g)$ and a right- sided original $f(x) \in O(R_+)$ with growth index $\sigma_-(f)$ form the hyperfunction $h(x) = g(x) + f(x)$ whose support is now the entire real axis. If $\hat{g}(s) = L[g(x)](s)$, $Rs<\sigma_+(g)$ and $\hat{f}(s) = L[f(x)](s)$, $Rs>\sigma_-(f)$ we may add the two image functions provided they have a common strip of convergence, i.e. $\sigma_-(f)<\sigma_+(g)$ holds.

Definition 2.11[1]: With $g(x) \in O(R_-)$,$f(x) \in O(R_+)$,$h(x) = g(x) + h(x)$,$L[h(x)](s) = \hat{g}(s) + \hat{f}(s)$, $\sigma_-(f)< Rs < \sigma_+(g)$, provided $\sigma_-(f) < \sigma_+(g)$.

Definition 2.12[1]: Hyperfunctions of the subclass $O(R_+)$ are of bounded exponential growth as $x \to \infty$ and hyperfunctions of the subclass $O(R_-)$ are of bounded exponential growth as $x \to -\infty$.

An ordinary function $f(x)$ is said to be of bounded exponential growth as $x \to \infty$, if there exists some real numbers $M'>0$ and σ' such that $|f(x)| \le M'e^{\sigma' x}$ for sufficiently large x. It is said to be of bounded exponential growth as $x \to -\infty$, if there exists some real numbers $M'' > 0$ and σ'' such that $|f(x)| \le M'' e^{\sigma'' x}$, for sufficiently negative large x

A function or a hyperfunction is said to be of bounded exponential growth, if it is of bounded exponential growth for $x \to -\infty$ as well as for $x \to \infty$ .Thus a hyperfunction or ordinary function $f(x)$ has a Laplace transform, if it is of bonded exponential growth, and if $\sigma_-(f) < \sigma_+(f)$

Proposition [1]: If $f(x) = [F(z)]$ is a hyperfunction of bounded exponential growth which is holomorphic at $x = c$, then

$$-\int_{-\infty}^{(c+)} e^{-sz}F(z)dz = \int_{-\infty}^{(c)} e^{-sx}f(x)dx,$$

$$-\int_{\infty}^{(c+)} e^{-sz}F(z)dz = \int_{c}^{\infty} e^{-sx}f(x)dx, \text{ thus}$$

$$-\left\{\int_{-\infty}^{(c+)} e^{-sz}F(z)dz + \int_{\infty}^{(c+)} e^{-sz}F(z)dz\right\}$$
$$= \int_{-\infty}^{\infty} e^{-sx}f(x)dx$$

Proposition [1]: Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and holomorphic at some point $x = c$ with an arbitrary support. If in addition

$\sigma_- = \sigma_-(\chi_{(0,\infty)}f(x)) < \sigma_+ = \sigma_+(\chi_{(-\infty,0)}f(x))$, then its Laplace transform is given by

$$L[f(x)](s) = L[\chi_{(-\infty,c)}f(x)](s) + L[\chi_{(c,\infty)}f(x)](s)$$
$$= \int_{-\infty}^{(c)} e^{-sx}f(x)dx + \int_{c}^{\infty} e^{-sx}f(x)dx$$
$$= \int_{-\infty}^{\infty} e^{-sx}f(x)dx$$

Definition 2.13 [2]: The conventional generalized Weierstrass transform of $f(x)$ with parameter t is defined by

$$g(s,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x)dx, t > 0$$

III. WEIERSTRASS TRANSFORMATION OF HYPERFUNCTIONS

Using the Laplace transform of hyperfunction Urs Graf defined the Weierstrass transform of a hyperfunction in his book [1].Here we are considering the Weierstrass transforms of hyperfunctions having bounded exponential growth.

Definition 3.1: Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth. The generalized Weierstrass transform of $f(x)$ with parameter $t > 0$ is defined by

$$W_t[f(x)](s) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x)dx$$
$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} F(z)dz$$

Proposition 3.2: $W_t[f(x)](s)$ is holomorphic

Proof: Kernel of the transformation $e^{\frac{-(s-z)^2}{4t}}$ is holomorphic for all $t>0$. Hence the integrand becomes a hyperfunction with integral depending on s holomorphically.

Proposition 3.3: The Weierstrass transform of a hyperfunction is injective

Proposition 3.4: Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth. Then

$$W_t[f(x)](s) = \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} L\left[e^{\frac{-x^2}{4t}} f(x)\right]\left(\frac{-s}{2t}\right),$$

if $\sigma_-(f) < R(\frac{-s}{2t}) < \sigma_+(f), t>0$

Proof:

$$W_t[f(x)](s) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x)dx$$
$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\left(\frac{-s^2}{4t} + \frac{xs}{2t} - \frac{x^2}{4t}\right)} f(x)dx$$
$$= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(-s)x}{2t}} e^{\left(-\frac{x^2}{4t}\right)} f(x)dx$$
$$= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} L\left[e^{\frac{-x^2}{4t}} f(x)\right]\left(\frac{-s}{2t}\right),$$

if $\sigma_-(f) < R(\frac{-s}{2t}) < \sigma_+(f), t>0$

Proposition 3.5: Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth, holomorphic at $x = c$ having an arbitrary support, $t>0$, satisfies

$$\sigma_- = \sigma_-\left(\chi_{(0,\infty)} e^{-\frac{x^2}{4t}} f(x)\right) < \sigma_+ = \sigma_+\left(\chi_{(-\infty,0)} e^{-\frac{x^2}{4t}} f(x)\right)$$

Then
$W_t[f(x)](s) =$
$W_t[\chi_{(-\infty,c)}f(x)](s) +$
$W_t[\chi_{(c,\infty)}f(x)](s)$

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Proof: If

$$\sigma_- = \sigma_-\left(\chi_{(0,\infty)}e^{-\frac{x^2}{4t}}f(x)\right) < \sigma_+ = \sigma_+\left(\chi_{(-\infty,0)}e^{-\frac{x^2}{4t}}f(x)\right)$$

Then

$$L\left[e^{\frac{-x^2}{4t}}f(x)\right]\left(\frac{-s}{2t}\right) = L\left[\chi_{(-\infty,c)}e^{\frac{-x^2}{4t}}f(x)\right]\left(\frac{-s}{2t}\right) + L\left[\chi_{(c,\infty)}e^{\frac{-x^2}{4t}}f(x)\right]\left(\frac{-s}{2t}\right)$$

Using this relation and the previous proposition the result follows.

Proposition 3.6: Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth, analytic at the points x_i, $-\infty<x_1<x_2<\ldots<x_n<\infty$,with an arbitrary support, such that $L(f(x)) = \hat{f}(s)$ has $\sigma_-(f) < R(\frac{-s}{2t}) < \sigma_+(f)$, for $t > 0$ as the strip of convergence, then

$$W_t[f(x)](s)$$

$$= \frac{1}{\sqrt{4\pi t}}\left[\int_{-\infty}^{x_1} e^{\frac{-(s-x)^2}{4t}}f(x)dx + \int_{x_1}^{x_2} e^{\frac{-(s-x)^2}{4t}}f(x)dx + \cdots + \int_{x_n}^{\infty} e^{\frac{-(s-x)^2}{4t}}f(x)dx\right]$$

Proof: Using proposition 3.5 in[1], if $f(x) = [F(z)]$ is a hyperfunction having bounded exponential growth having arbitrary support and analytic at the points x_i, $-\infty < x_1 < x_2 < \ldots < x_n < \infty$,such that $\hat{f}$ having the strip of convergence $\sigma_- < Rs < \sigma_+$ then its Laplace transform is given by

$$L[f(x)](s) = \int_{-\infty}^{x_1} e^{-sx}f(x)dx + \int_{x_1}^{x_2} e^{-sx}f(x)dx + \cdots + \int_{x_n}^{\infty} e^{-sx}f(x)dx]$$

Using this result and proposition 3.4 result follows.

Proposition 3.7: Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth. Then

$$Wt[f(-x)](-s) = Wt[f(x)](s)$$

Proposition 3.8: Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth, holomorphic at x=0 with an arbitrary support. Let $f_1(x) = \chi_{(-\infty,0)}e^{-\frac{x^2}{4t}}f(x) \in O(R-)$ and $f_2(x) = \chi_{(0,\infty)}e^{-\frac{x^2}{4t}}f(x) \in O(R+)$ with $\sigma_- = \sigma_-(f_2(x))$ and $\sigma_+ = \sigma_+(f_1(x))$.If $\sigma_-(f) < \sigma_-(f)$ then $W_t[f(x)](s) = W_t[f_1(-x)](s) + W_t[f_2(x)](s)$ with $-\sigma_+ < R(\frac{s}{4t}), \sigma_- < R\left(\frac{-s}{4t}\right), t > 0$

Proof: Follows from Proposition 3.6[1] and proposition 3.4

Proposition 3.9: For $t > 0$,

$W_t[\overline{f(x)}](s) = \overline{Wt[f(x)](\bar{s})}$ where $f(x) = [F(z)]$ is a hyperfunction of bounded exponential growth

Proof:

$$W_t[\overline{f(x)}](s) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}}\overline{f(x)}dx$$

$$= \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}}L\left[e^{-\frac{x^2}{4t}}f(x)\right](\frac{-s}{2t})$$

$$= \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}}\overline{L\left[e^{-\frac{x^2}{4t}}f(x)\right](\frac{-\bar{s}}{2t})}$$

$$= \overline{Wt[f(x)](\bar{s})}$$

Example 3.10: Weierstrass transform of Dirac's delta function as a hyperfunction is

$$W_t[\delta(x)](s)$$

$$= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}}\delta(x)dx$$

$$= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}}\frac{-1}{2\pi iz}dz$$

$$= \frac{-1}{2\pi i\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}}\frac{1}{z}dz$$

$$= \frac{-1}{2\pi i\sqrt{4\pi t}}Res_{z=0}\left[\frac{e^{\frac{-(s-z)^2}{4t}}}{z}\right]$$

$$= \frac{-1}{2\pi i\sqrt{4\pi t}}e^{\frac{-s^2}{4t}}$$

IV. OPERATIONAL PROPERTIES

Proposition 4.1: Let $f_1(x)$ and $f_2(x)$ are any two hyperfunctions having bounded exponential growth with Weierstrass transforms $W_t[f_1(x)](s)$ *and* $W_t[f_2(x)](s)$. If the two image functions have a non empty intersection then for constants c_1, c_2,

$Wt[c1f1(x) + c2f2(x)](s) = c1Wt[f1(x)](s) + c2Wt[f2(x)](s)$, where s belongs to the common strip of convergence

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Proof:

$W_t[c_1 f_1(x) + c_2 f_2(x)](s)$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} (c_1 f_1(x) + c_2 f_2(x)) dx$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} c_1 f_1(x) dx + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} (c_2 f_2(x)) dx$$

$$= W_t[c_1 f_1(x)](s) + W_t[c_2 f_2(x)](s)$$

Proposition 4.2: If a hyperfunction $f(x) = [F(z)]$ has Weierstrass transform $W_t[f(x)](s)$ and has the canonical splitting $f(x) = f_1(x) + f_2(x)$ and then

$W_t[f(x)](s) = W_t[f_1(x)](s) + W_t[f_2(x)](s)$

Proposition 4.3: Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth and c be a constant, then

$W_t[e^{cx} f(x)](s) = W_t[f(x-c)](s)$,

if $\sigma_-(f) + \mathrm{R}c < \mathrm{R}(\frac{-s}{2t}) < \sigma_+(f) + \mathrm{R}c$

Proposition 4.4: Let c be a non zero constant real number and $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth having the Weierstrass transformation $W_t[f(x)](s)$ and $\sigma_-(f) < \mathrm{R}s < \sigma_+(f)$ then $W_t[f(cx)](s) = \frac{1}{|c|} W_t[f(x)](s)$.The strip of convergence for $c > 0$ is $c\sigma_-(f) < \mathrm{R}s < c\sigma_+(f)$ and for $c < 0$ is $c\sigma_+(f) < \mathrm{R}s < c\sigma_-(f)$

Proof:

$W_t[f(cx)](s)$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(cx) dx$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} F(cz) dz$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} F(\psi) \frac{1}{c} d\psi$$

$$= \frac{1}{|c|\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} F(\psi) d\psi$$

$$= \frac{1}{|c|} W_t[f(x)](s)$$

Proposition 4.5: Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth and let c be a constant then $W_t[f(x+c)](s) = W_t[f(x)](s)$

Proposition 4.6: Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth having Weierstrass transform $W_t[f(x)](s)$.Then

$$W_t\left[e^{\frac{x^2}{4t}} f^n(x)\right](s) = (\tfrac{-s}{2t})^n W_t\left[e^{\frac{x^2}{4t}} f(x)\right](s)$$

Proof:

$W_t\left[e^{\frac{x^2}{4t}} f^n(x)\right](s)$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} e^{\frac{z^2}{4t}} F^n(z) dz$$

$$= \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}} L[f^n(x)](\tfrac{-s}{2t})$$

$$= \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}} (\tfrac{-s}{2t})^n L[f(x)](\tfrac{-s}{2t})$$

$$= (\tfrac{-s}{2t})^n \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t}} L[f(x)](\tfrac{-s}{2t})$$

$$= (\tfrac{-s}{2t})^n W_t\left[e^{\frac{x^2}{4t}} f(x)\right](s)$$

V. INVERSE WEIERSTRASS TRANSFORMATION OF HYPERFUNCTIONS

In [7] V. Karunakarn and T. Venugopal gave inversion formula for Weierstrass Transform for a subclass of generalized functions. Using this Weierstrass transform inversion formula of Hyperfunction can be defined as follows:

If $\tilde{f}(s) = W_t[f(x)](s)$ is the Weierstrass transform of $f(x) = [F(z)]$, a hyperfunction having bounded exponential growth then

$$f(x) = [F(z)] = \lim_{t\to 1} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s+iz)^2}{4t}} \tilde{f}(is) ds$$

VI. CONCLUSION

Using these ideas Weierstrass transform can be used to find the solution of partial differential equation problems having hyperfunction solution and also for initial value problems having initial value a hyperfunction.

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