Computation of Chromatic Numbers for New Class of Graphs and its Applications

B. Logeshwary Natarajan

Abstract*: In this paper, we represent the importance of several kinds of colouring techniques of graphs with illustrations. In this paper, we determine the graph colouring parameters like chromatic, achromatic and pseudoachromatic number for various graphs. In this paper, we compute the various results on chromatic and achromatic numbers for new class of graphs like central graph of cycle graph $C(C_n)$ and central graph of jelly fish $J(m,n)$. This research paper deals with the applications of various colouring techniques of graph in areas like automated differentiation, mobile network, optical network, medical data mining, game theory and radio network.

Index Terms*: Graph colouring, central graph, chromatic number, achromatic number.

I. INTRODUCTION

Graph is a pictorial representation of collection of objects that are linked by lines. The objects are called as vertices and the lines are called as edges. Graph theory acts as a tool to model a problem in varied field including network analysis, economics, chemistry, engineering, satellite navigation. The graph colouring problem is a well known problem used to solve conflicts in everyday life and real time situations [1], [2],[3].

II. MOTIVATION AND SIGNIFICANCE OF GRAPH COLOURING

Theory of graph colouring embarks with the task of assigning colours to the countries of a geographical map in such a manner that any two countries with the common border do not receive the same colour. If we mark the countries with points in the plane and join each pair of points by a line then we obtain a graph which is planar which is a famous Four Colour Problem that can be coloured with 4 colours. Suppose that there is a cost factor incurred each time when we add a new colour to the map then a map that uses n colours is more expensive than a map that (n-1) colours and so on. Mainly, theory of graph colouring is about resolving conflicts. In resolving the conflict problem, vertices that are adjacent must avoid getting same colours which in turn that they are in a permanent conflict. Hence the four colour problem was the real motivating factor for our research. [9].

III. COLOURING TECHNIQUES OF GRAPHS

Graph colouring is a technique used to solve the conflicts in the real life. In graph theory, graph colouring is a particular case of graph labelling. A proper colouring of a graph is an assignment of colours to its elements of a graph such that no two adjacent elements have the same colour. An improper colouring or pseudocolouring of a graph is an assignment of colours to the elements of the graph such that adjacent elements may share a common colour. A pseudocomplete colouring is not necessarily proper colouring if for every pair of distinct colours i,j there exists an edge (u,v) such that vertex u is coloured i and the vertex v is coloured j. A graph colouring is said to be complete if it is both proper and pseudocomplete [9],[10],[16].

Graph colouring is broadly classified into three types such as vertex colouring, edge colouring and face colouring [2],[16],[17].

Vertex colouring: Vertex colouring is the assignment of colours to the vertices of the graph such adjacent vertices are coloured with two different colours.

Edge Colouring: Edge colouring is the assignment of colours to the vertices of the graph such adjacent vertices are coloured with two different colours.

Face Colouring: Face colouring is the assignment of colours to the faces of the planar graph such adjacent faces are coloured with two different colours such that no two faces share a boundary with same colour.

IV. COLOURING PARAMETERS OF GRAPH WITH ILLUSTRATION

Chromatic number: The minimum number of colours used in complete colouring is called chromatic number of G. It is denoted by χ (G).

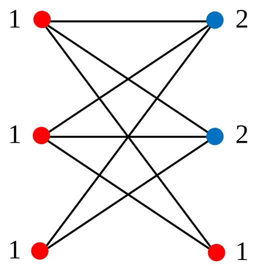

Figure 1: Chromatic number of $G(K_{3,3}-e)$, $\chi(K_{3,3}-e) = 2$

Achromatic number: The maximum number of colours used in complete colouring is called chromatic number of G. It is denoted by α(G).

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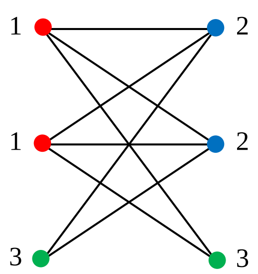

Figure 2: Achromatic number of G($K_{3,3}$–e), $\alpha(K_{3,3}$–e) = 3

V. COMPUTATION OF CHROMATIC NUMBERS FOR NEW CLASS OF GRAPHS

In this paper, we compute the chromatic for new class of graphs like central graphs of cycle graph C_n and jelly fish graph J(m,n) [16],[17].

Central graph: Let G be a undirected graph with no loops and parallel edges. The graph is formed by subdividing the each edge exactly once and joining all the non-adjacent vertices of the graph is called the central graph. It is denoted by the symbol C(G).

Theorem 5.1: Prove that the chromatic number for central graph of cycle of length n is 4. (i.e) $\chi(C(C_n))$ =4.

Proof:

Construction of Central graph of cycle C_n, $C(C_n)$: The cycle graph is denoted by the symbol C_n. The vertex set $V(C_n) = \{v_1, v_2, v_3, ..., v_n\}$ and edge set $E(C_n) = \{q_1, q_2, q_3, …, q_n\}$ where $q_i = v_i, v_{i+1}$, $1 \leq i \leq n-1$, $q_n = v_n, v_1$. The central graph of cycle of length n is denoted by $C(C_n)$. The central $C(C_n)$ is formed by subdividng each edge v_i, v_{i+1}, $1 \leq i \leq n-1$ of C_n exactly once by adding a new vertex u_i and subdividing v_n, v_1 by u_n and joining v_i with v_j, $1 \leq i,j \leq n$, $i \neq j$ and $v_i, v_j \notin E(C_n)$. The new vertex set formed is $V(C(C_n)) = \{ V_1 U U_1\}$ where $V_1 = \{v_1, v_2, v_3, ..., v_n\}$ and $U_1 = \{u_1, u_2, u_3, …, u_n\}$. The new edge set formed is $E(C(C_n)) = \{E_1 U E_2\}$ such that $E_1 = \{e_1', e_2', e_3', …, e_n'\}$ where $e_k' = v_i, v_j$; $1 \leq i,j \leq n$, k =1, 2, …, n; $i \neq j$; $v_i, v_j \notin E(C_n)$ and $E_2 = \{e_1'', e_2'', e_3'', …, e_n''\}$ where $e_i'' = u_i, v_{i+1}$ ¥ $i \neq j$; $e_i = v_i, u_j$ for i = j.

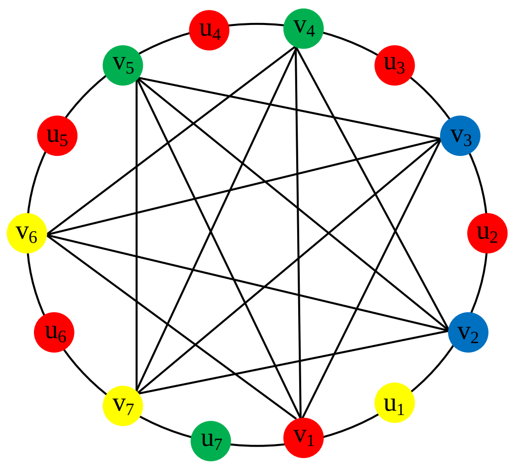

Figure 3: Central graph of cycle graph C_n,$C(C_n)$

Define the mapping ψ : $V(C(C_n))$ -> {1, 2, 3, 4} such that $\psi(v_1) = 1$, $\psi(v_2) = \psi(v_3) = 2$, $\psi(v_4) = \psi(v_5) = 3$, $\psi(v_6) = \psi(v_7) =$ and $\psi(u_2) = \psi(u_3) = \psi(u_4) = \psi(u_5) = \psi(u_6) = 1$; $\psi(u_1) = 2$, $\psi(u_7)$ = 3. Clearly it is easy to check that it is a proper colouring on vertices and hence we have $\chi(C(C_n)) \geq 4$. Suppose assume that $\psi(C(C_n))$ = 5 by some optimal colouring β. Then the colouring β assigns distinct colours to higher degree non-adjacent vertices. Therefore colours on v_1, v_2, v_4, v_7 must be distinct. Now the fifth colour must appear on any of the remaining vertices. If this happens, then there exists any one pair of colour with non-adjacent vertices does not have edge between them which is a contradiction. So we have $\psi(C(C_n)) \leq 4$. Therefore from the inequalities we arrive the result. Hence the chromatic number for central graph of cycle of length n is 4.(i.e) $\chi(C(C_n))$ = 4.

Theorem 5.2: Prove that the chromatic number for central graph of jelly fish graph 4. (i.e) $\chi(C(J(m,n)))$ =4.

Proof:

Construction of Central graph of jelly fish graph, C(J(m,n)): The Jelly fish graph is denoted by J(m,n). The vertex set V(Jm,n) = $\{x_1, x_2, x_3, x_4$ U $u_1, u_2, …\ u_m$ U $v_1, v_2, v_3, …\ v_n\}$ and edge set $E(C_n) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1), (x_1, x_3)$ U (x_4, u_i) U $(x_2, v_j)\}$ where i = 1, 2, 3 … n; j = 1, 2, 3 … m}. The central graph of jelly fish graph is denoted by the symbol C(J(m,n)). The central graph of C(J(m,n)) is formed by subdividng each edge x_i, x_{i+1}, $1 \leq i \leq n$ exactly once by adding a new vertex c_i and joining x_i with x_{i+2}; i = 1, 2, 3, 4 and inclusion of vertex c_5 between the edge (x_1, x_3). Also subdividing the pendant vertices connected by x_4 to p_i and x_2 to q_j and joining x_4 with p_i and x_2 with q_j where i = 1, 2, 3 … m; j = 1, 2, 3 … n. The new vertex set formed is V(C(J(mn))) = $\{x_1, x_2, x_3, x_4$ U c_1, c_2, c_3, c_4, c_5 U p_i U u_i U q_j U $v_j\}$. The new edge set is E(C(J(m,n)) = $\{\{(x_1, c_1), (x_1, c_2), (c_2, x_2), (x_2, x_3), (c_3, x_3), (c_4, x_3), (c_4, x_1), (c_4, x_4), (x_1, c_5), (x_3, c_5)$ U (x_4, p_i) U (p_i, u_i) U (x_2, q_j) U $(q_j, v_j)\}$ where i = 1, 2, 3, …, n; j = 1, 2, 3, …, m.

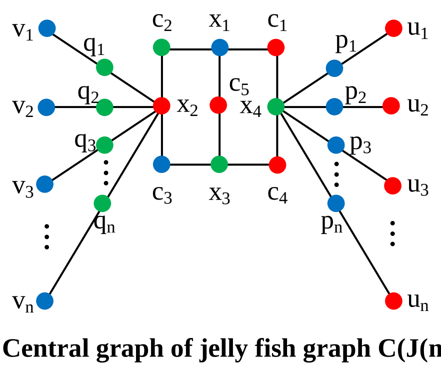

Figure 4: Central graph of jelly fish graph C(J(m,n))

Define the mapping ψ : V(C(Jm,n)) -> {1, 2, 3} such that $\psi(x_1) = 1$, $\psi(x_2) = 2$, $\psi(x_3) = 3$. The remaining vertices can be assigned any one of these colours with the proper colouring without loss of generality. Clearly it is easy to check that it is a proper vertex colouring on vertices and hence we have $\chi(C(Jm,n)) \geq 3$. Suppose assume that $\psi(C(Jm,n)) = 4$ by some optimal colouring μ. Then the colouring μ assigns different colours to higher degree non-adjacent vertices. Therefore colours on x_1, x_2, x_3 must be distinct. Now the fourth colour must appear on any of

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the remaining vertices. If this happens, then there exists any one pair of colour which does not have edge between them which is a contradiction to our assumption. So we have $\psi(C(J_n)) \leq 3$. Hence from the inequalities we arrive the required result. Hence the chromatic number for central graph of jellyfish graph is 3. (i.e) $\chi(C(Jm,n)) = 3$.

VI. A TOUR ON SELECTED KINDS OF GRAPH COLOURING TECHNIQUES WITH APPLICATIONS IN DIVERSIFIED ARENA

In this paper, we discuss about several kinds of colouring techniques of graphs, graph colouring parameters like chromatic and achromatic numbers with its applications in various fields. Graph colouring is one of the most essential concept in graph theory which has enormous applications in everyday life. In this paper, we discuss about several variants of graph colouring ideas and its real time applications in diversified fields.

6.1. Acyclic colouring and its Application in Automated Differentiation

A proper vertex colouring of a graph is said to be acyclic if adjacent vertices are coloured with two different colours and there exists no two coloured cycle in G. The least number of colours in an acyclic colouring is called the acyclic chromatic number. It is denoted by the symbol $\chi_A(G)$.

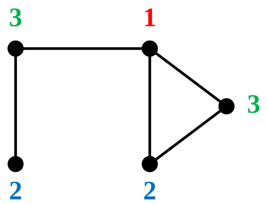

Figure 5: Acyclic chromatic number, $\chi_A(G) = 3$

The idea of acyclic colouring is widely applied in the field of computer science. Especially acyclic colouring is applied in the software technique of automated differentiation to compute sparse hessians. Automatic differentiation is a tool to transmute program from one domain into program of other domain that calculates the derivatives of that function. The result obtained from the execution of tool is the automated differentiation data. The technique is similar to divided differences that is, instead of evaluating the original data rather it builds a new data where it computes the derivatives along with original data. There are several modes in the process of automated differentiation like forward mode, reverse mode and combination mode. The implementation of program process through various methods like source to source transformation and operator overloading [1].

6.2. R-distinguishing colouring and its Application in Game Theory

A proper vertex colouring of a graph R-distinguishing colouring if no automorphism of the graph preserves all of the vertex colours or it destroys all of the nontrivial symmetries of the graph. The distinguishing chromatic number of a graph G is the minimum R such that graph G has a proper R-distinguishing colouring. It is denoted by the symbol $\chi_D(G)$.

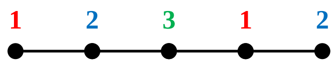

Figure 6: R-distinguishing chromatic number, $\chi_D(P_5) = 3$

The R-distinguishing colouring is applied in the field of game theory. Game theory is a self-governing discipline which is widely used in different areas applied science. The objective of game is to understand the rational behaviour. The distinguishing game is a game of two players with incompatible goals. The two players are represented as vertices with the a distinguishing colouring of a graph which assigns colours to the vertices of the graph which is not necessary proper that destroys all of the non-trivial symmetries of the graph. The games finishes when all the vertices have been coloured and the strategy of winning depends upon who commences the game [6],[13].

6.3. B-colouring and its Application in Medical Data Mining

A proper vertex colouring of a graph is said to be B-colouring such that each colour class contains a vertex that has a neighbour in all other colour classes. The largest number of colours in B-colouring is called the B-chromatic number. It is denoted by the symbol $\chi_b(G)$.

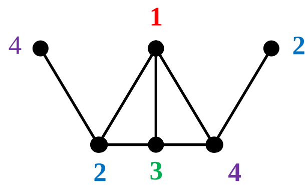

Figure 7: B-colouring chromatic number, $\chi_B(G) = 4$

The B-colouring idea is applied in the field of medical data mining especially in the area of clustering. Medical Data mining is the method of extracting hidden or concealed patterns from medical data. Clustering is the process of dividing the data into subsets where each entities in the subsets are likely to be dissimilar. B-colouring based clustering is partition problem of reducing the data sets into subclasses to the minimal colouring problem where each colour represents separate class and number of colours used to be optimised. This method inclines to build a partition of the data set with compact clusters where there is a less weightage to cluster separation [5],[12].

6.4. Fractional colouring and its Application in Optical Networks:

A fractional colouring of G is a non-negative real function f : I $\rightarrow$ I(G, u) where I is the independent set such that for any vertex u in G satisfies $\sum_{S\in I(G,u)} f(S) \geq 1$ where the sum of the values of f is called its weight. The minimum weight is called the fractional chromatic number. It is denoted by the symbol $\chi_f(G)$.

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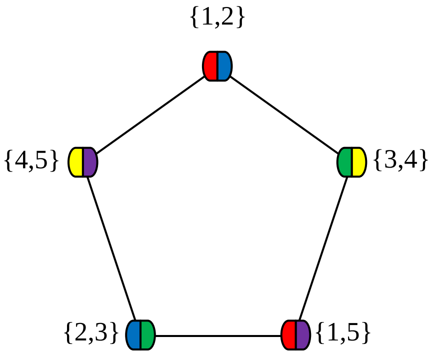

Figure 8: Fractional chromatic number $\chi_f(C_5) = 5/2$

The fractional colouring is applied in the field of optical networks. The graph colouring problem is an optimization problem consists of minimizing the colour set. In other words, it is the problem of finding the minimum cost. An optical network consists of n number of nodes and a collection of sets with an end to end communication requests. One must assign a each request a light path or link and each link a distinct colour to be assigned such that there is no conflicts with the light path. The objective is to optimize the colour set. Fractional colouring determines the result with natural linear programming for directed graph since optical transmissions are one way direction. This problem is referred as wavelength assignment problem [4],[8].

6.5. List colouring and its Application in Mobile Network

A list colouring is a proper vertex colouring such that every vertex *v* is assigned a colour from the list L(*v*) where L(*v*) is given list of colours for each vertex *v*. A graph is called *k* list colourable if it has a proper list colouring. The least number of colours in list colouring is called the list chromatic number. It is denoted by $\chi_l(G)$.

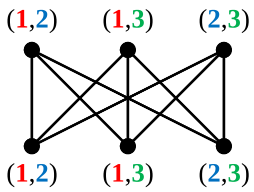

Figure 9: List chromatic number, $\chi_l(K_{3,3}) = 3$

The list colouring is applied in the field of mobile networks. The graph colouring problem in mobile network is the problem of colouring the graph with minimum number of colours. The mobile network deals with the problem of frequency assignment of network communication. In the list colouring problem each vertex of the graph is assigned a colour associated with a given list of colours such that each vertex is coloured with one of the colours in the list which is a proper vertex colouring of the graph. The frequency assignment problem is a interference graph where the graph represents the transmitter and edge represents an interference between transmitters. The mobile cellular network is divided into small regions called cells. Each cell acts as a service station connected to the network. The objective of the problem is to minimize the maximum frequency channel used in communication network [10],[11],[15].

6.6. Radio colouring and its Application in Radio Network

A proper vertex colouring of a graph G is said to be radio colouring such that two colours i and j can be assigned to two distinct vertices u and v only if it satisfies the conditions, $|f(x) - f(y)| \geq 1$, if $d(x, y) = 2$ and $|f(x) - f(y)| \geq 2$, if $d(x, y) = 1$. The least number of colours in radio colouring is called the radio chromatic number. It is denoted by $\chi_r(G)$.

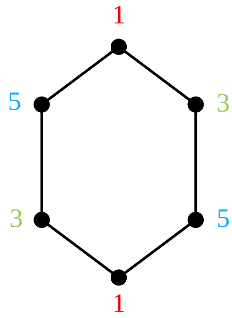

Figure 10: Radio chromatic number, $\chi_r(C_6) = 3$

The radio colouring idea is applied in the field of radio network. In radio colouring problem, the transmitters which are at distance two are assigned different channels. The potential source of interference is a channel separation. The interference constraints can be defined as a set of distances. The constraints of interference graph provides a minimum allowed distance for separation of channels assigned to each pair of transmitters. In radio networks, signal interference can be avoided by operating spatially adjacent transmitters at separated frequency channels. The objective of the radio colouring problem is to minimize the number of distinct colours used in radio colouring that is to minimize the maximum frequency channel used between adjacent transmitters. The frequency assignment problem in radio networks is similar to the graph colouring problem in the interference graph with the minimum number of colours used in radio colouring [7],[14].

VII. CONCLUSION

In this paper, we have explored the various types of graph colouring techniques with detailed explanation through different figures which becomes a driving force for young researchers. In this paper, we compute the various results on chromatic and achromatic numbers for new class of graphs like central graph of cycle graph $C(C_n)$ and central graph of jelly fish J(m,n). In this paper we determine various graph colouring parameters which makes the reader interesting. Our article emphasize importance in applying the graph colouring techniques on how to resolve the conflict problem in real life. This paper acts as a motivating factor for future readers in the field of graph theory.

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