Characterization Theorem for Fuzzy Functions on Time Scales Under Generalized Nabla Hukuhara Difference

R. Leelavathi, G.Suresh Kumar

Abstract: ***We establish a class of new derivative called nabla gH-derivative (generalized-Hukuhara) for fuzzy functions on time scales under generalized Hukuhara difference (gH-difference). We present a characterization theorem for multivalued fuzzy function G_λ which are generalized nabla (∇^{gh}) differentiable under gH-difference.***

Index Terms: Fuzzy functions, Generalized Hukuhara difference, Time scales.

I. INTRODUCTION

The primary approach to the Hukuhara-differentiability (H-differentiability) for fuzzy functions was based on the Hukuhara difference (H-difference)[4]. This H-difference exists only under certain conditions. To overcome this, Stefanini intoduced gH-difference for fuzzy valued functions that exists under less restrictive conditions [10]. On the other hand, stefan Hilger [10], developed the theory of time scales, which build-up the results for both theories simultaneously one for difference and differential equations[1]. Hong [3] studied the H-differentiability for multi-valued functions on time scales. Under gH-difference. Lupulescu [9], studied the interval differential equations on time scales (IDEs) using gH-difference. Fard [2], introduced delta-Hukuhara derivative (Δ_{gH}-derivative) and studied fuzzy functions on time scales using gH-difference. The author in [12]-[13] studied dynamic equation on time scales under different fuzzy derivatives.

Leela et al.[8] developed the theory for generalized nabla derivative (∇^g-derivative) for fuzzy functions on time scales using Hukuhara-difference. Recently, Leela et al.[6]-[7] studied first and second type nabla-Hukuhara derivative(∇_H-derivative). Nabla integrals for fuzzy functions on time scales was studied in [5]. In this context, we established nabla generalized Hukuhara derivative for fuzzy functions on time scales and study their properties. This paper is organized as follows. In section 2, we present some definitions, properties and fundamental results relating to calculus of fuzzy functions, fuzzy sets, and time scales calculus. Section 3 introduces nabla generalized Hukuhara (gh)-derivative of fuzzy functions on time scales and we proved a characterization theorem for (gh)-derivative of fuzzy functions on time scales.

II. PRELIMINARIES

Let $\Re_{\mathbb{F}}$ be the family of fuzzy real numbers defined as in [2] whose λ-level sets $[v]_\lambda = \{x \in \Re: \mu_V(x) \geq \lambda\}$, for $\lambda \in (0,1]$ and for $\lambda = 0$, $[V]_0 = cl\{x \in \Re; \mu_V(x) > 0\}$. Each λ-level set of a fuzzy set V is defined by its upper and lower end points $V^\lambda = [v_-^\lambda, v_-^\lambda] \subset \Re$.

A fuzzy set V is completely defined by any pair $V = [v_-, v_+]$ of functions $v_-, v_+ : [0,1] \to \Re$, defining the endpoints of the λ-level sets, and the follwwing holds:

$v_- : \lambda \to v_-^\lambda \in \Re$ is a left-continuous bounded monotonically non-decreasing function $\forall\ \lambda \in (0,1]$ and right-continuous for $\lambda = 0$; $v_+ : \lambda \to v_+^\lambda \in \Re$ is a left-continuous bounded monotonically non-increasing function $\forall\ \lambda \in (0,1]$ and right-continuous for $\lambda = 0$; $v_-^1 \leq v_+$ for $\lambda = 1$, which implies $v_-^\lambda \leq v_+^\lambda, \forall\ \lambda \in [0,1]$.

For any $S, T \in \Re_{\mathbb{F}}$, the Hukuhara-difference is defined by $S \ominus T = U \Leftrightarrow S = T \oplus U$. If $S \ominus T$ exists, its λ-level sets are $[S \ominus T]^\lambda = [x_-{}^\lambda - y_-{}^\lambda, x_+^\lambda - y_+{}^\lambda]$.

Definition 2.1.[11] The difference between two fuzzy sets $S, T \in \Re_{\mathbb{F}}$ is defined as

$$S \ominus_{gh} T = U \Leftrightarrow \begin{pmatrix} (a) S = T \oplus U, \quad (or) \\ (b) T = S \oplus (-1) \odot U, \end{pmatrix}$$

is called the generalized Hukuhara-difference (gh-difference) of S, T. If both (a) and (b) are valid then if U is an ordinary number. In terms of λ-level sets we have

$$[S \ominus_{gH} T]^\lambda = [\min\{x_-^\lambda - y_-^\lambda, x_+^\lambda - y_+^\lambda\}, \max\{x_-^\lambda - y_-^\lambda, x_+^\lambda - y_+^\lambda\}], = \begin{pmatrix} [x_-^\lambda - y_-^\lambda, x_+^\lambda - y_+^\lambda], \quad (or) \\ , \quad \forall \quad \lambda \in [0,1]. \end{pmatrix}$$

Now, we present some fundamental definitions and results on time scales.

Definition 2.7. [8] A Time scales is defined as a nonempty closed subset of (Real numbers) $\Re$ and is denoted by T.

(a) $\rho : \mathrm{T} \to \mathrm{T}$ is the jump operator and $\nu : \mathrm{T} \to \mathrm{P}^+$, the graininess operator are defined by

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$\rho(\theta)=\sup\{\theta_0\in T:\theta_0<\theta\}, \quad \nu(\theta)=\theta-\rho(\theta), \quad for \quad \theta\in T,$

(b) If $\theta > \inf T$ and $\rho(\theta)=\theta$, then θ is left-dense otherwise left-scattered.

(c) $T_k = T-\{m\}$, if T has a right-scattered minimum m. Otherwise $T_k = T$.

(d) A mapping $h^\rho : T\to\Re$ defined by where $h:T\to\Re$ is a function.

(e) The interval in time scale T is defined by

$$T^{[a,b]}=\{\theta\in T: a\le\theta\le b\}=[a,b]\cap T$$

and

$$T_k^{[a,b]}=\begin{cases}T^{[a,b]}, & \text{if a is right - dense}\\ T^{[\sigma(a),b]}, & \text{if a is right - scattered.}\end{cases}$$

Definition 2.8.[3]: Let $g:T\to\Re$ be a function and $\theta\in T_k$ Then $g^\nabla(\theta)$ exists as a number provided for any given $\varepsilon>0$, $\exists$ a neighbourhood N_δ of θ (i.e., $N_\delta=(\theta-\delta,\theta+\delta)\cap T$ for some $\delta>0$) such that

$$|[g(\rho(\theta))-g(\theta_0)]-g^\nabla(\theta)[\rho(\theta)-\theta_0]| \le \varepsilon|\rho(\theta)-\theta_0|,$$

for all $\theta_0\in N_\delta$. Thus, $g^\nabla(\theta)$ is called the nabla derivative of g at θ. g is said to be nabla-differentiable on T_k, if $g^\nabla(\theta)$ exists $\forall$ $\theta\in T_k$. The function $g^\nabla: T_k\to\Re$ is then called the nabla-derivative of g on T_k.

Definition 2.9. [3] A mapping $g:T\to\Re$ is said to be regulated if its left-sided limits exists and are finite at all ld-points in T and its right-sided limits exists and are finite at all rd-points in T.

Definition 2.10. [3] Let $g:T\to\Re$ be a function. g is said to be ld-continuous, if it is continuous at each ld-point in T and $\lim_{\theta\to 0^+} g(\theta)$ exists as a finite number for all rd-points in T.

For basic definitions, notations, results in time scales, we refer to [1].

II. PROPOSED METHODOLOGY

We proposed to develop the characterization theorem for fuzzy functions on time scales using gH-difference under nabla derivative. Upto now the results are in delta derivative. Since nabla derivative is more preferable than delta derivative and nabla derivative has many applications in cellular networks, production-inventory economical models, adaptive controls etc. In this context, we proposed to study the results under generalized Hukuhara difference methodology in nabla settings. These results are novel in the field of fuzzy difference and differential equations.

IV. RESULT AND DISCUSSIONS

Now, we present the definition of ∇_{gh}-derivative as well as the characterization results for fuzzy function.

Definition 3.1. Let $G:\mathbb{T}\to\Re_{\mathbb{F}}$ be a fuzzy function and $s\in\mathbb{T}_k$. Let $\nabla_{gH}G(t)$ be an element of $\Re_{\mathbb{F}}$ with the property that given any $\epsilon>0$, exists a neighborhood $P_{\mathbb{T}}$ of t for some $\delta>0$ $\ni$

$D_h[(G(t+h)\ominus_{gh} G(\rho(t)), G^{\nabla_{gh}}(t)(h+\nu(t))]\le\epsilon(h+\nu(t)),$

$D[(G(\rho(t))\ominus_{gh} G(t-h), \nabla_{gH}G(t)(h-\nu(t)]\le\epsilon(h-\nu(t)),$

for all $t+h, t-h\in P_{\mathbb{T}}$ with $0<h<\delta$. Then G is said to be ∇_{gh} differentiable at $t\in\mathbb{T}_k$.

Theorem 3.1.Let $[G(t)]^\lambda=[G_-^\lambda(t), G_+^\lambda(t)]$and $G:\mathbb{T}\to\Re_{\mathbb{F}}$ be a fuzzy function, for each $\lambda\in[0,1]$. If G is differentiable-∇_{gh}, then G_-^λ and G_+^λ are ∇-differentiable functions and

$$[G^{\nabla_{gh}}(t)]^\lambda=\begin{pmatrix}\left[G_-^{\nabla^\lambda}(t), G_+^{\nabla^\lambda}(t)\right],\\ (or)\\ \left[G_+^{\nabla^\lambda}(t), G_-^{\nabla^\lambda}(t)\right].\end{pmatrix}$$

, where the end-point functions are nabla differentiable.

proof: case(i): If G is differentiable-∇_{gh} at $t\in\mathbb{T}_k$ and t is left-scattered, for any $\lambda\in[0,1]$, we get

$$[G(\rho(t))\ominus_{gh} G(t)]^\lambda = \begin{cases}[G_-^\lambda(\rho(t))-G_-^\lambda(t), G_+^\lambda(\rho(t))-G_+^\lambda(t)](or)\\ [G_+^\lambda(\rho(t))-G_+^\lambda(t), G_-^\lambda(\rho(t))-G_-^\lambda(t)].\end{cases}$$

Now, Multipliying the above equation with $\frac{-1}{\nu(t)}$, we get

$$\frac{1}{\nu(t)}\odot\left[G(t+h)\ominus_{gh} G(\rho(t))\right]^\lambda$$

$$=\frac{1}{\nu(t)}\odot\left[G_-^\lambda(t+h)-G_-^\lambda(\rho(t)), G_+^\lambda(t+h)-G_+^\lambda(\rho(t))\right]$$

$$=\left[\frac{G_-^\lambda(t+h)-G_-^\lambda(\rho(t))}{\nu(t)}, \frac{G_+^\lambda(t+h)-G_+^\lambda(\rho(t))}{\nu(t)}\right]$$

$$=\left[\nabla G_-^\lambda, \nabla G_+^\lambda\right]$$

or

$$\frac{1}{\nu(t)}\odot\left[G(t+h)\ominus_{gh} G(\rho(t))\right]^\lambda$$

$$=\frac{1}{\nu(t)}\odot\left[G_+^\lambda(t+h)-G_+^\lambda(\rho(t)), G_-^\lambda(\rho(t))-G_-^\lambda(t)\right]$$

$$=\left[\frac{G_+^\lambda(t+h)-G_+^\lambda(\rho(t))}{\nu(t)}, \frac{G_-^\lambda(t+h)-G_-^\lambda(\rho(t))}{\nu(t)}\right]$$

$$=\left[\nabla G_+^\lambda, \nabla G_-^\lambda\right]$$

case(ii): If t is left-dense and G is differentiable-∇^{gh}. Then for any $\lambda\in[0,1]$, we get

$$[G(\rho(t)) \ominus_{gh} G(t)]^\lambda = \begin{cases}[G_-^{\lambda}(\rho(t)) - G_-^{\lambda}(t), G_+^{\lambda}(\rho(t)) - G_+^{\lambda}(t)]\,(or)\\ [G_+^{\lambda}(\rho(t)) - G_+^{\lambda}(t), G_-^{\lambda}(\rho(t)) - G_-^{\lambda}(t)].\end{cases}$$

Now, Multipliying the above equation with $\frac{-1}{\nu(t)}$, we get

$$\frac{1}{h} \odot \left[G(t-h)) \ominus_{gh} G(t)\right]^\lambda$$

$$= \frac{1}{h} \odot \left[G_-^{\lambda}(t-h) - G_-^{\lambda}(t), G_+^{\lambda}(t-h) - G_+^{\lambda}(t)\right]$$

$$= \left[\frac{G_-^{\lambda}(t-h) - G_-^{\lambda}(t)}{h}, \frac{G_+^{\lambda}(t-h) - G_+^{\lambda}(t)}{h)}\right]$$

$$= \left[\nabla G_-^{\lambda}, \nabla G_+^{\lambda}\right]$$

or

$$\frac{1}{h} \odot \left[G(t-h) \ominus_{gh} G(t)\right]^\lambda$$

$$= \frac{1}{h} \odot \left[G_+^{\lambda}(t-h) - G_+^{\lambda}(t), G_-^{\lambda}(t-h) - G_-^{\lambda}(t)\right]$$

$$= \left[\frac{G_+^{\lambda}(t-h) - G_+^{\lambda}(t)}{h}, \frac{G_-^{\lambda}(\rho(t)) - G_-^{\lambda}(t)}{h}\right]$$

$$= \left[\nabla G_+^{\lambda}, \nabla G_-^{\lambda}\right]$$

V CONCLUSION

In this paper, we established a new class of derivative called nabla generalized Hukuhara (gH)-derivative for fuzzy functions on time scales using (gH)-difference and we proved a characterization theorem for this new nabla generalized Hukuhara (gH-derivative) using gH-difference. This characterization theorem is useful to transform fuzzy derivative to crisp derivative. In our future work, we propose to study fuzzy nabla generalized integrals on time scales. Further, these concepts can be applied to study the fuzzy dynamic equations on time scales under generalized derivative.

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