Derivative based two-point Gauss Legendre rule for the Riemann-Stieltjes integral

P.M.Mohanty, S.N. Mohapatra, M.Acharya

Abstract: In this paper, derivative based two point Gauss-Legendre rule for the Riemann-Stieltjes integral is presented which uses derivative value in order to approximate the Riemann-Stieltjesintegral $\int_{-1}^{1} f(t)\,d\,g(t)$ *.This integral rule increases the order of the precision over the two point Gauss-Legendre rule meant for the Riemann-Stieltjes integration and the error term for the approximation is investigated.*

Key words: Riemann-Stieltjes integral; Gauss two point rule; error term.

Mathematics subject classification: 65D30, 65D32

I. INTRODUCTION

In mathematics Riemann-Stieltjes integral is a kind of generalization of the classical Riemann integral.It has applications in several fields like Operator theory, Probability and Statistics, Functional Analysis, Complex Analysis and others. Till now in comparison to the number of rules meant for the Riemann integrals, a very few number of quadrature rules are proposed for numerical evaluation of Riemann-Stieltjes integrals. In recent time, the approximation problems of Riemann-Stieltjes integrals have been taken attention by many researchers. Particularly,the quadrature rules by Asanov[1], Alomari[2], Mersor [3,4], Dragomir [5,6] and Zaho[10] are noteworthy.

Two point Gauss-Legendre rule is used to evaluate integrals of Riemann kind is having degrees of precision three whereas the two point Gauss Legendre Quadrature rule R_A derived by Alomary[2] to evaluate a Riemann–Stieltjes integral is

$$\int_{-1}^{1} f(t)\,d\,g(t) \approx R_A = Af\left(-\frac{\sqrt{3}}{3}\right) + Bf\left(\frac{\sqrt{3}}{3}\right) \quad (1)$$

where

$$\left.\begin{aligned} A &= \left(\frac{\sqrt{3}-3}{2\sqrt{3}}\right) g(1) - \left(\frac{\sqrt{3}+3}{2\sqrt{3}}\right) g(-1) + \frac{3}{2\sqrt{3}} \int_{-1}^{1} g(t)\,dt \\ B &= \left(\frac{\sqrt{3}+3}{2\sqrt{3}}\right) g(1) - \left(\frac{\sqrt{3}-3}{2\sqrt{3}}\right) g(-1) - \frac{3}{2\sqrt{3}} \int_{-1}^{1} g(t)\,dt \end{aligned}\right\} \quad (2)$$

is having degrees of precision one. Some authors like Zaho ([9], [11]) derived technique to enhance the degrees of precision of the quadrature rule by the use of the derivative term of the function. So in the present paper, our objective is to modify the two-point Gauss-Legendre rule meant for numerical evaluation of Riemann-Stieltjes integral by Alomari[2] so that the degree of precision of the rule will be three which is same as that of two-point Gauss-Legendre rule meant for Riemann integral and as the special case the rule will reduce to two-point Gauss-Legendre rule for the evaluation of Riemann integral if $g(x) = x$. The error calculation and the numerical verification of the derived rule is also consider.

II. DERIVATIVE BASED TWO-POINT GAUSS-LEGENDRE RULE FOR RIEMANN-STIELTJES INTEGRAL

In this section modified Alomary rule R_{MA} which is a derivative based two-point Gauss-Legendre rule for Riemann-Stieltjes integral is presented.

Theorem 1: Suppose that $f'(t)$ and $g(t)$ are continuous on $[-1,1]$ and $g(t)$ is increasing in $[-1,1]$. Then the proposedderivative based Gauss-Legendre two-point rule for Riemann-Stieltjes integral is

$$\int_{-1}^{1} f(t)\,d\,g(t) \approx R_{MA}(f) = Af\left(-\frac{\sqrt{3}}{3}\right) + B\,f\left(\frac{\sqrt{3}}{3}\right) + C\,f''(s) \quad (3)$$

where

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$$A=\left(\frac{\sqrt{3}-3}{2\sqrt{3}}\right)g(1)-\left(\frac{\sqrt{3}+3}{2\sqrt{3}}\right)g(-1)+\frac{3}{2\sqrt{3}}\int_{-1}^{1}g(t)\,dt$$

$$B=\left(\frac{\sqrt{3}+3}{2\sqrt{3}}\right)g(1)-\left(\frac{\sqrt{3}-3}{2\sqrt{3}}\right)g(-1)-\frac{3}{2\sqrt{3}}\int_{-1}^{1}g(t)\,dt$$

$$C=\frac{1}{3}g(1)-\frac{1}{3}g(-1)-\int_{-1}^{1}g(t)\,dt+\int_{-1}^{1}\int_{-1}^{t}g(x)\,dx\,dt$$

and

$$s=\frac{1}{\frac{1}{3}g(1)-\frac{1}{3}g(-1)-\int_{-1}^{1}g(t)\,dt+\int_{-1}^{1}\int_{-1}^{t}g(x)\,dx\,dt}\left(\frac{1}{9}g(1)+\frac{1}{9}g(-1)-\frac{4}{9}\int_{-1}^{1}g(t)\,dt+6\int_{-1}^{1}\int_{-1}^{t}g(x)\,dx\,dt-6\int_{-1}^{1}\int_{-1}^{t}\int_{-1}^{y}g(x)\,dx\,dy\,dt\right)$$

Proof: For the derivative based two-point Gauss-Legendre rule we want to find the values of *A*,*B*,*C* and *s* such that $\int_{-1}^{1}f(t)\,d\,g(t)=R_{MA}(f)$ for $f(t)=1,\ t,\ t^2$ and t^3. That is

$$\left.\begin{aligned}
&\int_{-1}^{1}1\,d\,g(t)=A+B\\
&\int_{-1}^{1}t\,d\,g(t)=-\frac{\sqrt{3}A}{3}+\frac{\sqrt{3}B}{3}\\
&\int_{-1}^{1}t^2\,d\,g(t)=\frac{A}{3}+\frac{B}{3}+2C\\
&\int_{-1}^{1}t^3\,d\,g(t)=-\frac{\sqrt{3}A}{9}+\frac{\sqrt{3}B}{9}+6sC
\end{aligned}\right\}\quad(4)$$

Solving equation(4) we get the system of equation in *A*,*B*,*C* and *s* as:

$$\left.\begin{aligned}
&\Rightarrow A+B=g(1)-g(-1)\\
&-\frac{\sqrt{3}A}{3}+\frac{\sqrt{3}B}{3}=g(1)+g(-1)-\int_{-1}^{1}g(t)\,dt\\
&\frac{A}{3}+\frac{B}{3}+2C=g(1)-g(-1)-2\int_{-1}^{1}g(t)\,dt+2\int_{-1}^{1}\int_{-1}^{t}g(x)\,dx\,dt\\
&-\frac{\sqrt{3}A}{9}+\frac{\sqrt{3}B}{9}+6s\,C=g(1)+g(-1)-3\int_{-1}^{1}g(t)\,dt+6\int_{-1}^{1}\int_{-1}^{t}g(x)\,dx\,dt-6\int_{-1}^{1}\int_{-1}^{t}\int_{-1}^{y}g(x)\,dx\,dy\,dt
\end{aligned}\right\}\quad(5)$$

Solving the above system of equation for *A*,*B*,*C* and *s* we get

$$\left.\begin{aligned}
&A=\left(\frac{\sqrt{3}-3}{2\sqrt{3}}\right)g(1)-\left(\frac{\sqrt{3}+3}{2\sqrt{3}}\right)g(-1)+\frac{3}{2\sqrt{3}}\int_{-1}^{1}g(t)\,dt\\
&B=\left(\frac{\sqrt{3}+3}{2\sqrt{3}}\right)g(1)-\left(\frac{\sqrt{3}-3}{2\sqrt{3}}\right)g(-1)-\frac{3}{2\sqrt{3}}\int_{-1}^{1}g(t)\,dt\\
&C=\frac{1}{3}g(1)-\frac{1}{3}g(-1)-\int_{-1}^{1}g(t)\,dt+\int_{-1}^{1}\int_{-1}^{t}g(x)\,dx\,dt
\end{aligned}\right\}\quad(6)$$

and

$$s=\frac{1}{\frac{1}{3}g(1)-\frac{1}{3}g(-1)-\int_{-1}^{1}g(t)\,dt+\int_{-1}^{1}\int_{-1}^{t}g(x)\,dx\,dt}\left(\frac{1}{9}g(1)+\frac{1}{9}g(-1)-\frac{4}{9}\int_{-1}^{1}g(t)\,dt+6\int_{-1}^{1}\int_{-1}^{t}g(x)\,dx\,dt-6\int_{-1}^{1}\int_{-1}^{t}\int_{-1}^{y}g(x)\,dx\,dy\,dt\right)$$

Hence, we derive the derivative based Gauss-Legendre two-point rule R_{MA} for the Riemann-Stieltjes integral. It is pertinent to note that the

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first two equations of equation (4) and as solution of that the values of *A* and *B* in (6) are exactly the same as of quadrature rule by Alomary[2].

Corollary 1: When $g(t)=t$ then the rule R_{MA} in equation (3) reduces to classical Gauss-Legendre two -point rule from the evaluation of Riemann integral.

Proof: For $g(t)=t$ it can be easily derived from equation (6) that *A*=*B*=1 and *C*=0, so the rule for the numerical evaluation of classical Riemann integral is

$$\int_{-1}^{1} f(t)\,dt \approx f\left(-\frac{\sqrt{3}}{3}\right)+f\left(\frac{\sqrt{3}}{3}\right)$$

Corollary 2: The degree of precision for the derivative based Gauss-Legendre two-point rule R_{MA} for the Riemann-Stieltjes integral is three.

Proof: From the derivation of *A*,*B*,*C* and *s* it is clear that the degree of precision for the derivative based Gauss-Legendre two-point rule R_{MA} for the Riemann-Stieltjes integral is three and it is clearly shown in section 3 that the equality does not holds for $f(t)=t^4$.

III. THE ERROR CALCULATION FOR THE DERIVATIVE BASED TWO-POINT GAUSS-LEGENDRE RULE

In numerical analysis error calculation is an important part and can be calculated in many different ways [5-7]. In this section, we derived the error term of the derivative based two-point Gauss-Legendre rule by the concepts of degrees of precision [8-9]. The error term is calculated as the difference between the value of rule for the monomial $\frac{t^{n+1}}{(n+1)!}$ and exact value of the integral $\frac{1}{(n+1)!}\int_{-1}^{1} t^{n+1} d\,g(t)$ where n is the degrees of precision of the derived rule.

Theorem 2: Suppose that $f'(t)$ and $g(t)$ are continuous on $[-1,1]$ and $g(t)$ is increasing in $[-1,1]$. Then the derivative based Gauss-Legendre two-point rule R_{MA} with an error is

$$\int_{-1}^{1} f(t)\,d\,g(t) \approx R_{MA}(f) = A\,f\left(-\frac{\sqrt{3}}{3}\right)+B\,f\left(\frac{\sqrt{3}}{3}\right) + C\,f''(s) + \left[\frac{2-9s^2}{54}g(1) + \frac{9s^2-2}{54}g(-1) + \frac{3s^2-1}{6}\int_{-1}^{1} g(t)\,dt + \frac{1-s^2}{2}\int_{-1}^{1}\int_{-1}^{t} g(x)\,dx\,dt - \int_{-1}^{1}\int_{-1}^{t}\int_{-1}^{y} g(x)\,dx\,dy\,dt + \int_{-1}^{1}\int_{-1}^{t}\int_{-1}^{z}\int_{-1}^{y} g(x)\,dx\,dy\,dz\,dt\right] f^{(4)}(\xi)\,g'(\eta)$$

Where, the last term of the right side is the error term of the derived rule and the values of *A*,*B*,*C* and *s* aregive in (6).

Proof: Let $f(t)=\frac{t^4}{4!}$ then

$$\frac{1}{4!}\int_{-1}^{1} t^4 d\,g(t) = \frac{1}{24}\left[g(1)-g(-1)\right] - \frac{1}{6}\int_{-1}^{1} g(t)\,dt + \frac{1}{2}\int_{-1}^{1}\int_{-1}^{t} g(x)\,dx\,dt - \int_{-1}^{1}\int_{-1}^{t}\int_{-1}^{y} g(x)\,dx\,dy\,dt + \int_{-1}^{1}\int_{-1}^{t}\int_{-1}^{z}\int_{-1}^{y} g(x)\,dx\,dy\,dz\,dt$$

But by the present derived rule

$$\frac{1}{4!}\int_{-1}^{1} t^4 d\,g(t) \approx R_{MA}(f) = \frac{1}{24}\left[A\left(-\frac{\sqrt{3}}{3}\right)^4 + B\left(\frac{\sqrt{3}}{3}\right)^4 + 12\,C\,s^2\right]$$

Substituting the values of *A*,*B*,*C* and *s* from (6) we getthe error term as

$$\frac{1}{4!}\int_{-1}^{1} t^4 d\,g(t) - R_{MA}(f) = \frac{2-9s^2}{54}g(1) + \frac{9s^2-2}{54}g(-1) + \frac{3s^2-1}{6}\int_{-1}^{1} g(t)\,dt + \frac{1-s^2}{2}\int_{-1}^{1}\int_{-1}^{t} g(x)\,dx\,dt + \int_{-1}^{1}\int_{-1}^{t}\int_{-1}^{y} g(x)\,dx\,dy\,dt + \int_{-1}^{1}\int_{-1}^{t}\int_{-1}^{z}\int_{-1}^{y} g(x)\,dx\,dy\,dz\,dt \qquad (7)$$

IV. NUMERICAL VERIFICATION AND CONCLUSION

In this section for the propose of the numerical verification of the derived derivative based two-point Gauss-Legendre rule R_{MA} we consider some values for $f(t)$ and $g(t)$, the absolute value of error is compared with the absolute value of error due to R_A Alomari[2], and R_M Mercer[3] in the following table and we briefly summarize our main result of this paper as follows:

- The derivative based two-point Gauss-Legendre rule R_{MA} for the evaluation of Riemann-Stieltjes integral is presented whose degree of precision is three.
- The error term of the derived quadrature rule is calculated and numerically the rule R_{MA} gives better result than Alomari[2], and Mercer[3].
- It is observed that the derived derivative based two-point Gauss Legendre rule for the evaluation of Riemann-Stieltjes integral is reduces to classical two-point Gauss-Legendre rule when

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$g(t)=t$ with same degrees of precision.

Table-I Error calculations

$f(t)$	$g(t)$	Error due to		
		R_A Alomari [2]	R_M Mercer [3]	R_{MA}
e^t	t^3	2.9396×10^{-1}	4.4951×10^{-1}	2.7291×10^{-2}
	t^5	4.1917×10^{-1}	3.2429×10^{-1}	3.8215×10^{-2}
$\cos(t)$	t^3	2.4102×10^{-1}	3.5419×10^{-1}	2.5645×10^{-2}
	t^5	3.4506×10^{-1}	2.5016×10^{-1}	3.5895×10^{-2}
$\cos(t)$	$\sin(t)$	4.4492×10^{-2}	5.4535×10^{-1}	3.1350×10^{-3}

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