

Ordered Neutrosophic Fuzzy Convergence Bitopological Spaces

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Abstract: In this paper we introduce the new concept of an ordered neutrosophic fuzzy convergence bitopological spaces (ONFCONVGBTS). Besides giving some interesting propositions. Tietze extension theorem for pairwise ordered NFlim extremally disconnected space (PNFLIMEDS) is established.

Keywords: ONFCONVGBTS, PNFLIMEDS, ordered τ_{lim_i} NFlimCF, lower (resp. upper) τ_{lim_i} NFlimCF, (i=1 or 2).

I. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [12]. Fuzzy sets have applications in many fields such as information [8] and control [9]. The concept of an ordered L-fuzzy bitopological spaces was introduced and studied in [2]. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy set, vague fuzzy set, interval-valued fuzzy sets, etc.

Tomasz Kubiaz[10,11] extended the Urysohn lemma and Tietze extension theorem for the L-fuzzy normal spaces. After the introduction of intuitionistic fuzzy sets and its topological spaces by Atanassov[1] and Coker[3], the concept of imprecise data called neutrosophic sets was introduced by Smarandache[4].

Later A. A. Salama [6,7] studied the neutrosophic topological space and neutrosophic filters. The concepts of separation axiom in an ordered neutrosophic bitopological space was studied by R.Narmada Devi[5].

In this paper, the concept of an ordered NF convergence bitopological space and pairwise ordered NFlim extremally disconnected spaces are introduced and studied. Some interesting propositions are discussed. Tietze extension theorem for pairwise ordered NFlim extremally disconnected spaces has been established.

II. PRELIMINARIES

Definition 2.1. [4] Let X be a nonempty set. A neutrosophic set A in X is defined as an object of the form $A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X \}$ such that $T_A, I_A, F_A : X \rightarrow [0, 1]$, and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

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Definition 2.2. [4] Let $A = \langle x, T_A(x), I_A(x), F_A(x) \rangle$ and $B = \langle x, T_B(x), I_B(x), F_B(x) \rangle$ be any two neutrosophic fuzzy sets in X . Then

- (i) $A \cup B = \langle x, T_{A \cup B}(x), I_{A \cup B}(x), F_{A \cup B}(x) \rangle$ where $T_{A \cup B}(x) = T_A(x) \vee T_B(x)$, $I_{A \cup B}(x) = I_A(x) \vee I_B(x)$ and $F_{A \cup B}(x) = F_A(x) \wedge F_B(x)$.
- (ii) $A \cap B = \langle x, T_{A \cap B}(x), I_{A \cap B}(x), F_{A \cap B}(x) \rangle$ where $T_{A \cap B}(x) = T_A(x) \wedge T_B(x)$, $I_{A \cap B}(x) = I_A(x) \wedge I_B(x)$ and $F_{A \cap B}(x) = F_A(x) \vee F_B(x)$.
- (iii) $A \subseteq B$ if $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x)$, for all $x \in X$.
- (iv) The complement of A is defined as $C(A) = \langle x, T_{C(A)}(x), I_{C(A)}(x), F_{C(A)}(x) \rangle$ where $T_{C(A)}(x) = 1 - T_A(x)$, $I_{C(A)}(x) = 1 - I_A(x)$ and $F_{C(A)}(x) = 1 - F_A(x)$.
- (v) $0_N = \{ \langle x, 0, 0, 1 \rangle : x \in X \}$ and $1_N = \{ \langle x, 1, 1, 0 \rangle : x \in X \}$.

III. VIEW ON NF CONVERGENCE BITOPOLOGICAL SPACES

Notation 3.1.

- (i) Y : nonempty set
- (ii) $\mathfrak{F}, \mathcal{G}$: neutrosophic fuzzy filters on Y
- (iii) $\mathcal{U}(Y)$: set of all neutrosophic fuzzy ultrafilters on X .
- (iv) $\mathcal{NF}_p(Y)$: set of all neutrosophic fuzzy prime filters on X .
- (v) $\mathcal{NF}_m(\mathfrak{F})$: set of all minimal prime neutrosophic fuzzy filters finer than \mathfrak{F} .
- (vi) ζ^Y : set of all neutrosophic fuzzy sets on Y .
- (vii) $\text{lim}\mathfrak{F}, \text{lim}\mathcal{G}, \text{lim}\mathfrak{h}, \text{lim}\mathcal{M}, \text{lim}\mathfrak{K}, \text{lim}\mathcal{L}$: neutrosophic fuzzy sets on Y .
- (viii) $\zeta^{\mathbb{R}}$: set of all neutrosophic fuzzy sets on the real line \mathbb{R} .
- (ix) $\mathcal{NF}(Y)$: set of all neutrosophic fuzzy filters on Y .

Definition 3.1. A nonempty collection \mathfrak{F} of elements in the lattice ζ^Y is called a neutrosophic fuzzy filter on Y provided

- (i) $0_N \notin \mathfrak{F}$,
- (ii) $D, E \in \mathfrak{F}$ implies $D \cap E \in \mathfrak{F}$,
- (iii) If $D \in \mathfrak{F}$ and $E \in \zeta^Y$ with $D \subseteq E$, then $E \in \mathfrak{F}$.

Definition 3.2. A neutrosophic fuzzy filter \mathfrak{F} is said to be neutrosophic fuzzy prime filter (or) prime neutrosophic fuzzy filter whenever, $L \cup M \in \mathfrak{F}$ implies $L \in \mathfrak{F}$ or $M \in \mathfrak{F}$.

Definition 3.3. A base for a neutrosophic fuzzy filter is a nonempty subset \mathcal{B} of ζ^Y obeying (i) $0_N \notin \mathcal{B}$, (ii) $L, M \in \mathcal{B}$ implies $L \cap M \supseteq H$, for some $H \in \mathcal{B}$.



The neutrosophic fuzzy filter generated by \mathcal{B} is denoted by $[\mathcal{B}] = \{ B \in \zeta^Y : D \supseteq E, D \in \mathcal{B} \}$.

Definition 3.4. Let H be any neutrosophic fuzzy set on Y . The characteristic $*$ set of $\mathfrak{F} \in \mathbb{NF}(Y)$ is defined by $c(\mathfrak{F}) = \cup_{H \in \mathfrak{F}} H$. Moreover, $c(\mathfrak{F}) = c(f\mathfrak{F})$.

Definition 3.5. Let $x_{r,p,s}$ be any *NFP*. A *NF* characteristic function of $x_{r,p,s}$ is denoted by

$$1_{x_{r,p,s}} = \langle x, T_{x_{r,p,s}}, I_{x_{r,p,s}}, F_{x_{r,p,s}} \rangle.$$

Definition 3.6. Any *NFS* $\alpha 1_{x_{r,p,s}}$ is of the form $\alpha 1_{x_{r,p,s}} = \langle y, \alpha T_{x_{r,p,s}}(y), \alpha I_{x_{r,p,s}}(y), \alpha F_{x_{r,p,s}}(y) \rangle$, where $r + p + s \leq 3$ and $\alpha \in (0, 1)$.

Notation 3.2. The collection of all neutrosophic fuzzy set of the form $\alpha 1_{x_{r,p,s}}$ is denoted by \mathcal{E} .

Definition 3.7. Any *NF* filter generated by \mathcal{E} is denoted and defined by $[\mathcal{E}] = \{ B \in \zeta^X : B \supseteq A, A \in \mathcal{E} \}$, however $[\mathcal{E}]$ is written as $\alpha 1_{x_{r,p,s}}$.

Definition 3.8. Given a nonempty set X , the pair (X, \lim) is said to be a neutrosophic fuzzy convergence space (*NFCONVGS*) where $\lim : \mathbb{NF}(X) \rightarrow \zeta^X$ provided:

- (i) For every $\mathfrak{F} \in \mathbb{NF}(X)$, $\lim \mathfrak{F} = \bigcap_{\mathfrak{G} \in \mathbb{NF}_m(\mathfrak{F})} \lim \mathfrak{G}$,
- (ii) For every $\mathfrak{F} \in \mathbb{NF}_p(X)$, $\lim \mathfrak{F} \subseteq c(\mathfrak{F})$,
- (iii) For every $\mathfrak{F}, \mathfrak{G} \in \mathbb{NF}_p(X)$, $\mathfrak{F} \subseteq \mathfrak{G} \Rightarrow \lim \mathfrak{G} \subseteq \lim \mathfrak{F}$.
- (iv) For every $\alpha \in (0,1)$ and $\alpha 1_{x_{r,p,s}} \in \mathcal{E}$,

$$\lim(\alpha 1_{x_{r,p,s}}) \supseteq \alpha 1_{x_{r,p,s}}.$$

Definition 3.9. Let (X, \lim) be a *NFCONVGS*. An operator *NFint* : $\zeta^X \rightarrow \zeta^X$ is defined by $NFint(\lim \mathfrak{F}) = \cup \{ \lim \mathfrak{G} : \lim \mathfrak{G} \subseteq \lim \mathfrak{F}, \text{ for every } \mathfrak{G} \in \mathbb{U}(X) \text{ and } \mathfrak{F} \in \mathbb{NF}(X) \}$.

Definition 3.10. Let (X, \lim) be any *NFCONVGS*. Then *NFlim* topology is defined by $\tau_{\lim} = \{ \lim \mathfrak{F} / NFint(\lim \mathfrak{F}) = \lim(\mathfrak{F}), \text{ for every } \mathfrak{F} \in \mathbb{NF}(X) \}$.

The pair (X, τ_{\lim}) is said be a neutrosophic fuzzy convergence topological space (*NFCONVGTs*). Every member of *NFlim* topology is a neutrosophic fuzzy *lim* open set (*NFlimOS*).

The complement of a neutrosophic fuzzy *lim* open set is a neutrosophic fuzzy *lim* closed set (*NFlimCS*).

Definition 3.11. Let (X, τ_{\lim}) be any *NFCONVGTs* and $\lim \mathfrak{F} \in \zeta^X$. Then the *NFlim* closure and *NFlim* interior of $\lim \mathfrak{F}$ are denoted and defined by

- (i) $NFlimcl(\lim \mathfrak{F}) = \bigcap \{ \lim \mathfrak{G} : \lim \mathfrak{G} \text{ is a } NFlimCS \text{ in } X \text{ and } \lim \mathfrak{F} \subseteq \lim \mathfrak{G} \}$,
- (ii) $NFlimint(\lim \mathfrak{F}) = \bigcup \{ \lim \mathfrak{G} : \lim \mathfrak{G} \text{ is a } NFlimOS \text{ in } X \text{ and } \lim \mathfrak{G} \subseteq \lim \mathfrak{F} \}$, $\forall \mathfrak{G}, \mathfrak{F} \in \mathbb{NF}(X)$.

Definition 3.12. An ordered neutrosophic fuzzy convergence topological space (*ONFCONVGTs*) is a triple (X, τ_{\lim}, \leq) where τ_{\lim} is a *NFlim* topology on X equipped with a partial order \leq .

Definition 3.13. Let (X, τ_{\lim}, \leq) be an *ONFCONVGTs*. Then a neutrosophic fuzzy set $\lim \mathfrak{F}$ in (X, τ_{\lim}, \leq) is said to be an

- (i) increasing neutrosophic fuzzy set (*increasing NFS*) if $x \leq y$ implies $\lim \mathfrak{F}(x) \subseteq \lim \mathfrak{F}(y)$.
That is, $T_{\lim \mathfrak{F}}(x) \leq T_{\lim \mathfrak{F}}(y)$, $I_{\lim \mathfrak{F}}(x) \leq I_{\lim \mathfrak{F}}(y)$ and $F_{\lim \mathfrak{F}}(x) \geq F_{\lim \mathfrak{F}}(y)$.
- (ii) decreasing neutrosophic fuzzy set (*decreasing NFS*) if $x \leq y$ implies $\lim \mathfrak{F}(x) \supseteq \lim \mathfrak{F}(y)$.
That is, $T_{\lim \mathfrak{F}}(x) \geq T_{\lim \mathfrak{F}}(y)$, $I_{\lim \mathfrak{F}}(x) \geq I_{\lim \mathfrak{F}}(y)$ and $F_{\lim \mathfrak{F}}(x) \leq F_{\lim \mathfrak{F}}(y)$.

Definition 3.14. Let (X, τ_{\lim}, \leq) be an *ONFCONVGTs* and $\lim \mathfrak{F}$ be any *NFS* in (X, τ_{\lim}, \leq) .

Then we define the increasing-*NF* closure, decreasing-*NFS* closure, increasing-*NFS* interior and decreasing-*NFS* interior of $\lim \mathfrak{F}$ respectively as follows:

- (i) $IncrNFcl(\lim \mathfrak{F}) = \bigcap \{ \lim \mathfrak{G} : \lim \mathfrak{G} \text{ is an increasing } NFlimCS \text{ in } X \text{ and } \lim \mathfrak{F} \subseteq \lim \mathfrak{G} \}$,
- (ii) $DecrNFcl(\lim \mathfrak{F}) = \bigcap \{ \lim \mathfrak{G} : \lim \mathfrak{G} \text{ is a decreasing } NFlimCS \text{ in } X \text{ and } \lim \mathfrak{F} \subseteq \lim \mathfrak{G} \}$,
- (iii) $IncrNFint(\lim \mathfrak{F}) = \bigcup \{ \lim \mathfrak{G} : \lim \mathfrak{G} \text{ is an increasing } NFlimOS \text{ in } X \text{ and } \lim \mathfrak{F} \subseteq \lim \mathfrak{G} \}$,
- (iv) $DecrNFint(\lim \mathfrak{F}) = \bigcup \{ \lim \mathfrak{G} : \lim \mathfrak{G} \text{ is a decreasing } NFlimOS \text{ in } X \text{ and } \lim \mathfrak{G} \subseteq \lim \mathfrak{F} \}$,

for every $\mathfrak{F}, \mathfrak{G} \in \mathbb{NF}(X)$.

Proposition 3.1. For any neutrosophic fuzzy set $\lim \mathfrak{F}$ in an *ONFCONVGTs*, the following statements holds

- (i) $C(IncrNFcl(\lim \mathfrak{F})) = DecrNFint(C(\lim \mathfrak{F}))$,
- (ii) $C(DecrNFcl(\lim \mathfrak{F})) = IncrNFint(C(\lim \mathfrak{F}))$,
- (iii) $C(IncrNFint(\lim \mathfrak{F})) = DecrNFcl(C(\lim \mathfrak{F}))$,
- (iv) $C(DecrNFint(\lim \mathfrak{F})) = IncrNFcl(C(\lim \mathfrak{F}))$.

Proof:

We shall prove (i) only, (ii),(iii) and (iv) can be proved in a similar manner.

(i) Let $\lim \mathfrak{F}$ be any neutrosophic fuzzy set in X . Then $IncrNFcl(\lim \mathfrak{F}) = \bigcap \{ \lim \mathfrak{G} : \lim \mathfrak{G} \text{ is an increasing } NFlimCS \text{ in } X \text{ and } \lim \mathfrak{F} \subseteq \lim \mathfrak{G} \}$, for every $\mathfrak{G}, \mathfrak{F} \in \mathbb{NF}(X)$.

Taking complement on both sides, we have $C(IncrNFcl(\lim \mathfrak{F})) = \bigcup \{ C(\lim \mathfrak{G}) : C(\lim \mathfrak{G}) \text{ is a decreasing } NFlimOS \text{ in } X \text{ and } C(\lim \mathfrak{G}) \subseteq C(\lim \mathfrak{F}) \} = DecrNFint(C(\lim \mathfrak{F}))$, for every $\mathfrak{G}, \mathfrak{F} \in \mathbb{NF}(X)$.

Definition 3.15. An ordered neutrosophic fuzzy convergence bitopological space (*ONFCONVGBTs*)



is an *OrderedNFCONVGBTS*

$(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ where τ_{lim_1} and τ_{lim_2} are the *NFlim* topologies on X equipped with a partial order \leq .

IV. CHARACTERIZATION OF *PONFlimEDS*

Definition 4.1. Let $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ be an *NFCONVGBTS*. Let $lim\mathfrak{F}$ be any τ_{lim_1} increasing (resp. decreasing) *NFlimOS* in X . If $IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F})$ (resp. $DecrNFcl_{\tau_{lim_2}}(lim\mathfrak{F})$) is τ_{lim_2} increasing (resp. decreasing) *NFlimOS*.

Then $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is said to be τ_{lim_1} upper (resp. lower) *NFlim* extremally disconnected space. Similarly we can define τ_{lim_2} upper (resp. lower) *NFlim* extremally disconnected space.

Definition 4.2. An *NFCONVGBTS* $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is said to be pairwise upper *NFlim* extremally disconnected (*PUpperNFlimEDS*) if it is both τ_{lim_1} upper *NFlim* extremally disconnected and τ_{lim_2} upper *NFlim* extremally disconnected.

Similarly we can define the pairwise lower *NFlim* extremally disconnected space (*PLowerNFlimEDS*).

Definition 4.3. An *NFCONVGBTS* $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is said to be pairwise ordered *NFlim* extremally disconnected (*PONFlimEDS*) if it is both *PUpperNFlimEDS* and *PLowerNFlimEDS*.

Proposition 4.1. For *NFCONVGBTS* $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$, the following statements are equivalent:

- (i) $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is *PUpperNFlimEDS*.
- (ii) $DecrNFint_{\tau_{lim_2}}(lim\mathfrak{F})$ is a τ_{lim_2} decreasing *NFlimCS*, for each τ_{lim_1} decreasing *NFlimCS* $lim\mathfrak{F}$. Similarly, $DecrNFint_{\tau_{lim_1}}(lim\mathfrak{F})$ is a τ_{lim_1} decreasing *NFlimCS*, for each τ_{lim_2} decreasing *NFlimCS* $lim\mathfrak{F}$.

$$(iii) DecrNFcl_{\tau_{lim_2}}(C(IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F}))) = DecrNFint_{\tau_{lim_2}}(C(lim\mathfrak{F})),$$

for each τ_{lim_1} increasing *NFlimOS* $lim\mathfrak{F}$.

$$\text{Similarly, } DecrNFcl_{\tau_{lim_1}}(C(IncrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}))) = DecrNFint_{\tau_{lim_1}}(C(lim\mathfrak{F})),$$

for each τ_{lim_2} increasing *NFlimOS* $lim\mathfrak{F}$.

- (v) For each pair of τ_{lim_1} increasing *NFlimOS* $lim\mathfrak{F}$ and τ_{lim_2} decreasing *NFlimOS* $lim\mathfrak{G}$ with $IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F}) = C(lim\mathfrak{G})$, $DecrNFcl_{\tau_{lim_2}}(lim\mathfrak{G}) = C(IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F}))$.

Similarly, for each pair of τ_{lim_2} increasing *NFlimOS* $lim\mathfrak{F}$ and τ_{lim_1} decreasing *NFlimOS* $lim\mathfrak{G}$ with $IncrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}) = C(lim\mathfrak{G})$, $DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{G}) = C(IncrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}))$.

Proof:

(i) \Rightarrow (ii) Let $lim\mathfrak{F}$ be any τ_{lim_1} decreasing *NFlimCS*. Then $C(lim\mathfrak{F})$ is τ_{lim_1} increasing *NFlimOS*.

By assumption(i) $IncrNFcl_{\tau_{lim_2}}(C(lim\mathfrak{F}))$ is τ_{lim_2} increasing *NFlimOS*.

Since, $IncrNFcl_{\tau_{lim_2}}(C(lim\mathfrak{F})) = C(DecrNFint_{\tau_{lim_2}}(lim\mathfrak{F}))$.

Thus $DecrNFint_{\tau_{lim_2}}(lim\mathfrak{F})$ is τ_{lim_2} decreasing *NFlimCS*.

(ii) \Rightarrow (iii) Let $lim\mathfrak{F}$ be any τ_{lim_1} increasing *NFlimOS*. Then $C(lim\mathfrak{F})$ is τ_{lim_1} decreasing *NFlimCS*.

By assumption(ii) $DecrNFint_{\tau_{lim_2}}(lim\mathfrak{F})$ is τ_{lim_2} decreasing *NFlimCS*. Consider

$$\begin{aligned} DecrNFcl_{\tau_{lim_2}}(C(IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F}))) \\ = DecrNFcl_{\tau_{lim_2}}(DecrNFint_{\tau_{lim_2}}(C(lim\mathfrak{F}))) \\ = DecrNFint_{\tau_{lim_2}}(C(lim\mathfrak{F})). \end{aligned}$$

(iii) \Rightarrow (iv) For each pair of τ_{lim_1} increasing *NFlimOS* $lim\mathfrak{F}$ and τ_{lim_2} decreasing *NFlimOS* $lim\mathfrak{G}$ with $IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F}) = C(lim\mathfrak{G})$.

By assumption (iii),

$$\begin{aligned} DecrNFcl_{\tau_{lim_2}}(C(IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F}))) \\ = DecrNFint_{\tau_{lim_2}}(C(lim\mathfrak{G})). \end{aligned}$$

By using Proposition 3.1 and by the hypothesis, we have $DecrNFcl_{\tau_{lim_2}}(lim\mathfrak{G})$

$$\begin{aligned} = DecrNFcl_{\tau_{lim_2}}(C(IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F}))) \\ = DecrNFint_{\tau_{lim_2}}(C(lim\mathfrak{F})) \\ = C(IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F})). \end{aligned}$$

(iv) \Rightarrow (i) Let $lim\mathfrak{F}$ be any τ_{lim_1} increasing *NFlimOS*. Put $lim\mathfrak{G} = C(IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F}))$. Clearly, $lim\mathfrak{G}$ is τ_{lim_2} decreasing *NFlimOS*.

By assumption (iv), it follow that

$$DecrNFcl_{\tau_{lim_2}}(lim\mathfrak{G}) = C(IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F})).$$

That is, $C(IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F}))$ is τ_{lim_2} decreasing *NFlimCS*. This implies that $IncrNFcl_{\tau_{lim_2}}(lim\mathfrak{F})$ is τ_{lim_2} increasing *NFlimOS*. Thus, $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is τ_{lim_1} upper *NFlim* extremally disconnected space.

Similarly, we can show that $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is τ_{lim_2} upper *NFlim* extremally disconnected space.



Hence $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is $\mathcal{P}UpperNFlim\mathcal{E}DS$.

Proposition 4.2. An $\mathcal{ONFC}ONVGBTS (X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is $\mathcal{PONFlim}\mathcal{E}DS$ iff for each τ_{lim_1} decreasing $NFlimOS$ $lim\mathfrak{F}$ and τ_{lim_2} decreasing $NFlimCS$ $lim\mathfrak{G}$ such that $lim\mathfrak{F} \subseteq lim\mathfrak{G}$, we have $DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}) \subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})$.

Proof:

Suppose $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is $\mathcal{P}UpperNFlim\mathcal{E}DS$. Let $lim\mathfrak{F}$ be any τ_{lim_1} decreasing $NFlimOS$ and $lim\mathfrak{G}$ be any τ_{lim_2} decreasing $NFlimCS$ such that $lim\mathfrak{F} \subseteq lim\mathfrak{G}$. Then by (ii) of Proposition 4.1, $DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})$ is τ_{lim_1} decreasing $NFlimCS$.

Also, since $lim\mathfrak{F}$ is τ_{lim_1} decreasing $NFlimOS$ and $lim\mathfrak{F} \subseteq lim\mathfrak{G} \subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})$. It follows that $DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}) \subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})$.

Conversely, let $lim\mathfrak{G}$ be any τ_{lim_2} decreasing $NFlimCS$. By Definition 3.14, $DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})$ is τ_{lim_1} decreasing $NFlimOS$ and $DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}) \subseteq lim\mathfrak{G}$.

By assumption,

$$DecrNFcl_{\tau_{lim_1}}(DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})) \subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}).$$

Also we know that

$$DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}) \subseteq DecrNFcl_{\tau_{lim_1}}(DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})).$$

Thus $DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})$

$$= DecrNFcl_{\tau_{lim_1}}(DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})).$$

Therefore, $DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})$ is τ_{lim_1} decreasing $NFlimCS$. Hence by (ii) of Proposition 4.1, it follows that $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is τ_{lim_1} upper $NFlim$ extremally disconnected space. Similarly we can prove the other cases.

Therefore, $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is $\mathcal{PONFlim}\mathcal{E}DS$.

Notation 4.1. A NFS which is both increasing (resp. decreasing) $NFlimOS$ and increasing (resp. decreasing) $NFlimCS$ is increasing (resp. decreasing) $NFlim$ clopen set.

Remark 4.1. Let $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ be a $\mathcal{P}UpperNFlim\mathcal{E}DS$. Let $\{lim\mathfrak{F}_i, C(lim\mathfrak{G}_i): i \in \mathbb{N}\}$ be a collection such that $lim\mathfrak{F}_i$'s are τ_{lim_1} decreasing $NFlimOS$ s and $lim\mathfrak{G}_i$'s are τ_{lim_2} decreasing $NFlimCS$ s.

Let $lim\mathfrak{F}$ and $C(lim\mathfrak{G})$ be τ_{lim_1} decreasing $NFlimOS$ and τ_{lim_2} increasing $NFlimOS$ respectively.

If $lim\mathfrak{F}_i \subseteq lim\mathfrak{F} \subseteq C(lim\mathfrak{G})$ and $lim\mathfrak{G}_i \subseteq C(lim\mathfrak{G}) \subseteq lim\mathfrak{G}$ for all $i, j \in \mathbb{N}$, then there exists a τ_{lim_1} and τ_{lim_2} decreasing $NFlim$ clopen sets $lim\mathfrak{H}$ such that

$$DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}_i) \subseteq lim\mathfrak{H} \subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}_j),$$

for all $i, j \in \mathbb{N}$.

Proof: By Proposition 4.2,

$$\begin{aligned} DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}_i) &\subseteq DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}) \cap DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}) \\ &\subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}_j), \end{aligned}$$

for all $i, j \in \mathbb{N}$.

Letting $lim\mathfrak{H} = DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}) \cap DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G})$ in the above, we have $lim\mathfrak{H}$ is τ_{lim_1} and τ_{lim_2} decreasing $NFlim$ clopen set satisfying the required condition.

Proposition 4.3. Let $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ be a $\mathcal{PONFlim}\mathcal{E}DS$. Let $\{lim\mathfrak{F}_q\}_{q \in \mathbb{Q}}$ and $\{lim\mathfrak{G}_q\}_{q \in \mathbb{Q}}$ be monotone increasing collections of τ_{lim_1} decreasing $NFlimOS$ s and τ_{lim_2} decreasing $NFlimCS$ s of $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ respectively.

Suppose that $lim\mathfrak{F}_{q_1} \subseteq lim\mathfrak{G}_{q_2}$ whenever $q_1 < q_2$ (\mathbb{Q} is the set of all rational numbers). Then there exists a monotone increasing collection $\{lim\mathfrak{H}_q\}_{q \in \mathbb{Q}}$ of τ_{lim_1} and τ_{lim_2} decreasing $NFlim$ clopen sets of $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ such that

$$\begin{aligned} DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}_{q_1}) &\subseteq lim\mathfrak{H}_{q_2} \text{ and} \\ lim\mathfrak{H}_{q_1} &\subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}_{q_2}) \end{aligned}$$

whenever $q_1 < q_2$.

Proof:

Let us arrange all rational numbers into a sequence $\{q_n\}$ (without repetitions). For every $n \geq 2$, we shall define inductively a collection $\{lim\mathfrak{H}_i / 1 \leq i \leq n\} \subset \zeta^X$ such that

$$DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}_q) \subseteq lim\mathfrak{H}_{q_i} \text{ if } q < q_i \text{ and}$$

$$lim\mathfrak{H}_{q_i} \subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}_q) \text{ if } q_i < q, \text{ for all } i < n \text{ (S}_n\text{)}$$

By Proposition 4.2, the countable collections $\{DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}_q)\}$ and $\{DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}_q)\}$ satisfy $DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}_{q_1}) \subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}_{q_2})$ if $q_1 < q_2$.

By Remark 4.1, there exists a τ_{lim_1} and τ_{lim_2} decreasing $NFlim$ clopen set $lim\mathfrak{R}_1$ such that

$$\begin{aligned} DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}_{q_1}) &\subseteq lim\mathfrak{R}_1 \subseteq \\ DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}_{q_2}) &). \text{ Letting } lim\mathfrak{H}_{q_i} = lim\mathfrak{R}_1, \text{ we get (S}_2\text{).} \end{aligned}$$

Assume that τ_{lim_1} NFS s $lim\mathfrak{H}_{q_i}$ are already defined for $i < n$ and satisfy (S_n) .

Define $\lim\mathcal{L} = \cup \{ \lim\mathfrak{S}_{q_i} \mid i < n, q_i < q_n \} \cup \lim\mathfrak{S}_{q_n}$ and
 $\lim\mathfrak{M} = \cap \{ \lim\mathfrak{S}_{q_j} \mid j < n, q_j > q_n \} \cap \lim\mathfrak{S}_{q_n}$.

Then we have that

$$\begin{aligned} & DecrNFcl_{\tau_{lim_1}}(\lim\mathfrak{S}_{q_i}) \\ & \subseteq DecrNFcl_{\tau_{lim_1}}(\lim\mathcal{L}) \subseteq DecrNFint_{\tau_{lim_1}}(\lim\mathfrak{S}_{q_j}) \\ & \text{and} \\ & DecrNFcl_{\tau_{lim_1}}(\lim\mathfrak{S}_{q_i}) \\ & \subseteq DecrNFint_{\tau_{lim_1}}(\lim\mathfrak{M}) \subseteq DecrNFcl_{\tau_{lim_1}}(\lim\mathfrak{S}_{q_j}) \end{aligned}$$

whenever $q_i < q_n < q_j$ ($i, j < n$).

As well as $\lim\mathfrak{S}_{q_i} \subseteq DecrNFcl_{\tau_{lim_1}}(\lim\mathcal{L}) \subseteq \lim\mathfrak{S}_{q'}$ and
 $\lim\mathfrak{S}_{q_j} \subseteq DecrNFint_{\tau_{lim_1}}(\lim\mathfrak{M}) \subseteq \lim\mathfrak{S}_{q'}$ whenever $q < q_n < q'$.

This shows that the countable collections

$\{ \lim\mathfrak{S}_{q_i} \mid i < n, q_i < q_n \} \cup \{ \lim\mathfrak{S}_{q_n} \mid q < q_n \}$ and
 $\{ \lim\mathfrak{S}_{q_j} \mid j < n, q_j > q_n \} \cup \{ \lim\mathfrak{S}_{q_n} \mid q > q_n \}$ together with
 $\lim\mathcal{L}$ and $\lim\mathfrak{M}$ fulfil the conditions of Remark 4.1.

Hence, there exists a τ_{lim_1} and τ_{lim_2} decreasing *NFlim* clopen set $\lim\mathfrak{R}_n$ such that

$$\begin{aligned} & DecrNFcl_{\tau_{lim_1}}(\lim\mathfrak{R}_n) \subseteq \lim\mathfrak{S}_{q_i} \text{ if } q_n < q, \\ & \lim\mathfrak{S}_{q_j} \subseteq DecrNFint_{\tau_{lim_1}}(\lim\mathfrak{R}_n) \text{ if } q < q_n, \quad DecrNFcl_{\tau_{lim_1}}(\lim\mathfrak{S}_{q_i}) \\ & \subseteq DecrNFint_{\tau_{lim_1}}(\lim\mathfrak{R}_n) \text{ if } q_i < q_n, \\ & DecrNFcl_{\tau_{lim_1}}(\lim\mathfrak{R}_n) \subseteq DecrNFint_{\tau_{lim_1}}(\lim\mathfrak{S}_{q_i}) \text{ if } q_n < q_j \\ & \text{where } 1 \leq i, j \leq n-1. \end{aligned}$$

Letting $\lim\mathfrak{S}_{q_n} = \lim\mathfrak{R}_n$ we obtain a τ_{lim_1} *NFSs* $\lim\mathfrak{S}_{q_1}, \lim\mathfrak{S}_{q_2}, \lim\mathfrak{S}_{q_3}, \dots, \lim\mathfrak{S}_{q_n}$ that satisfy **(S_n+1)**.

Therefore, the collection $\{ \lim\mathfrak{S}_{q_i} \mid i = 1, 2, \dots \}$ has the required property. This completes the proof.

Definition 4.4. Let $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ and $(Y, \sigma_{lim_1}, \sigma_{lim_2}, \leq)$ be any two *ONFCONVGBTS*.

Let $f: (X, \tau_{lim_1}, \tau_{lim_2}, \leq) \rightarrow (Y, \sigma_{lim_1}, \sigma_{lim_2}, \leq)$ be any *NF* function. Then f is said to be a

(i) τ_{lim_1} increasing *NFlim* continuous function (τ_{lim_1} increasing *NFlimCF*) if for every σ_{lim_1} (or) σ_{lim_2} *NFlimOS* $\lim\mathfrak{S}$ in $(Y, \sigma_{lim_1}, \sigma_{lim_2}, \leq)$, $f^{-1}(\lim\mathfrak{S})$ is a τ_{lim_1} increasing *NFlimOS* in $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$.

(ii) τ_{lim_1} decreasing *NFlim* continuous function (τ_{lim_1} decreasing *NFlimCF*) if for every σ_{lim_1} (or) σ_{lim_2} *NFlimOS* $\lim\mathfrak{S}$ in $(Y, \sigma_{lim_1}, \sigma_{lim_2}, \leq)$, $f^{-1}(\lim\mathfrak{S})$ is a τ_{lim_1} decreasing *NFlimOS* in $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$.

(iii) If f is both τ_{lim_1} increasing *NFlimCF* and τ_{lim_1} decreasing *NFlimCF* then it is said to be an ordered τ_{lim_1}

NFlim continuous function ($\mathcal{O}\tau_{lim_1}$ *NFlimCF*). Similarly we can define $\mathcal{O}\sigma_{lim_2}$ *NFlimCF*.

V. TIETZE EXTENSION THEOREM FOR *PONFCONVGBTS*

Definition 5.1. A *NF* real line $\mathbb{R}_I(I)$ is the set of all monotone decreasing *NFS* $A \in \zeta^{\mathbb{R}}$ satisfying

$$\cup \{ A(t) : t \in \mathbb{R} \} = 1^{\mathbb{N}} \text{ and } \cap \{ A(t) : t \in \mathbb{R} \} = 0^{\mathbb{N}}$$

after the identification of an *NFSs* $A, B \in \mathbb{R}_I(I)$ if and only if

$A(t-) = B(t-)$ and $A(t+) = B(t+)$ for all $t \in \mathbb{R}$ where
 $A(t-) = \cap \{ A(s) : s < t \}$ and $A(t+) = \cup \{ A(s) : s > t \}$.

The *NF* unit interval $\mathbb{I}_I(I)$ is a subset of $\mathbb{R}_I(I)$ such that $[A] \in \mathbb{I}_I(I)$ if the degrees of membership, indeterminate-membership and nonmembership of A are defined by

$$T_A(t) = \begin{cases} 1, & t < 0 \\ 0, & t > 1 \end{cases}, \text{ and } I_A(t) = \begin{cases} 1, & t < 0 \\ 0, & t > 1 \end{cases}, \quad F_A(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 1 \end{cases}$$

respectively.

The natural *NF* topology on $\mathbb{R}_I(I)$ is generated from the subbasis $\{ L_t^I, R_t^I : t \in \mathbb{R} \}$ where $L_t^I, R_t^I : \mathbb{R}_I(I) \rightarrow \mathbb{I}_I(I)$ are given by $L_t^I[A] = C(A(t-))$ and $R_t^I[A] = A(t+)$ respectively.

Definition 5.2. Let $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ be any *ONFCONVGBTS*. A function $f: X \rightarrow \mathbb{R}_I(I)$ is said to be

(i) lower τ_{lim_1} *NFlimCF* if $f^{-1}(R_t^I)$ is τ_{lim_1} increasing or decreasing *NFlimOS*,
(ii) upper τ_{lim_1} *NFlimCF* if $f^{-1}(L_t^I)$ is τ_{lim_1} increasing or decreasing *NFlimOS*, for each $t \in \mathbb{R}$.

Similarly we can define lower τ_{lim_1} *NFlimCF* and upper τ_{lim_2} *NFlimCF* respectively.

Notation 5.1. Let X be any nonempty set and $\lim\mathfrak{S} \in \zeta^X$. Then for $x \in X$, $\langle T_{\lim\mathfrak{S}}(x), I_{\lim\mathfrak{S}}(x), F_{\lim\mathfrak{S}}(x) \rangle$ is denoted by $\lim\mathfrak{S}^N$.

Proposition 5.1. Let $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ be any *ONFCONVGBTS* and let $\lim\mathfrak{S}$ be *NFS* in X .

$$\text{Let } f: X \rightarrow \mathbb{R}_I(I) \text{ be such that } f(x)(t) = \begin{cases} 1^{\mathbb{N}} & \text{if } t < 0 \\ \lim\mathfrak{S}^N & \text{if } 0 \leq t \leq 1 \\ 0^{\mathbb{N}} & \text{if } t > 1 \end{cases}$$



for all $x \in X$ and $t \in \mathbb{R}$. Then f is lower (resp. upper) τ_{lim_1} $NFlimCF$ iff $lim\mathfrak{F}$ is τ_{lim_1} increasing or decreasing $NFlimOS$ ($NFlimCS$).

Proof:

$$f^{-1}(R^{\mathbb{I}}_t)(t) = \begin{cases} 1_N & \text{if } t < 0 \\ lim\mathfrak{F} & \text{if } 0 \leq t \leq 1 \\ 0_N & \text{if } t > 1 \end{cases}$$

implies that f is lower τ_{lim_1} $NFlimCF$ continuous function iff $lim\mathfrak{F}$ is τ_{lim_1} increasing or decreasing $limOS$.

$$f^{-1}(C(L^{\mathbb{I}}_t))(t) = \begin{cases} 1_N & \text{if } t < 0 \\ lim\mathfrak{F} & \text{if } 0 \leq t \leq 1 \\ 0_N & \text{if } t > 1 \end{cases}$$

implies that f is upper τ_{lim_1} $NFlimCF$ iff $lim\mathfrak{F}$ is τ_{lim_1} increasing or decreasing $NFlimCS$. Hence the proof is complete.

Definition 5.3. A NF characteristic function of a NFS $lim\mathfrak{F}$ in X is a map $\psi_{lim\mathfrak{F}} : X \rightarrow \mathbb{I}_{\mathbb{I}}(I)$ is defined by $\psi_{lim\mathfrak{F}}(x) = lim\mathfrak{F}^N$, for each $x \in X$.

Remark 5.1. Let $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ be any $ONFCONVGBTS$. Let $\psi_{lim\mathfrak{F}}$ be NF characteristic function of a NFS $lim\mathfrak{F}$ in X . Then $\psi_{lim\mathfrak{F}}$ is lower (resp. upper) τ_{lim_1} $NFlimCF$ iff $lim\mathfrak{F}$ is τ_{lim_1} increasing or decreasing $NFlimOS$ ($NFlimCS$).

Proof: The proof follows Proposition 5.1.

Proposition 5.2. Let $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ be any $ONFCONVGBTS$. Then the following conditions are equivalent

- (i) $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ is $\mathcal{PONFlimEDS}$.
- (ii) If $g, h : X \rightarrow \mathbb{R}_{\mathbb{I}}(I)$, g is lower τ_{lim_1} $NFlimCF$, h is upper τ_{lim_2} $NFlimCF$ and $g \subseteq h$, then there exists increasing τ_{lim_1} and τ_{lim_2} $NFlimCF$ $f : X \rightarrow \mathbb{R}_{\mathbb{I}}(I)$ such that $g \subseteq f \subseteq h$.
- (iii) If $C(lim\mathfrak{F})$ is increasing τ_{lim_2} $NFlimOS$ and $lim\mathfrak{G}$ is decreasing τ_{lim_1} $NFlimOS$ such that $lim\mathfrak{G} \subseteq lim\mathfrak{F}$, then there exists an increasing τ_{lim_1} and τ_{lim_2} $NFlimCF$ $f : X \rightarrow \mathbb{R}_{\mathbb{I}}(I)$ such that $lim\mathfrak{G} \subseteq f^{-1}(C(L^{\mathbb{I}}_1)) \subseteq f^{-1}(R^{\mathbb{I}}_0) \subseteq lim\mathfrak{F}$.

Proof:

(i) \Rightarrow (ii) Define $lim\mathfrak{F}_r = h^{-1}(L^{\mathbb{I}}_r)$ and $lim\mathfrak{G}_r = g^{-1}(C(R^{\mathbb{I}}_r))$, for all $r \in \mathbb{Q}$ (\mathbb{Q} is the set of all rationals). Clearly, $\{lim\mathfrak{F}_r\}_{r \in \mathbb{Q}}$ and $\{lim\mathfrak{G}_r\}_{r \in \mathbb{Q}}$ are monotone increasing families of decreasing τ_{lim_1} $NFlimOS$ s and decreasing τ_{lim_2} $NFlimCS$ s respectively. Moreover $lim\mathfrak{F}_r \subseteq lim\mathfrak{G}_s$ if $r < s$.

By Proposition 4.3, there exists a monotone increasing family $\{lim\mathfrak{F}_r\}_{r \in \mathbb{Q}}$ of τ_{lim_1} and τ_{lim_2} decreasing $NFlim$ clopen sets of $(X, \tau_{lim_1}, \tau_{lim_2}, \leq)$ such that $DecrNFcl_{\tau_{lim_1}}(lim\mathfrak{F}_r) \subseteq lim\mathfrak{G}_s$ and

$lim\mathfrak{F}_r \subseteq DecrNFint_{\tau_{lim_1}}(lim\mathfrak{G}_s)$ whenever $r < s$ ($r, s \in \mathbb{Q}$).

Letting $lim\mathfrak{V}_t = \bigcap_{r < t} \{C(lim\mathfrak{F}_r)\}$ for all $t \in \mathbb{R}$, we define a monotone decreasing family $\{lim\mathfrak{V}_t \mid t \in \mathbb{R}\} \subseteq \zeta^X$.

Moreover we have

$IncrNFcl_{\tau_{lim_1}}(lim\mathfrak{V}_t) \subseteq IncrNFint_{\tau_{lim_1}}(lim\mathfrak{V}_s)$ whenever $s < t$.

We have,

$$\begin{aligned} \bigcup_{t \in \mathbb{R}} lim\mathfrak{V}_t &= \bigcup_{t \in \mathbb{R}} \bigcap_{r < t} C(lim\mathfrak{F}_r) \\ &\supseteq \bigcup_{t \in \mathbb{R}} \bigcap_{r < t} C(lim\mathfrak{F}_r) \\ &= \bigcup_{t \in \mathbb{R}} \bigcap_{r < t} g^{-1}(R^{\mathbb{I}}_r) \\ &= \bigcup_{t \in \mathbb{R}} g^{-1}(C(L^{\mathbb{I}}_t)) = g^{-1}(\bigcup_{t \in \mathbb{R}} C(L^{\mathbb{I}}_t)) = 1_N \end{aligned}$$

Similarly, $\bigcap_{t \in \mathbb{R}} lim\mathfrak{V}_t = 0_N$. Now define a function $f : X \rightarrow \mathbb{R}_{\mathbb{I}}(I)$ possessing required conditions.

Let $f(x)(t) = lim\mathfrak{V}_t(x)$, for all $x \in X$ and $t \in \mathbb{R}$. By the above discussion, it follows that f is well defined.

To prove f is increasing τ_{lim_1} and τ_{lim_2} $NFlimCF$. Observe that $\bigcup_{s > t} lim\mathfrak{V}_s = \bigcup_{s > t} IncrNFint_{\tau_{lim_1}}(lim\mathfrak{V}_s)$ and

$$\begin{aligned} \bigcap_{s < t} lim\mathfrak{V}_s &= \bigcap_{s < t} IncrNFcl_{\tau_{lim_1}}(lim\mathfrak{V}_s). \text{ Then } f^{-1}(R^{\mathbb{I}}_t) \\ &= \bigcup_{s > t} lim\mathfrak{V}_s = \bigcup_{s > t} IncrNFint_{\tau_{lim_1}}(lim\mathfrak{V}_s) \text{ is } \tau_{lim_1} \\ &\text{increasing or decreasing } NFlimOS \text{ and } f^{-1}(L^{\mathbb{I}}_t) = \bigcap_{s < t} lim\mathfrak{V}_s \\ &= \bigcap_{s < t} IncrNFcl_{\tau_{lim_1}}(lim\mathfrak{V}_s) \text{ is } \tau_{lim_1} \text{ increasing or decreasing} \end{aligned}$$

$NFlimCS$. Hence f is increasing τ_{lim_1} $NFlimCF$.

Similarly, we can prove f is increasing τ_{lim_2} $NFlimCF$ in same manner. Therefore, f is increasing τ_{lim_1} and τ_{lim_2} $NFlimCF$.

To conclude the proof it remains to show that $g \subseteq f \subseteq h$. That is $g^{-1}(C(L^{\mathbb{I}}_t)) \subseteq f^{-1}(C(L^{\mathbb{I}}_t)) \subseteq h^{-1}(C(L^{\mathbb{I}}_t))$ and $g^{-1}(R^{\mathbb{I}}_t) \subseteq f^{-1}(R^{\mathbb{I}}_t) \subseteq h^{-1}(R^{\mathbb{I}}_t)$ for each $t \in \mathbb{R}$. We have,

$$\begin{aligned} g^{-1}(C(L^{\mathbb{I}}_t)) &= \bigcap_{s < t} g^{-1}(C(L^{\mathbb{I}}_s)) \\ &= \bigcap_{s < t} \bigcap_{r < s} g^{-1}(R^{\mathbb{I}}_r) \\ &= \bigcap_{s < t} \bigcap_{r < s} C(lim\mathfrak{G}_r) \\ &\subseteq \bigcap_{s < t} \bigcap_{r < s} C(lim\mathfrak{F}_r) \end{aligned}$$



$$= \bigcap_{s < t} \lim \mathfrak{A}_s = f^{-1}(C(L_t^I))$$

and

$$\begin{aligned} f^{-1}(C(L_t^I)) &= \bigcap_{s < t} \lim \mathfrak{A}_s \\ &= \bigcap_{s < t} \bigcap_{r < s} C(\lim \mathfrak{F}_r) \\ &\subseteq \bigcap_{s < t} \bigcap_{r < s} C(\lim \mathfrak{F}_r) \\ &= \bigcap_{s < t} \bigcap_{r < s} h^{-1}(C(L_r^I)) \\ &= \bigcap_{s < t} h^{-1}(C(L_s^I)) = h^{-1}(C(L_t^I)) \end{aligned}$$

Similarly,

$$\begin{aligned} g^{-1}(R_t^I) &= \bigcup_{s > t} g^{-1}(R_s^I) \\ &= \bigcup_{s > t} \bigcup_{r > s} g^{-1}(R_r^I) \\ &= \bigcup_{s > t} \bigcup_{r > s} C(\lim \mathfrak{G}_r) \\ &\subseteq \bigcup_{s > t} \bigcap_{r < s} C(\lim \mathfrak{F}_r) \\ &= \bigcup_{s > t} \lim \mathfrak{A}_s = f^{-1}(R_t^I) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(R_t^I) &= \bigcup_{s > t} \lim \mathfrak{A}_s \\ &= \bigcup_{s > t} \bigcap_{r < s} C(\lim \mathfrak{F}_r) \\ &\subseteq \bigcup_{s > t} \bigcup_{r > s} C(\lim \mathfrak{F}_r) \\ &= \bigcup_{s > t} \bigcup_{r > s} h^{-1}(C(L_r^I)) \\ &= \bigcup_{s > t} h^{-1}(R_s^I) = h^{-1}(R_t^I) \end{aligned}$$

Hence, the condition (ii) is proved.

(ii)⇒(iii) Let $C(\lim \mathfrak{F})$ be an increasing τ_{\lim_2} *NFlimOS* and $\lim \mathfrak{G}$ be a decreasing τ_{\lim_1} *NFlimOS* such that $\lim \mathfrak{G} \subseteq \lim \mathfrak{F}$. Then $\psi \lim \mathfrak{G} \subseteq \psi \lim \mathfrak{F}$ where $\psi \lim \mathfrak{G}$ and $\psi \lim \mathfrak{F}$ are lower τ_{\lim_1} *NFlimCF* and upper τ_{\lim_2} *NFlimCF* respectively.

By (ii), there exists increasing τ_{\lim_1} and τ_{\lim_2} *NFlimCF* $f : X \rightarrow \mathbb{R}_I(I)$ such that $\psi \lim \mathfrak{G} \subseteq f \subseteq \psi \lim \mathfrak{F}$. Clearly, $f(x) \in \mathbb{R}_I(I)$ for all $x \in X$ and $\lim \mathfrak{G} = \psi^{-1} \lim \mathfrak{G}(C(L_1^I)) \subseteq f^{-1}(C(L_1^I)) \subseteq f^{-1}(R_0^I) \subseteq \psi^{-1} \lim \mathfrak{F}(R_0^I) = \lim \mathfrak{F}$.

Therefore, $\lim \mathfrak{G} \subseteq f^{-1}(C(L_1^I)) \subseteq f^{-1}(R_0^I) \subseteq \lim \mathfrak{F}$.

(iii)⇒(i) Since $f^{-1}(C(L_1^I))$ and $f^{-1}(R_0^I)$ are decreasing τ_{\lim_1} *NFlimCS* and decreasing τ_{\lim_2} *NFlimOS* respectively. By Proposition 4.2, $(X, \tau_{\lim_1}, \tau_{\lim_2}, \leq)$ is *PONFlimEDS*.

Notation 5.2. Let (X, τ_{\lim}) be any *NFCONVGTs*. Let $A \subset X$. Then $NFlim \chi_A$ is of the form $\langle x, \chi_A(x), \chi_A(x), 1 - \chi_A(x) \rangle$.

Proposition 5.3. Let $(X, \tau_{\lim_1}, \tau_{\lim_2}, \leq)$ is *PONFlimEDS*.

Let $A \subset X$ such that $\lim \chi_A$ is τ_{\lim_1} and τ_{\lim_2} *NFlimOS* in X respectively.

Let $f : (A, \tau_{\lim_1}/A, \tau_{\lim_2}/A) \rightarrow \mathbb{R}_I(I)$ be an increasing τ_{\lim_1} and τ_{\lim_2} *NFlimCF*. Then f has an increasing τ_{\lim_1} and τ_{\lim_2} *NFlim* continuous extension over $(X, \tau_{\lim_1}, \tau_{\lim_2}, \leq)$.

Proof:

Let $g, h : X \rightarrow \mathbb{R}_I(I)$ be such that $g = f = h$ on A and $g(x) = 0^N, h(x) = 1^N$ if $x \notin A$. For every $t \in \mathbb{R}$, We have,

$$g^{-1}(R_t^I) = \begin{cases} \lim \mathfrak{F}_t \bigcap \lim \chi_A, t \geq 0 \\ I_N, t < 0 \end{cases}$$

where $\lim \mathfrak{F}_t$ is τ_{\lim_1} increasing or decreasing *NFlimOS* such that $\lim \mathfrak{F}_t/A = f^{-1}(R_t^I)$ and

$$h^{-1}(L_t^I) = \begin{cases} \lim \mathfrak{G}_t \bigcap \lim \chi_A, t \leq 1 \\ I_N, t > 1 \end{cases}$$

where $\lim \mathfrak{G}_t$ is τ_{\lim_1} increasing or decreasing *NFlimOS* such that $\lim \mathfrak{G}_t/A = f^{-1}(L_t^I)$. Thus g is lower τ_{\lim_1} *NFlimCF* and h is upper τ_{\lim_2} *NFlimCF* such that $g \subseteq h$.

Hence by Proposition 5.2, there exists an increasing τ_{\lim_1} and τ_{\lim_2} *NFlimCF* $F : X \rightarrow \mathbb{R}_I(I)$ such that $g(x) \subseteq F(x) \subseteq h(x)$ for all $x \in X$. Hence for all $x \in A, f(x) \subseteq F(x) \subseteq f(x)$. So that F is the required extension of f over X .

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